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Non-renormalization theorems of Supersymmetric QED in the Wess–Zumino gauge

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Abstract

The non-renormalization theorem of chiral vertices and the generalized non-renormalization theorem of the photon self energy are derived in SQED on the basis of algebraic renormalization. For this purpose the gauge coupling is extended to an external superfield. This extension already provides detailed insight into the divergence structure. Moreover, using the local supercoupling together with an additional external vector multiplet that couples to the axial current, the model becomes complete in the sense of multiplicative renormalization, with two important implications. First, a Slavnov–Taylor identity describing supersymmetry, gauge symmetry, and axial symmetry including the axial anomaly can be established to all orders. Second, from this Slavnov–Taylor identity we can infer a Callan–Symanzik equation expressing all aspects of the non-renormalization theorems. In particular, the gauge β-function appears explicitely in the closed form.

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1 Introduction

Supersymmetric theories have always been famous for their extraordinary renormalization properties. Particular non-renormalization theorems state the absence of divergences to the superpotential [1, 2], and the β-functions of supersymmetric gauge theories have been given in a closed form as a function of a one-loop coefficient and the anomalous dimension of the matter fields [3, 4]. However, non-renormalization theorems have been derived in general only as a consequence of the Feynman rules in superspace. Outside of superspace, the non-renormalization theorems have not been proven. And the expressions for the gauge β-function can be derived perturbatively only by constructing the supercurrent in the manifestly supersymmetric gauge and in a strict sense they are only available in supersymmetric QED (SQED) [5, 6]. In this construction, the relation to the divergence structure of underlying diagrams is not apparent.

Owing to the increasing number of phenomenological calculations carried out in the Wess–Zumino gauge the situation is not very satisfactory. From explicit calculations in the Wess–Zumino gauge it was apparent that the non-renormalization theorems for the superpotential do not hold in the form derived in superspace [7]. However, it was argued that their consequences for gauge independent quantities should hold in the Wess–Zumino gauge as well. For example, the β-functions of chiral couplings and masses in the Callan–Symanzik equation should be restricted in the same way as in superspace, where they are related to each other by a gauge independent field renormalization. Since a transition from the supersymmetric gauge to the Wess–Zumino gauge cannot be performed in perturbation theory, it is important to derive the results directly in the Wess–Zumino gauge by using algebraic properties of the supersymmetric actions.

Such an algebraic derivation of the non-renormalization theorems has been performed in a foregoing paper in the context of the Wess–Zumino model [8]. In the present paper we will apply the same analysis to the supersymmetric extension of QED in the Wess–Zumino gauge [7] and derive both the non-renormalization theorems of chiral vertices and the closed form of the β-function.

The algebraic analysis is based on the following observations: Supersymmetric Lagrangians can be written as the highest component of a supermultiplet, and so they are related to lower dimensional field monomials by supersymmetry transformations. The implications of this multiplet structure for Green functions can be worked out by extending the coupling constant to an external superfield. Differentiation with respect to the local supercoupling yields Green functions with one insertion of the supermultiplet of the interaction Lagrangian. For constant coupling these Green functions are the ones of the original model. But now they are related to Green functions with lower-dimensional vertex insertions by supersymmetry. In the context of the Wess-Zumino model it was shown that these

relations imply an improvement for the power-counting degree of divergence for chiral Green functions.

In the algebraic approach non-renormalization theorems can be first considered on the basis of invariant counterterms. Whenever invariant counterterms are forbidden for reasons of symmetry, the corresponding Green functions are related to non-local expressions. Hence, absence of counterterms can be viewed as a manifestation of underlying non-renormalization theorems. These non-renormalization theorems can be worked out by relating the respective Green functions to the non-local expressions.

When we extend the gauge coupling of SQED to an external superfield, it turns out that independent counterterms to chiral vertices and counterterms to the photon self energy from two-loop order onwards are absent. However, in the Wess–Zumino gauge the non-linear supersymmetry permits individual field renormalizations for all matter fields. Therefore, we obtain the non-renormalization theorem of chiral vertices similar to the one present in superspace but modified by these gauge dependent field renormalizations.

By working out the non-renormalization theorems in explicit expressions, we find that the non-renormalization theorem of chiral vertices and the generalized non-renormalization theorem of the photon self energy are of a different nature: Chiral Green functions are superficially convergent up to gauge-dependent field redefinition; in contrast, the photon self energy is related to linearly divergent Green functions, which become meaningful only in the course of renormalization. Their local divergent part, however, is uniquely determined from the non-local one by gauge invariance, and it is for this reason that counterterms representing the independent divergences cannot appear in higher orders.

In the past it could only be suggested from the closed expression of the gauge β-function that there is an underlying generalized non-renormalization theorem for the photon self energy. Vice versa, we have to prove in the present context, how the generalized non-renormalization theorem of the photon self energy gives rise to the closed form of the gauge β function. For this purpose we have to derive the Callan–Symanzik equation of SQED with local gauge coupling.

The Callan–Symanzik equation can only be derived if all invariant counterterms can be understood as field and coupling redefinitions — which is not the case in the presence of local couplings. For this reason we introduce an axial vector multiplet whose vector component couples to the axial current and gives rise to an (anomalous) axial Ward identity. Combining axial symmetry and local gauge coupling, the construction of the theory is remarkable on both sides: For local couplings, the model is multiplicatively renormalizable only when the axial vector multiplet is introduced; for axial symmetry the Adler–Bardeen anomaly can be

absorbed into the Slavnov–Taylor operator by means of the local coupling and the model can be constructed by algebraic renormalization in the presence of the anomaly. Even the non-renormalization of the Adler–Bardeen anomaly [12] is a simple consequence of the local coupling.

By using the extended action with the gauged axial current and the local coupling the Callan–Symanzik equation can be derived and describes consistently the scaling of the local coupling and of the axial vector multiplet in presence of the anomaly. The closed form of the gauge β-function is the result of the algebraic construction of the Callan–Symanzik equation as a linear combination of symmetric operators with respect to the anomalous Slavnov–Taylor identity.

According to the general outline the paper is divided into two parts: In the first part of the paper (section 2–5) we derive the non-renormalization theorems of chiral vertices and the photon self energy: In section 2 SQED is extended to a supersymmetric theory with local coupling, in section 3 we outline the renormalization of SQED with local couplings in the Wess–Zumino gauge. In section 4 we construct the invariant counterterms and find the non-renormalization theorems implicitly as absence of counterterms to chiral vertices and to the photon self-energy from 2-loop order onwards. In section 5 the analysis is continued to the explicit construction of the corresponding non-local expressions.

The second part (section 6–9) is devoted to the derivation of the Callan-Symanzik equation and, in particular, of the closed form of the β-function: We introduce the axial vector multiplet in section 6 and construct the Slavnov–Taylor identity in presence of the Adler–Bardeen anomaly in section 7. Finally, in section 8 we derive the Callan–Symanzik equation, and we find the implications of non-renormalization theorems as restrictions on the Callan–Symanzik coefficients. In section 9 we derive an interesting relation between the axial-current Green functions and the photon self energy. This relation explains the appearance of terms of higher orders to the gauge β-function and can be identified as the analogue to the Konishi anomaly [5, 13] in the Wess–Zumino gauge. In the appendices we give the notations and conventions, the BRS transformations and the transformations of fields under discrete symmetries.

2 SQED with local couplings

In the manifestly supersymmetric gauge it is obvious that the Lagrangian of an invariant action is the highest component of a supermultiplet. As will be shown here, the gauge invariant parts of the Lagrangian of SQED in the Wess–Zumino gauge are the highest components of ordinary supermultiplets, too. This observation is the basis for the derivation of the non-renormalization theorems. Technically it is exploited by extending the gauge coupling to a space-time dependent

external field, the local coupling. In order to maintain supersymmetry, the local gauge coupling has to be taken as the lowest component of a constrained real superfield. The invariant action then includes as additional terms the complete chiral and antichiral multiplet of the gauge and supersymmetric invariant kinetic Lagrangian of the photon multiplet, whose lowest component is the photino mass term.

2.1 The multiplet structure of the gauge invariant action

The classical action of the supersymmetric extension of QED (SQED) [7] extends the gauge invariant action of ordinary QED to a supersymmetric action. In the Wess–Zumino gauge it contains the vector multiplet $(A^\mu, \lambda^\alpha, \overline{\lambda}^{\dot\alpha}, D)$ and the left and right handed chiral multiplets $(\varphi_L, \psi_L^\alpha, F_L)$ and $(\varphi_R, \psi_R^\alpha, F_R)$ and the respective complex conjugate antichiral multiplets.[2] The matter fields are charged with the electric charge $Q_L = -1$ and $Q_R = 1$.

The invariant action can be decomposed into the invariant kinetic part of the photon and photino, the matter part containing the interaction of the matter fields with the photon multiplet and the supersymmetric mass term of matter:

$$\begin{aligned}\Gamma_{\rm SQED} &= \Gamma_{\rm kin} + \Gamma_{\rm matter} + \Gamma_{\rm mass} \\ &= \int d^4x \, \Big(\frac{1}{2}(L_{\rm kin} + \bar{L}_{\rm kin}) + L_{\rm matter} + m(L_{\rm mass} + \bar{L}_{\rm mass})\Big) \end{aligned} \tag{1}$$

The corresponding Lagrangians are defined by the following expressions:

$$L_{\rm kin} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\lambda^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\overline{\lambda}^{\dot\alpha} + \frac{1}{8}D^2 - \frac{i}{8}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \tag{2}$$

$$L_{\rm mass} = F_L\varphi_R + F_R\varphi_L - \psi_L\psi_R \tag{3}$$

and respective expressions for their complex conjugates. $L_{\rm matter}$ can be split into a left- and right-handed part:

$$L_{\rm matter} = L_{\rm matter,L} + L_{\rm matter,R} \tag{4}$$

where $(A = L, R)$

$$\begin{aligned} L_{\rm matter,A} = {} & \frac{1}{2}D^\mu\overline{\varphi}_A D_\mu\varphi_A - \frac{1}{4}\overline{\varphi}_A D^\mu D_\mu\varphi_A - \frac{1}{4}\varphi_A D^\mu D_\mu\overline{\varphi}_A \\ & + \frac{i}{2}\psi_A^\alpha\sigma_{\alpha\dot\alpha}D_\mu\overline{\psi}_A^{\dot\alpha} - \frac{i}{2}D_\mu\psi_A^\alpha\sigma^\mu_{\alpha\dot\alpha}\overline{\psi}_A^{\dot\alpha} + \overline{F}_A F_A \\ & + ieQ_A\sqrt{2}(\lambda\psi_A\overline{\varphi}_A - \overline{\lambda\psi}_A\varphi_A) + \frac{1}{2}eQ_A D\overline{\varphi}_A\varphi_A \end{aligned} \tag{5}$$

[2] For the purpose of the present section we keep the auxiliary fields D and F in the action, and we eliminate them when we proceed to quantization in section 3.

The field strength $F_{\mu\nu}$ and the gauge covariant derivative are given by

$$D_\mu \phi_A = (\partial_\mu + ieQ_A A_\mu)\phi_A \ , \qquad D_\mu \bar{\phi}_A = (\partial_\mu - ieQ_A A_\mu)\bar{\phi}_A \ ,$$
$$F_{\mu\nu}(A) = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{6}$$

In the Wess–Zumino gauge the algebra of supersymmetry transformations closes on translations only up to an abelian gauge transformation $\delta_\omega^{\text{gauge}}$:

$$\{\delta_\alpha, \bar{\delta}_{\dot{\alpha}}\} = 2i\sigma^\mu_{\alpha\dot{\alpha}}\big(\partial_\mu + \delta^{\text{gauge}}_{eA^\mu}\big) \tag{7}$$

When applied to gauge invariant expressions the algebra closes on translations. The Lagrangians above are gauge invariant and transform covariantly under supersymmetry transformations. Hence, the supersymmetry algebra implies that they can be written as a supersymmetry variation of lower dimensional field monomials: In fact, L_{kin} (2) and L_{mass} (3) are the highest components of chiral multiplets, and as such they are the second variations of their lowest components:

$$\mathcal{L}_{\text{kin}}(x,\theta) = -\frac{1}{2}\lambda^\alpha\lambda_\alpha - \frac{i}{2}\theta^\alpha(\sigma_\alpha^{\mu\nu\ \beta}\lambda_\beta F_{\mu\nu} + \lambda_\alpha D) + \theta^2 L_{\text{kin}}(x) \tag{8}$$

$$\mathcal{L}_{\text{mass}}(x,\theta) = \varphi_L\varphi_R + \sqrt{2}\theta^\alpha(\psi_{L\,\alpha}\varphi_R + \psi_{R\,\alpha}\varphi_L) + \theta^2 L_{\text{mass}}(x) \tag{9}$$

Their transformations under supersymmetry can be given in a closed form by using the superspace formulation (see (164 and (165) in appendix A for definitions and conventions):

$$\delta_\alpha \mathcal{L}(x,\theta) = \frac{\partial}{\partial\theta^\alpha}\mathcal{L}(x,\theta) \quad \bar{\delta}_{\dot{\alpha}}\mathcal{L}(x,\theta) = -2i(\theta\sigma)_{\dot{\alpha}}\partial_\mu\mathcal{L}(x,\theta) \tag{10}$$

$\bar{L}_{\text{kin}}$ and $\bar{L}_{\text{mass}}$ are the highest components of the respective antichiral multiplets. The matter part of the action is the highest component of a real supermultiplet and can be written even as a fourth variation of the 2-dimensional field monomial $\varphi_A\overline{\varphi}_A$ (see (48)).

In the following section we show that the chiral multiplet $\mathcal{L}_{\text{kin}}$ and its complex conjugate determine the supersymmetric extension of the local gauge coupling to a real superfield, which is composed of a chiral and antichiral superfield.

2.2 SQED with local gauge coupling

In a first step towards a supersymmetric action with local gauge coupling, we extend the coupling constant e in the matter Lagrangian (5) to an external field $e(x)$:

$$e \to e(x) \quad \text{and} \quad L_{\text{matter}}(e) \to L_{\text{matter}}(e(x)) \ . \tag{11}$$

It is gauge invariant with local coupling $e(x)$ under modified gauge transformations:

$$\delta^{\rm gauge}A_\mu = \frac{1}{e(x)}\partial_\mu\omega(x), \qquad \delta^{\rm gauge}\lambda = \delta^{\rm gauge}D = 0, \tag{12}$$
$$\delta^{\rm gauge}\phi_A = -iQ_A\omega(x)\phi_A, \quad \delta^{\rm gauge}\overline{\phi}_A = iQ_A\omega(x)\overline{\phi}_A, \qquad \phi = \varphi, \psi, F.$$

$\Gamma_{\rm matter}(e(x))$ is even invariant under supersymmetry transformations, if the supersymmetry transformations are modified to

$$\begin{aligned}
&\delta_\alpha(eA^\mu) = i\sigma^\mu_{\alpha\dot\alpha}e\overline{\lambda}^{\dot\alpha}\ , && \bar\delta_{\dot\alpha}(eA^\mu) = -ie\lambda^\alpha\sigma^\mu_{\alpha\dot\alpha}\ ,\\
&\delta_\alpha(e\lambda^\beta) = \frac{i}{2}\sigma^{\mu\nu}_\alpha{}^\beta F_{\mu\nu}(eA) + \frac{i}{2}\delta^\beta_\alpha eD, && \bar\delta_{\dot\alpha}(e\lambda_\beta) = 0\ ,\\
&\delta_\alpha(e\overline{\lambda}_{\dot\alpha}) = 0\ , && \bar\delta^{\dot\alpha}(e\overline{\lambda}_{\dot\beta}) = \frac{i}{2}\bar\sigma^{\mu\nu\ \dot\alpha}_{\ \ \dot\beta}F^{\mu\nu}(eA) - \frac{i}{2}\delta^{\dot\alpha}_{\dot\beta}eD,\\
&\delta_\alpha(eD) = 2\sigma^\mu_{\alpha\dot\alpha}\partial_\mu(e\overline{\lambda}^{\dot\alpha})\ , && \bar\delta_{\dot\alpha}(eD) = 2\partial_\mu(e\lambda^\alpha)\sigma^\mu_{\alpha\dot\alpha}\ ,
\end{aligned} \tag{13}$$

and $(A = L, R)$

$$\begin{aligned}
&\delta_\alpha\varphi_A = \sqrt{2}\psi_{A\,\alpha}\ , && \bar\delta_{\dot\alpha}\varphi_A = 0\ ,\\
&\delta_\alpha\psi^\beta_A = \sqrt{2}\delta^\beta_\alpha F_A\ , && \bar\delta_{\dot\alpha}\psi_{A\,\alpha} = \sqrt{2}i\sigma^\mu_{\alpha\dot\alpha}D_\mu\varphi_A\ ,\\
&\delta_\alpha F_A = 0\ , && \bar\delta_{\dot\alpha}F_A = \sqrt{2}iD_\mu\psi^\alpha_A\sigma^\mu_{\alpha\dot\alpha} + 2ieQ_A\overline{\lambda}_{\dot\alpha}\varphi_A\ .
\end{aligned} \tag{14}$$

Since the gauge transformation of the field $e(x)A^\mu$ takes the same form as the gauge transformation with constant coupling, the supersymmetry transformations in terms of the fields $(e(x)A^\mu, e(x)\lambda, e(x)\overline{\lambda}, e(x)D)$ take the same form as the usual supersymmetry transformations. These transformations are in agreement with the algebraic restrictions, since the supersymmetry transformations commute with the gauge transformations (12) and fulfill the supersymmetry algebra as given in (7).

$\Gamma_{\rm matter}$ depends on the coupling only via the product with a photon, photino or D-field. For this reason it is invariant under the supersymmetry transformations with local couplings no matter how the transformation of $e(x)$ itself is defined.

However, the extension of $\Gamma_{\rm kin}$ to a gauge invariant expression with local gauge coupling,

$$\int d^4x\ \Big(-\frac{1}{4e^2}F^{\mu\nu}(eA)F_{\mu\nu}(eA) + \frac{i}{2}(\lambda\sigma\partial\overline{\lambda} - \partial\lambda\sigma\overline{\lambda}) + \frac{1}{8}D^2\Big), \tag{15}$$

depends on the coupling $e(x)$ not only in the combination with the photon field and is not supersymmetric without further modifications. This is particularly transparent in the superspace formulation of (15),

$$\int dS\ \frac{1}{2e(x)^2}\mathcal{L}_{\rm kin} + \int d\bar S\ \frac{1}{2e(x)^2}\overline{\mathcal{L}}_{\rm kin}\,. \tag{16}$$

Therefore, in a second step we extend the coupling $e(x)$ to a supermultiplet of fields.

Indeed, a natural replacement of (16) is given by

$$\int dS\ \boldsymbol{\eta}\,\mathcal{L}_{\rm kin} + \int d\bar{S}\ \overline{\boldsymbol{\eta}}\,\overline{\mathcal{L}}_{\rm kin} \tag{17}$$

with a dimensionless chiral superfield $\boldsymbol{\eta}(x,\theta)$ and its complex conjugate $\overline{\boldsymbol{\eta}}(x,\overline{\theta})$, which are given by

$$\boldsymbol{\eta}(x,\theta) = \eta + \theta^\alpha \chi_\alpha + \theta^2 f\ , \qquad \overline{\boldsymbol{\eta}}(x,\overline{\theta}) = \overline{\eta} + \theta_{\dot\alpha}\overline{\chi}^{\dot\alpha} + \overline{\theta}^2\overline{f} \tag{18}$$

in the respective chiral and antichiral representation. The expression (17) is supersymmetric and gauge invariant with the transformations (13), (12). And (17) contains the gauge invariant expression (16) if we identify the square of the local coupling with the inverse of the real part of η and $\overline{\eta}$:

$$\frac{1}{e^2(x)} = \eta(x) + \overline{\eta}(x). \tag{19}$$

The relation (19) specifies $e(x)$ as the lowest component of a constrained real vector multiplet $E(x,\theta,\overline{\theta})$:

$$E(x,\theta,\overline{\theta}) = (\boldsymbol{\eta}(x,\theta,\overline{\theta}) + \overline{\boldsymbol{\eta}}(x,\theta,\overline{\theta}))^{-\frac{1}{2}} \tag{20}$$

(here $\boldsymbol{\eta}$ and $\overline{\boldsymbol{\eta}}$ have to be taken in the real representation), and thus determines the supersymmetry transformations of the local gauge coupling as

$$\delta_\alpha e = -\frac{1}{2}e^3\chi_\alpha \qquad \bar\delta_{\dot\alpha} e = -\frac{1}{2}e^3\overline{\chi}_{\dot\alpha}\ . \tag{21}$$

With this identification we find the following explicit expression for the gauge invariant and supersymmetric action (17):

$$\begin{aligned}\Gamma_{\rm kin} \quad = \int d^4x\ \Big(& -\frac{1}{4e^2}F^{\mu\nu}(eA)F_{\mu\nu}(eA) + \frac{i}{2}(\lambda\sigma\partial\overline{\lambda} - \partial\lambda\sigma\overline{\lambda}) + \frac{1}{8}D^2 \\ & + \frac{i}{2}(\eta-\overline{\eta})\partial_\mu(e^2\lambda\sigma^\mu\overline{\lambda}) - \frac{i}{8}(\eta-\overline{\eta})\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}(eA)F_{\rho\sigma}(eA) \\ & + \frac{i}{4}e(\chi\sigma^{\mu\nu}\lambda - \overline{\lambda}\bar\sigma^{\mu\nu}\overline{\chi})F_{\mu\nu}(eA) + \frac{i}{4}e^2(\chi\lambda - \overline{\chi}\overline{\lambda})D \\ & - \frac{1}{2}e^2 f\lambda\lambda - \frac{1}{2}e^2\overline{f\lambda\lambda}\Big).\end{aligned} \tag{22}$$

The complete action of SQED with local coupling is composed of $\Gamma_{\rm kin}$ (22), the matter action $\Gamma_{\rm matter}$ with the Lagrangian (11) and the supersymmetric mass term $\Gamma_{\rm mass}$ (see (1) with (3)). It is supersymmetric under the transformations (13), (14) and under transformations of the gauge coupling and its superpartners according to its definition (20). It is remarkable that the modifications of the local gauge coupling concern only the kinetic part of the action, which includes now the chiral multiplet $\mathcal{L}_{\rm kin}$ and its complex conjugate. The matter and mass terms remain unaffected by the superfield extension of the coupling.

For the derivation of non-renormalization theorems it is important to note that the classical action depends on the parity odd external field $\eta - \overline{\eta}$ only via a total derivative:

$$\int d^4x \, \Big(\frac{\delta}{\delta\eta} - \frac{\delta}{\delta\overline{\eta}}\Big)\Gamma_{\rm cl} = 0. \tag{23}$$

This identity constitutes a Ward identity expressing that the local coupling is the lowest component of a constrained real superfield as defined in (20), and it states the absence of purely chiral or antichiral interactions in SQED.

2.3 Chiral vertices

It is well-known that in the Wess–Zumino gauge the non-renormalization of chiral vertices is not manifest. It will turn out that there is an underlying non-renormalization theorem, but the non-renormalization properties are superposed by gauge-dependent field renormalizations of the matter fields. In order to disentangle the effect of the unphysical field renormalization from the non-renormalization properties, it is useful to introduce a second chiral vertex into the model. This can be done easily using a further external chiral and an antichiral field multiplet with dimension 1:

$$\mathbf{q} = q + \theta^\alpha q_\alpha + \theta^2 q_F \ , \qquad \overline{\mathbf{q}} = \overline{q} + \overline{\theta}_{\dot\alpha}\overline{q}^{\dot\alpha} + \overline{\theta}^2 \overline{q}_F \ . \tag{24}$$

It can be coupled to the mass term as follows:

$$\begin{aligned}\Gamma_{\rm q} = \int d^4x \, \Big(& qL_{\rm mass} + \bar{q}\bar{L}_{\rm mass} \\ & - \frac{1}{\sqrt{2}}\big(q^\alpha(\psi_{L\alpha}\varphi_R + \psi_{R\alpha}\varphi_L) + \overline{q}_{\dot\alpha}(\overline{\psi}^{\dot\alpha}_L\overline{\varphi}_R + \overline{\psi}^{\dot\alpha}_R\overline{\varphi}_L)\big) \\ & + q_F\varphi_L\varphi_R + \bar{q}_F\overline{\varphi}_L\overline{\varphi}_R\Big) \ . \end{aligned} \tag{25}$$

The action of SQED with local coupling being enlarged by $\Gamma_{\rm q}$, it contains two chiral vertex functions $\Gamma_{\psi\psi}$ and $\Gamma_{q\psi\psi}$, and we will obtain non-trivial non-renormalization theorems relating the divergences appearing in $\Gamma_{q\psi\psi}$ and $\Gamma_{\psi\psi}$.

Furthermore it will turn out that the **q**-multiplet and its complex conjugate take an important role when we gauge softly broken axial symmetry and when we construct the Callan–Symanzik equation.

3 Quantization of SQED with local coupling

3.1 The Slavnov–Taylor identity

For quantizing SQED with local gauge coupling in the Wess–Zumino gauge all symmetries, gauge symmetry, supersymmetry and translational symmetry, are included into an enlarged Slavnov–Taylor identity [15, 16]. As in ordinary SQED [14], all transformation parameters are replaced by the respective ghosts. So we introduce the Faddeev–Popov ghost $c(x)$ for gauge transformations, and the space-time independent supersymmetry and translation ghosts $\epsilon^\alpha, \overline{\epsilon}^{\dot\alpha}$ and ω^ν.

Then we eliminate the auxiliary fields D and $F_L, F_R, \bar F_L, \bar F_R$ by their equations of motions,

$$\begin{aligned} D &= -ie^2(\chi\lambda - \overline{\chi}\overline{\lambda}) - 2eQ_L(\varphi_L\overline{\varphi}_L - \varphi_R\overline{\varphi}_R) \ , \\ \overline{F}_L &= -(q+m)\varphi_R \ , \qquad F_L = -(\overline{q}+m)\overline{\varphi}_R \ , \\ \overline{F}_R &= -(q+m)\varphi_L \ , \qquad F_R = -(\overline{q}+m)\overline{\varphi}_L \ , \end{aligned} \tag{26}$$

and get an action in terms of physical fields.

All symmetries and the structure constants of the algebra are summarized in the BRS transformations listed in appendix C. After the elimination of the auxiliary fields the BRS transformations are nilpotent only up to equations of motions.

Owing to the nilpotency of the BRS transformations it is straightforward to add a BRS invariant gauge fixing and Faddeev–Popov field part to the action. We choose it in such a way that the gauge fixing is stable under renormalization and results in the usual gauge fixing for constant coupling. Using the auxiliary field B, the Faddeev-Popov anti-ghost $\overline{c}$ and the local gauge parameter $\xi(x) + \xi$ it reads:

$$\Gamma_{\rm g.f} + \Gamma_{\phi\pi} = {\rm s}\int d^4x\ (\frac{1}{2}(\xi(x)+\xi)\overline{c}B + \frac{1}{e}\overline{c}\partial(eA)) \ . \tag{27}$$

Working this out with the BRS transformations of the appendix we get:

$$\Gamma_{\rm g.f.}+\Gamma_{\phi\pi}=\int d^4x\,\Big(\frac{1}{e}B\partial^\mu(eA_\mu)+\frac{1}{2}(\xi(x)+\xi)B^2-\frac{1}{e}\bar{c}\Box(ec)+\frac{1}{2}\chi_\xi\bar{c}B$$
$$-\frac{1}{e}\bar{c}\partial^\mu(i\epsilon\sigma_\mu e\overline{\lambda}-ie\lambda\sigma_\mu\bar{\epsilon})+(\xi(x)+\xi)i\epsilon\sigma^\nu\bar{\epsilon}(\partial_\nu\bar{c})\bar{c}\Big)\ . \qquad (28)$$

We want to note that the local gauge fixing field $\xi(x)$ is introduced for later use in the Callan–Symanzik equation. For a space-time dependent gauge fixing parameter $\xi(x)$ supersymmetry and translational invariance enforces to introduce its BRS partner $\chi_\xi(x)$. BRS variations of the gauge parameter have already been introduced in ordinary gauge theories for controlling gauge parameter dependence of Green functions [18] and are an important tool for identifying gauge invariant quantities [19].

For the formulation of the Slavnov–Taylor identity one introduces the external field part, which also contains the bilinear part for absorbing the equations of motion terms violating the nilpotency of BRS transformations [20]:

$$\Gamma_{\rm ext}=\int d^4x\,\Big(Y^\alpha_\lambda s\lambda_\alpha+Y_{\overline{\lambda}\dot{\alpha}}s\overline{\lambda}^{\dot{\alpha}}$$
$$+Y_{\varphi_L}s\varphi_L+Y_{\overline{\varphi}_L}s\overline{\varphi}_L+Y^\alpha_{\psi_L}s\psi_{L\alpha}+Y_{\overline{\psi}_L\dot{\alpha}}s\overline{\psi}^{\dot{\alpha}}_L+(_{L\to R})$$
$$+\frac{1}{2}(Y_\lambda\epsilon-\bar{\epsilon}Y_{\overline{\lambda}})^2-2(Y_{\psi_L}\epsilon)(\bar{\epsilon}Y_{\overline{\psi}_L})-2(Y_{\psi_R}\epsilon)(\bar{\epsilon}Y_{\overline{\psi}_R})\Big)\ . \qquad (29)$$

It coincides in its structure with the one of ordinary SQED [14]. The complete classical action

$$\Gamma_{\rm cl}=\Gamma_{\rm SQED}+\Gamma_{\rm g.f.}+\Gamma_{\phi\pi}+\Gamma_{\rm ext}\,, \qquad (30)$$

satisfies the Slavnov–Taylor identity:

$$\mathcal{S}(\Gamma_{\rm cl})=0\ . \qquad (31)$$

The Slavnov–Taylor operator acting on a general functional $\mathcal{F}$ is defined as

$$\mathcal{S}(\mathcal{F})=\int d^4x\,\Big(sA^\mu\frac{\delta\mathcal{F}}{\delta A^\mu}+\frac{\delta\mathcal{F}}{\delta Y_{\lambda\alpha}}\frac{\delta\mathcal{F}}{\delta\lambda^\alpha}+\frac{\delta\mathcal{F}}{\delta Y^{\dot{\alpha}}_{\overline{\lambda}}}\frac{\delta\mathcal{F}}{\delta\overline{\lambda}_{\dot{\alpha}}}$$
$$+sc\frac{\delta\mathcal{F}}{\delta c}+sB\frac{\delta\mathcal{F}}{\delta B}+s\bar{c}\frac{\delta\mathcal{F}}{\delta\bar{c}}+s\xi\frac{\delta\mathcal{F}}{\delta\xi}+s\chi_\xi\frac{\delta\mathcal{F}}{\delta\chi_\xi}$$
$$+\frac{\delta\mathcal{F}}{\delta Y_{\varphi_L}}\frac{\delta\mathcal{F}}{\delta\varphi_L}+\frac{\delta\mathcal{F}}{\delta Y_{\overline{\varphi}_L}}\frac{\delta\mathcal{F}}{\delta\overline{\varphi}_L}+\frac{\delta\mathcal{F}}{\delta Y_{\psi_L\alpha}}\frac{\delta\mathcal{F}}{\delta\psi^\alpha_L}+\frac{\delta\mathcal{F}}{\delta Y^{\dot{\alpha}}_{\overline{\psi}_L}}\frac{\delta\mathcal{F}}{\delta\overline{\psi}_{L\dot{\alpha}}}+(_{L\to R})\Big)$$
$$+s\eta^i\frac{\delta\mathcal{F}}{\delta\eta^i}+s\overline{\eta}^i\frac{\delta\mathcal{F}}{\delta\overline{\eta}^i}+sq^i\frac{\delta\mathcal{F}}{\delta q^i}+s\overline{q}^i\frac{\delta\mathcal{F}}{\delta\overline{q}^i}+\Big)+s\omega^\nu\frac{\partial\mathcal{F}}{\partial\omega^\nu}\ . \qquad (32)$$

In comparison to usual SQED the Slavnov–Taylor operator contains in addition the BRS transformations of the chiral multiplet $\eta^i = (\eta, \chi^\alpha, f)$ and its complex conjugate antichiral multiplet $\bar{\eta}^i$, which define the local coupling (19), the BRS transformation of the chiral multiplet $q^i = (q, q^\alpha, q_F)$ and its complex conjugate $\overline{q}^i$ (24) and the BRS transformations of the gauge parameter doublet.

The full Slavnov–Taylor operator and its linearized version have the nilpotency property

$$s_{\mathcal{F}} S(\mathcal{F}) = 0 \tag{33}$$

if the functional $\mathcal{F}$ satisfies the linear identity

$$i\epsilon\sigma^\mu \frac{\delta \mathcal{F}}{\delta Y_{\overline{\lambda}}} + i\frac{\delta \mathcal{F}}{\delta Y_\lambda}\sigma^\mu \overline{\epsilon} = \big(\frac{1}{2}e^2(\epsilon\chi - \overline{\chi\epsilon}) - i\omega^\nu \partial_\nu\big)(i\epsilon\sigma^\mu\overline{\lambda} - i\lambda\sigma^\mu\overline{\epsilon}) + 2i\epsilon\sigma_\nu\overline{\epsilon}\frac{1}{e}F^{\nu\mu}(eA)\ . \tag{34}$$

This linear identity guarantees the nilpotency properties on the linear transformations of the photon field A^μ: $s_{\mathcal{F}}^2 A^\mu = 0$. Eq. (34) is satisfied in particular by $\Gamma_{\rm cl}$ and can be maintained also in the course of renormalization.

3.2 Renormalization

The local gauge coupling and its superpartners (20) are considered as external fields which appear in the same way as ordinary external fields in the generating functional of 1PI Green functions Γ. In ordinary SQED the coupling constant is the perturbative expansion parameter. By a simple consideration of diagrams it can be seen that this property is true also for the local coupling $e(x)$. Indeed, the number of local couplings appearing in a specific diagram is related to the loop order l by a topological formula:

$$N_{e(x)} = N_{\rm amp.legs} + N_Y + 2N_f + 2N_\chi + 2N_{\eta-\overline{\eta}} + 2(l-1)\ . \tag{35}$$

Here $N_{\rm amp.legs}$ counts the number of external amputated legs with propagating fields ($A^\mu, \lambda, \varphi_A, \psi_A, c, \overline{c}$ and the respective complex conjugate fields), N_Y gives the number of BRS insertions, counted by the number of differentiations with respect the the external fields Y_ϕ. N_f, N_χ and $N_{\eta-\overline{\eta}}$ gives the number of insertions corresponding to the respective external fields. These are the $\eta - \overline{\eta}$ and higher components of the chiral and antichiral multiplets defining the local coupling. By the existence of a topological formula the local coupling is distinguished from

ordinary dimensionless external fields like the spurion fields [21, 22], which can appear in arbitrary orders in the Green functions of higher order perturbation theory.

We are able to renormalize the Green functions of SQED with local coupling as a simple extension of usual SQED. The Slavnov–Taylor identity of SQED with local couplings is — as for usual SQED — not anomalous, and one is able to establish the Slavnov–Taylor identity

$$\mathcal{S}(\Gamma) = 0 \tag{36}$$

and the linear equation (34) to all orders. Furthermore we require the linear gauge fixing function (see (28)) as normalization for the B-field and the linear ghost equations,

$$\frac{\delta\Gamma}{\delta c} = \frac{\delta\Gamma_{\rm cl}}{\delta c} \qquad \text{and} \qquad \frac{\partial\Gamma}{\partial\omega^\mu} = \frac{\partial\Gamma_{\rm cl}}{\partial\omega^\mu} \tag{37}$$

as normalization conditions for the ghosts. As in ordinary SQED the abelian gauge Ward identity is derived from the consistency equation of the ghost equation with the Slavnov–Taylor identity [14, 16]. With local gauge coupling it takes the form

$$\Big(\mathbf{w}^{\rm em} - \partial^\mu\big(\frac{1}{e}\frac{\delta}{\delta A^\mu}\big)\Big)\Gamma = \Box\big(\frac{1}{e}B\big) + \mathcal{O}(\omega)\ . \tag{38}$$

In the adiabatic limit the real superfield of the coupling becomes a constant

$$E(x,\theta,\overline{\theta}) \to e = \text{const}, \tag{39}$$

and one recovers the 1PI Green functions and Ward identities of ordinary SQED as defined in [14],

$$\lim_{E\to e}\Gamma = \Gamma^{\rm SQED}. \tag{40}$$

As a new ingredient one requires in addition to the above symmetries the identity (23):

$$\int d^4x\ \Big(\frac{\delta}{\delta\eta} - \frac{\delta}{\delta\overline{\eta}}\Big)\Gamma = 0\ . \tag{41}$$

It is valid in the classical approximation and can be extended to higher orders of perturbation theory. In combination with supersymmetry, eq. (41) is the crucial identity for the construction of non-renormalization theorems in SQED.

4 Improved renormalization behaviour

4.1 Invariant counterterms

The simplest, but somewhat indirect way to derive the improved renormalization behavior of 1PI Green functions can be carried out by an investigation of symmetric counterterms: Absence of symmetric counterterms to Green functions means that the corresponding Green functions are related to non-local expressions, which cannot appear with independent renormalizations. In this section we construct the invariant counterterms of higher orders and find that invariant counterterms to the photon self energy from two-loop order onwards and independent counterterms to the chiral vertices are absent. The algebraic reason for the absence of these counterterms is the identity (23), which characterizes the local gauge coupling as a constrained real superfield.

Due to the nilpotency properties of the Slavnov–Taylor operator the invariant counterterms in loop order l are restricted by the symmetries of the classical action, i.e.

$$s_{\Gamma_{\rm cl}}\Gamma^{(l)}_{\rm ct,inv}=0 \qquad \text{and} \qquad \int d^4x\,\Big(\frac{\delta}{\delta\eta}-\frac{\delta}{\delta\overline{\eta}}\Big)\Gamma^{(l)}_{\rm ct,inv}=0\ , \tag{42}$$

and $\Gamma_{\rm ct,inv}$ is invariant under the discrete symmetries C, P and R-parity (see appendix B). A further constraint on the counterterms is the topological formula (35), which determines the order in the local coupling.

All symmetric counterterms can be algebraically classified as non-BRS variations and BRS variations. Non-BRS variations comprise the renormalization of physical parameters, whereas BRS variations are connected with unphysical field renormalizations. In contrast to the non-BRS variations, BRS variations receive gauge-dependent coefficients, as can be easily derived using the BRS varying gauge parameter (cf. [18] for a detailed discussion).

We start the construction of invariant counterterms with the physically relevant ones, the non-BRS variations. The simplest and most obvious way to construct them is given by their superspace formulation. Therefore we use the multiplet structure of the gauge invariant Lagrangians of section 2.1 and eliminate the auxiliary fields in the end. Indeed, since the non-BRS variations are gauge-independent, they appear in the same way in the supersymmetric gauge and are not affected by the elimination of the auxiliary fields.

Particular supersymmetric and gauge invariant counterterms are the counterterms to the chiral vertices

$$\int dS\ \boldsymbol{\eta}^{-l}\mathcal{L}_{\rm mass}+\int d\bar{S}\ \boldsymbol{\eta}^{-l}\bar{\mathcal{L}}_{\rm mass},\quad \int dS\ \boldsymbol{\eta}^{-l}\mathbf{q}\mathcal{L}_{\rm mass}+\int d\bar{S}\ \boldsymbol{\eta}^{-l}\bar{\mathbf{q}}\bar{\mathcal{L}}_{\rm mass}, \tag{43}$$

and the counterterm to the kinetic term of the photon multiplet

$$\int dS\ \boldsymbol{\eta}^{-l+1}\mathcal{L}_{\rm kin} + \int d\bar{S}\ \overline{\boldsymbol{\eta}}^{-l+1}\bar{\mathcal{L}}_{\rm kin}\ . \tag{44}$$

Here the dependence on the loop order l is governed by the topological formula (35). However, the symmetric counterterms have to satisfy the second identity in (42), too. Indeed, (43) does not satisfy (42) for $l \neq 0$, and (44) does not satisfy (42) for $l \geq 2$. Accordingly, loop diagrams to these non-symmetric counterterms are not present since the classical action only depends the constrained real superfield $\boldsymbol{\eta} + \overline{\boldsymbol{\eta}}$ but not on the single chiral or antichiral fields. Thus, the only of the above counterterms that is in agreement with both identities in (42) is the 1-loop counterterm to the kinetic terms of the photon multiplet:

$$\Gamma^{(1)}_{\rm ct,kin} = \int d^4x\ \Big(F^{\mu\nu}(eA)F_{\mu\nu}(eA) + e^2 i\lambda\sigma^\mu\partial\overline{\lambda} + \frac{1}{8}e^2D^2\Big). \tag{45}$$

There exists a further field monomial which is a non-BRS variation:

$$\Gamma^{(l)}_{\rm ct,matter} = \frac{1}{16}\int dV\ E^{2l}\mathcal{L}_{\rm matter}\ , \tag{46}$$

where E^{2l} and $\mathcal{L}_{\rm matter}$ are the real multiplets corresponding to e^{2l} and $L_{\rm matter}$, respectively:

$$E^{2l}(x,\theta,\overline{\theta}) = \big(\boldsymbol{\eta}(x,\theta,\overline{\theta}) + \overline{\boldsymbol{\eta}}(x,\theta,\overline{\theta})\big)^{-l} \tag{47}$$

and

$$\begin{aligned}
&\mathcal{L}_{{\rm matter},A}(x,\theta,\overline{\theta}) = \\
&\quad \overline{\varphi}_A\varphi_A + \theta\sqrt{2}\psi_A\overline{\varphi}_A + \overline{\theta}\sqrt{2}\overline{\psi}_A\varphi_A + \theta^2F_A\overline{\varphi}_A + \overline{\theta}^2\bar{F}_A\varphi_A \\
&+\ \theta\sigma^\mu\overline{\theta}(i\varphi_AD_\mu\overline{\varphi}_A - i\overline{\varphi}_AD_\mu\varphi_A + \psi_A\sigma^\mu\overline{\psi}_A) \\
&+\ \overline{\theta}^2\theta\big(\sigma^\mu\frac{i}{\sqrt{2}}(\overline{\psi}_AD_\mu\varphi_A - D_\mu\overline{\psi}_A\varphi_A) - 2ieQ_A\lambda\varphi_A\overline{\varphi}_A + \sqrt{2}\psi\overline{F}_A\big) \\
&+\ \theta^2\overline{\theta}\big(\bar{\sigma}^\mu\frac{i}{\sqrt{2}}(\psi_AD_\mu\overline{\varphi}_A - D_\mu\psi_A\overline{\varphi}_A) + 2ieQ_A\overline{\lambda}\varphi_A\overline{\varphi}_A + \sqrt{2}\overline{\psi}F_A\big) \\
&+\ \theta^2\overline{\theta}^2L_{{\rm matter},A}\ .
\end{aligned} \tag{48}$$

The two counterterms $\Gamma_{\rm ct,kin}$ and $\Gamma_{\rm ct,matter}$ are the only gauge-independent counterterms, i.e.

$$\Gamma^{(l)}_{\rm ct,nonBRS} = z^{(1)}_{\rm kin}\Gamma^{(1)}_{\rm ct,kin}\delta_{l1} + z^{(l)}_m\Gamma^{(l)}_{\rm ct,matter}\ . \tag{49}$$

Finally we have to eliminate the auxiliary fields D and F from $\Gamma^{(1)}_{\mathrm{ct,kin}}$ and $\Gamma^{(l)}_{\mathrm{ct,matter}}$ by their equations of motion:

$$\frac{\delta}{\delta D}(\Gamma_{\mathrm{cl}}(D,F)+\Gamma^{(l)}_{\mathrm{ct,nonBRS}}(D,F))=0 \tag{50}$$

and respective expressions for the F-fields. When we insert the result into the action and put together all terms of loop order l, we obtain the $s_{\Gamma_{\mathrm{cl}}}$-invariant counterterms without auxiliary fields.

The explicit form of the corresponding symmetric counterterms can be found in a simpler way by expressing the invariant counterterms in terms of symmetric operators. For the counterterm (46), however, an invariant operator can only be constructed with the help of a further axial vector multiplet (see (130) and (131)). Nevertheless, it is possible and sufficient for our purposes to consider the limit to constant coupling here. For constant coupling, $\Gamma^{(1)}_{\mathrm{ct,kin}}$ corresponds to the usual coupling renormalization

$$\begin{aligned}\lim_{E\to e}\Gamma^{(1)}_{\mathrm{ct,kin}} = -\frac{1}{2}e^2\Big(&e\partial_e - N_A - N_c - N_\lambda + N_{Y_\lambda}\\ &+ N_B + N_{\bar c} - 2\xi\partial_\xi\Big)\Gamma^{\mathrm{SQED}}_{\mathrm{cl}}\ ,\end{aligned} \tag{51}$$

and $\Gamma_{\mathrm{ct,matter}}$ can be decomposed into a field and mass renormalization:

$$\lim_{E\to e}\Gamma^{(l)}_{\mathrm{ct,matter}} = \frac{1}{2}e^{2l}\big((N_\varphi + N_\psi - N_{Y_\psi} - N_{Y_\varphi}) - 2(N_{\mathbf{q}} + m\partial_m)\big)\Gamma^{\mathrm{SQED}}_{\mathrm{cl}}\ . \tag{52}$$

In (51) and (52) the operators N_ϕ denote the usual field counting operators including also the complex conjugates for complex fields. Eqs. (51) and (52) determine the corresponding invariant counterterms without auxiliary fields in the limit to constant coupling.

The bilinear form of the Slavnov–Taylor identity gives rise to three kinds of unphysical counterterms corresponding to field renormalizations of the matter fields:

$$\begin{aligned}&\psi_A \to z^{(l)}_\psi e^{2l}\psi_A\ , \qquad \varphi_A \to z^{(l)}_\varphi e^{2l}\varphi_A\ ,\\ &\psi_A \to z_{\psi\varphi}e^{2l+2}\chi\varphi_A\ .\end{aligned} \tag{53}$$

These field renormalizations are BRS variations, and the coefficients $z^{(l)}$ are therefore gauge dependent. In explicit calculations they indeed appear with non-vanishing coefficients in the Wess–Zumino gauge (see e.g. [14]). The invariant

counterterms corresponding to these field renormalizations are best given in the form of a symmetric operator acting on the classical action:

$$\begin{aligned}
\Gamma^{(l)}_{\mathrm{ct},\varphi} &= s_{\Gamma_{\mathrm{cl}}} \int d^4x\ e^{2l} f^{(l)}_{\varphi}(\tilde{\xi}) \big(Y_{\varphi_L}\varphi_L + Y_{\varphi_R}\varphi_R + Y_{\overline{\varphi}_L}\overline{\varphi}_L + Y_{\overline{\varphi}_R}\overline{\varphi}_R \big) \\
&= \int d^4x\ e^{2l} f^{(l)}_{\varphi}(\tilde{\xi}) \Bigg(\Big(\varphi_L \frac{\delta}{\delta\varphi_L} - Y_{\varphi_L}\frac{\delta}{\delta Y_{\varphi_L}} + \mathrm{c.c.} \Big) + (_{L\to R}) \Bigg) \Gamma_{\mathrm{cl}} \\
&\quad + \int d^4x\ s\big(e^{2l} f^{(l)}_{\varphi}(\tilde{\xi}) \big) \big(Y_{\varphi_L}\varphi_L + Y_{\varphi_R}\varphi_R + Y_{\overline{\varphi}_L}\overline{\varphi}_L + Y_{\overline{\varphi}_R}\overline{\varphi}_R \big) \Bigg) \\
&\equiv \mathcal{N}^{(l)}_{\varphi}\Gamma_{\mathrm{cl}} + \Delta^{(l)}_{Y,\varphi}\ , \qquad (54) \\
\Gamma^{(l)}_{\mathrm{ct},\psi} &= s_{\Gamma_{\mathrm{cl}}} \int d^4x\ e^{2l} f^{(l)}_{\psi}(\tilde{\xi}) \big(\psi_L Y_{\psi_L} + \psi_R Y_{\psi_R} + \overline{\psi}_L Y_{\overline{\psi}_L} + \overline{\psi}_R Y_{\overline{\psi}_R} \big) \\
&= \int d^4x\ e^{2l} f^{(l)}_{\psi}(\tilde{\xi}) \Bigg(\Big(\psi^{\alpha}_L \frac{\delta}{\delta\psi^{\alpha}_L} - Y^{\alpha}_{\psi_L}\frac{\delta}{\delta Y^{\alpha}_{\psi_L}} + \mathrm{c.c.} \Big) + (_{L\to R}) \Bigg) \Gamma_{\mathrm{cl}} \\
&\quad + \int d^4x\ s\big(e^{2l} f^{(l)}_{\psi}(\tilde{\xi}) \big) \big(\psi_L Y_{\psi_L} + \psi_R Y_{\psi_R} + \overline{\psi}_L Y_{\overline{\psi}_L} + \overline{\psi}_R Y_{\overline{\psi}_R} \big) \Bigg) \\
&\equiv \mathcal{N}^{(l)}_{\psi}\Gamma_{\mathrm{cl}} + \Delta^{(l)}_{Y,\psi}\ , \qquad (55) \\
\Gamma^{(l)}_{\mathrm{ct},\psi\varphi} &= s_{\Gamma_{\mathrm{cl}}} \int d^4x\ \sqrt{2} f^{(l)}_{\psi\varphi}(\tilde{\xi}) \big(\chi^{E^{(2l)}}\varphi_L Y_{\psi_L} + \overline{\chi}^{E^{(2l)}}\overline{\varphi}_L Y_{\overline{\psi}_L} + (_{L\to R}) \big) \\
&= -l \int d^4x\ \sqrt{2} e^{2(l+1)} f^{(l)}_{\psi\varphi}(\tilde{\xi}) \times \\
&\qquad\qquad \Bigg(\Big(\chi^{\alpha} \big(\varphi_L \frac{\delta}{\delta\psi^{\alpha}_L} - Y_{\psi_L\alpha}\frac{\delta}{\delta Y_{\varphi}} \big) + \mathrm{c.c.} \Big) + (_{L\to R}) \Bigg) \Gamma_{\mathrm{cl}} \\
&\quad -l \int d^4x\ s\big(\sqrt{2} e^{2(l+1)} f^{(l)}_{\psi\varphi}(\tilde{\xi}) \big) \big(\chi\varphi_L Y_{\psi_L} + \overline{\chi\varphi}_L Y_{\overline{\psi}_L} + (_{L\to R}) \big) \\
&\quad -l \int d^4x\ \sqrt{2} e^{2(l+1)} f^{(l)}_{\psi\varphi}(\tilde{\xi}) \big(s(\chi)\varphi_L Y_{\psi_L} + s(\overline{\chi})\overline{\varphi}_L Y_{\overline{\psi}_L} + (_{L\to R}) \big) \\
&\equiv \mathcal{N}^{(l)}_{\psi\varphi}\Gamma_{\mathrm{cl}} + \Delta^{(l)}_{Y,\psi\varphi}. \qquad (56)
\end{aligned}$$

4.2 Non-renormalization theorems

The complete action of symmetric counterterms $\Gamma_{\mathrm{ct,inv}}$ in loop order l is a linear combination of the single BRS invariant parts constructed above:

$$\begin{aligned}
\Gamma^{(l)}_{\mathrm{ct,inv}} &= z^{(1)}_{\mathrm{kin}}\Gamma^{(1)}_{\mathrm{ct,kin}}\delta_{l1} + z^{(l)}_{m}\Gamma^{(l)}_{\mathrm{ct,matter}} \\
&\quad + z^{(l)}_{\varphi}\Gamma^{(l)}_{\mathrm{ct},\varphi}(\xi) + z^{(l)}_{\psi}\Gamma^{(l)}_{\mathrm{ct},\psi}(\xi) + z^{(l)}_{\psi\varphi}\Gamma^{(l)}_{\mathrm{ct},\psi\varphi}(\xi)\ . \qquad (57)
\end{aligned}$$

Its independent coefficients are the 1-loop coefficient of the kinetic term of the photon and photino $z^{(1)}_{\rm kin}$, the coefficients of the matter part $z^{(l)}_m$, and the coefficients of the field renormalizations $z^{(l)}_\varphi, z^{(l)}_\psi$ and $z^{(l)}_{\psi\varphi}$.

Comparing this result with the counterterms of usual SQED with constant coupling yields the non-renormalization theorems.

Using eqs. (51) and (52) the symmetric counterterms of SQED are determined by the following expression:

$$\begin{aligned}\lim_{E\to e}\Gamma^{(l)}_{\rm ct,inv} = \Big(&\delta Z_e^{(1)}\big(e\partial_e - N_A - N_c - N_\lambda + N_{Y_\lambda} + N_B + N_{\overline{c}} - 2\xi\partial_\xi\big) \\ &+ \delta Z_m^{(l)}(N_q + N_{q^\alpha} + N_{q_F} + m\partial_m) \\ &+ \delta Z_\varphi^{(l)}(N_\varphi - N_{Y_\varphi}) + \delta Z_\psi^{(l)}(N_\psi - N_{Y_\psi})\Big)\Gamma^{\rm SQED}_{\rm cl} \ .\end{aligned} \tag{58}$$

Here we have defined the z-factors as power series in the coupling:

$$\begin{aligned}\delta Z_e^{(1)} &= -\frac{1}{2}z^{(1)}_{\rm kin}e^2 \ , \qquad \delta Z_m^{(l)} = -z_m^{(l)}e^{2l} \ , \\ \delta Z_\varphi^{(l)} &= e^{2l}(z_\varphi^{(l)}f_\varphi^{(l)}(\xi) + \frac{1}{2}z_m^{(l)}) \ , \\ \delta Z_\psi^{(l)} &= e^{2l}(\, z_\psi^{(l)}f_\psi^{(l)}(\xi) + \frac{1}{2}z_m^{(l)}) \ .\end{aligned} \tag{59}$$

These restrictions on the counterterms constitute the non-renormalization theorem of chiral vertex functions and the generalized non-renormalization theorem of the photon self energy from two-loop order onwards. Non-renormalization of chiral vertex functions is expressed by the common z-factor δZ_m of q-field and mass renormalization. Hence, in the Wess–Zumino gauge the non-renormalization of chiral vertices is hidden by the appearance of individual gauge-dependent field renormalizations to all matter fields.

As can be immediately seen from (59), the non-renormalization of chiral vertices would be manifest if the gauge dependent field renormalizations z_φ and z_ψ vanished and δZ_φ and δZ_ψ were related to δZ_m. But this is the case only in the manifestly supersymmetric gauge.

The absence of counterterms to the photon self energy from two-loop order onwards is a remarkable result of the construction with a local coupling. It implies that the photon self energy is related to non-local Green functions in the present construction. In section 8 the relation of this result to the well-known restrictions on the gauge β-function will be worked out.

5 Non-renormalization theorems in explicit expressions

In this section we explicitly work out the non-renormalization theorem of chiral vertices in the example of the $\Gamma_{\psi\psi}$ and $\Gamma_{q\psi\psi}$ vertices and the non-renormalization of the photon self energy in loop order $l \geq 2$.

The procedure we apply is similar to the one which has already been applied in the context of the Wess–Zumino model [8]: Due to local couplings one is able to extract a supersymmetry transformation $\bar{\delta}_{\dot{\alpha}}$ from the vertex functions of ordinary SQED in loop orders $l \geq 1$. Technically such an extraction is performed by working out $\frac{\delta S(\Gamma)}{\delta \overline{c}^{\alpha} \delta \overline{\chi}_{\alpha}(x)}$. The result yields

$$\lim_{E\to e} \int d^4x \, \Big(2\frac{\delta}{\delta\overline{\eta}} + e^2 A^{\mu}\frac{\delta}{\delta A^{\mu}}\Big)\Gamma = \lim_{E\to e} \frac{\partial}{\partial \overline{c}^{\dot{\alpha}}}\Big(s_{\Gamma}\big(-\int d^4x \, \frac{\delta}{\delta \overline{\chi}_{\dot{\alpha}}(x)}\Gamma\big)\Big). \tag{60}$$

For evaluating the l.h.s. of (60) the identity

$$\int d^4x \, \Big(\frac{\delta}{\delta\eta} - \frac{\delta}{\delta\overline{\eta}}\Big)\Gamma = 0 \tag{61}$$

becomes relevant. Using (19) it enables us to relate the integrated field differentiation with respect to $\overline{\eta}$ immediately to a differentiation with respect to the coupling:

$$\int d^4x \, \frac{\delta}{\delta\overline{\eta}}\Gamma = \int d^4x \, \frac{\delta}{\delta\eta}\Gamma = -\frac{1}{2}\int d^4x \, e^3(x)\frac{\delta}{\delta e}\Gamma \ . \tag{62}$$

Now we can use (40) in (60) and obtain

$$\Big(-e^3\partial_e + e^2\int d^4x \, A^{\mu}\frac{\delta}{\delta A^{\mu}}\Big)\Gamma^{\mathrm{SQED}} = \lim_{E\to e} \frac{\partial}{\partial \overline{c}^{\dot{\alpha}}}\Big(s_{\Gamma}\big(-\int d^4x \, \frac{\delta}{\delta \overline{\chi}_{\dot{\alpha}}(x)}\Gamma\big)\Big) \ . \tag{63}$$

This is the desired relation between SQED Green functions and the Green functions with a vertex insertion of the spinor component of the chiral multiplet $\bar{\mathcal{L}}_{\mathrm{kin}}$. In the following we evaluate the right-hand-side of eq. (63) and obtain the non-local expressions for the Green functions we are looking for.

5.1 The chiral vertex functions

First we derive explicit expressions for the chiral vertex functions $\Gamma_{\psi_L\psi_R}$ and $\Gamma_{q\psi_L\psi_R}$. These chiral vertex functions are superficially divergent in the Wess–Zumino gauge. But from the expression $\Gamma_{\mathrm{ct,inv}}$ (57) the common origin of their

divergences can be identified as the gauge dependent field redefinition $\Gamma_{\mathrm{ct},\psi}$, so both divergences arise with the same coefficient $z_\psi f_\psi(\xi)$. This structure can be made explicit by evaluating the identity (63) for the vertex functions $\Gamma_{\psi_L\psi_R}$ and $\Gamma_{q\psi_L\psi_R}$.

Differentiating the identity (63) with respect to $\psi_L^\beta(x)$ and $\psi_R^\gamma(y)$ yields

$$\begin{aligned} & e^3\partial_e\Gamma_{\psi_L^\beta\psi_R^\gamma}(p,-p) = \\ & \Gamma_{\overline{\epsilon}^{\dot\beta}\psi_L^\beta Y_{\overline{\varphi}_R}}(p,-p)\Gamma_{\overline{\chi}_{\dot\beta}\psi_R^\gamma\overline{\varphi}_R}(-p,p) - \Gamma_{\overline{\epsilon}^{\dot\beta}\psi_R^\gamma Y_{\overline{\varphi}_L}}(-p,p)\Gamma_{\overline{\chi}_{\dot\beta}\psi_L^\beta\overline{\varphi}_L}(p,-p) \\ - & \Gamma_{\overline{\epsilon}^{\dot\beta}\overline{\chi}_{\dot\beta}\psi_L^\beta Y_{\overline{\psi}_R^\alpha}}(p,-p)\Gamma_{\psi_R^\gamma\overline{\psi}_{R\alpha}}(-p,p) + \Gamma_{\overline{\epsilon}^{\dot\beta}\overline{\chi}_{\dot\beta}\psi_R^\gamma Y_{\overline{\psi}_L^\alpha}}(-p,p)\Gamma_{\psi_L^\beta\overline{\psi}_{L\alpha}}(p,-p) \\ - & \Gamma_{\overline{\epsilon}^{\dot\beta}\overline{\chi}_{\dot\beta}\psi_L^\beta Y_{\psi_{L\,\alpha}}}(p,-p)\Gamma_{\psi_R^\gamma\psi_L^\alpha}(-p,p) + \Gamma_{\overline{\epsilon}^{\dot\beta}\overline{\chi}_{\dot\beta}\psi_R^\gamma Y_{\psi_{R\,\alpha}}}(-p,p)\Gamma_{\psi_L^\beta\psi_R^\alpha}(p,-p)\ . \end{aligned} \tag{64}$$

Here all vertex functions are evaluated with constant gauge coupling. In particular the expression on the left-hand-side is nothing but the scalar part of the ordinary electron self energy, where the derivative can be evaluated using the topological formula:

$$e\partial_e\Gamma^{(l)}_{\psi_L^\beta\psi_R^\gamma}(p,-p) = 2l\,\Gamma^{(l)}_{\psi_L^\beta\psi_R^\gamma}(p,-p)\ . \tag{65}$$

Now we argue that the r.h.s. of (64) can be written as

$$e^3\partial_e\Gamma_{\psi_L^\beta\psi_R^\gamma}(p,-p) = 2\Sigma_{\psi,\mathrm{div}}(p^2)\Gamma_{\psi_L^\beta\psi_R^\gamma}(p,-p) + \epsilon_{\beta\gamma}mp^2\Sigma_{\psi,\mathrm{conv}}\ , \tag{66}$$

where $\Sigma_{\psi,\mathrm{conv}}$ is superficially convergent and $\Sigma_{\psi,\mathrm{div}}$ contains the superficial divergence. Indeed, the expressions $\Gamma_{\overline{\epsilon}^{\dot\beta}\overline{\chi}_{\dot\beta}\psi_L^\beta Y_{\psi_{L\,\alpha}}}(p,-p)$ (and similar for ψ_R) in the last line of (64) are logarithmically divergent and are of the same loop order as the left-hand-side. Due to their Lorentz structure we write

$$\Gamma_{\overline{\epsilon}^{\dot\beta}\overline{\chi}_{\dot\beta}\psi_L^\beta Y_{\psi_{L\,\alpha}}}(p,-p) = \delta^\alpha_\beta\Sigma_{\psi,\mathrm{div}}(p^2). \tag{67}$$

On the other hand, the Green functions $\Gamma_{\overline{\epsilon}^{\dot\beta}\psi_L^\beta Y_{\overline{\varphi}_R}}(p,-p)$ and $\Gamma_{\overline{\epsilon}^{\dot\beta}\overline{\chi}_{\dot\beta}\psi_L^\beta Y_{\overline{\psi}_R^\alpha}}(p,-p)$ have dimension zero, but due to their Lorentz structure they are superficially convergent with the degree of divergence being -1. Inspection of the diagrams shows that they vanish in 1-loop order and contribute only from 2-loop order onwards. In (64) these superficially convergent Green functions are multiplied with linearly divergent Green functions. In the l-loop expression the divergent Green functions appear therefore at most with loop order $l-2$. Products of superficially convergent functions with lower order divergent Green functions contribute to the left-hand-side as superficially convergent contributions and the divergences of lower orders are related to divergent subdiagrams. Using the Lorentz structure

we denote the contributions of the first two lines with $m\epsilon_{\beta\gamma}p^2\Sigma_{\psi,\mathrm{conv}}$, establishing (66).

Explicitly, in 1-loop order (66) reduces to

$$e^2\Gamma^{(1)}_{\psi_L^\beta\psi_R^\gamma}(p,-p) = \epsilon_{\beta\gamma}m\Sigma^{(1)}_{\psi,\mathrm{div}}(p^2)\ . \tag{68}$$

For working out the non-renormalization of chiral vertex functions we have to show that the function $\Sigma_{\psi,\mathrm{div}}$ is the only divergent contribution to the chiral vertex function $\Gamma_{q\psi_L\psi_R}$. Indeed, differentiating the identity (60) with respect to ψ_L, ψ_R and q and repeating the same steps as above, we obtain the identity

$$\begin{aligned} &e^3\partial_e\Gamma_{q\psi_L^\beta\psi_R^\gamma}(p_1,p_2,p_3) = \\ &(\Sigma_{\psi,\mathrm{div}}(p_2^2)+\Sigma_{\psi,\mathrm{div}}(p_3^2))\Gamma_{q\psi_L^\beta\psi_R^\gamma}(p_1,p_2,p_3) + \epsilon_{\beta\gamma}\Sigma_{q\psi,\mathrm{conv}}\ . \end{aligned} \tag{69}$$

While the convergent part $\Sigma_{q\psi,\mathrm{conv}}$ appearing here is different from $\Sigma_{\psi,\mathrm{conv}}$, the divergent part in (69) and (66) is the same.

In one-loop order one has for example

$$\epsilon_{\beta\gamma}\Sigma_{q\psi,\mathrm{conv}} = -m\Big(\Gamma^{(1)}_{\overline{\epsilon}^\beta\overline{\chi}_\beta q\psi_L^\beta Y^\gamma_{\psi_L}}(p_1,p_2,p_3) - \Gamma^{(1)}_{\overline{\epsilon}^\beta\overline{\chi}_\beta q\psi_R^\gamma Y^\beta_{\psi_R}}(p_1,p_3,p_2)\Big) \tag{70}$$

and therefore

$$\begin{aligned} 2e^2\Gamma^{(1)}_{q\psi_L^\beta\psi_R^\gamma}(p_1,p_2,p_3) &= (\Sigma^{(1)}_{\psi,\mathrm{div}}(p_2^2)+\Sigma^{(1)}_{\psi,\mathrm{div}}(p_3^2))\epsilon_{\beta\gamma} \\ &\quad - m\Big(\Gamma^{(1)}_{\overline{\epsilon}^\beta\overline{\chi}_\beta q\psi_L^\beta Y^\gamma_{\psi_L}}(p_1,p_2,p_3) - \Gamma^{(1)}_{\overline{\epsilon}^\beta\overline{\chi}_\beta q\psi_R^\gamma Y^\beta_{\psi_R}}(p_1,p_3,p_2)\Big) \end{aligned} \tag{71}$$

and similar expressions in higher orders. Therefore the only divergent contribution to the vertex function $\Gamma_{q\psi\psi}$ is again $\Sigma_{\psi,\mathrm{div}}(p^2)$ appearing in the chiral self energy of the electrons. For this reason the divergences of $\Gamma_{\psi\psi}$ and $\Gamma_{q\psi\psi}$ can be absorbed into the gauge-dependent part $z_\psi f_\psi(\xi)$ of the electron field redefinition (57). Equivalently, and as expressed by eq. (58), the renormalization constants for the mass parameter m and the field q have to be equal.

5.2 The non-renormalization of the photon self energy in higher orders

Since divergent counterterms to the photon self energy are absent in loop order $l \geq 2$, the photon self energy can be completely related to non-local, counterterm independent expressions. For working out this relation we will apply a similar procedure as in the previous section. The result differs from the corresponding expression for chiral vertex function in a remarkable way: The photon self energy is not related to superficially convergent Green functions but to expressions which are linearly divergent by power counting. By gauge invariance, however, the local divergent part of all these expressions is completely determined by the non-local part.

For the derivation we need two relations of SQED Green functions determined in a previous paper [14]. The first relation is the relation between the photon and photino self energy

$$\Gamma_{\overline{\lambda}^{\beta}\lambda^{\alpha}}(-p,p) = -\sigma^{\mu}_{\alpha\dot{\beta}}p_{\mu}(1+\Sigma_A(p^2)) \ , \tag{72}$$

where $\Sigma_A(p^2)$ is the self energy of the photon:

$$\Gamma_{A^{\mu}A^{\nu}}(-p,p) = -(\eta_{\mu\nu}p^2 - p_{\mu}p_{\nu})(1+\Sigma_A(p^2)) \ . \tag{73}$$

(We use the B-gauge, where the photon self energy is transversal.) Second we use that the supersymmetry transformation of the photino into the photon is local and determined by its tree expression:

$$\Gamma_{\epsilon^{\beta}A_{\mu}Y_{\lambda\alpha}}(p,-p) = p_{\rho}(\sigma^{\rho\mu})_{\beta}{}^{\alpha} \ , \qquad \Gamma_{\overline{\epsilon}^{\dot{\beta}}A_{\mu}Y_{\overline{\lambda}_{\dot{\alpha}}}}(p,-p) = -p_{\rho}(\overline{\sigma}^{\rho\mu})^{\dot{\alpha}}{}_{\dot{\beta}} \ . \tag{74}$$

In a first step we extract a supersymmetry transformation δ_{α} from the photon self energy by using $\frac{\delta S(\Gamma)}{\delta\epsilon^{\beta}\delta\chi_{\beta}}$ (which reproduces the complex conjugate identity to (60)). With eq. (62) and with the help of the topological formula we find the following result:

$$2(l-1)e^2\Gamma^{(l)}_{A^{\mu}A^{\nu}} = p^{\rho}\Gamma^{(l)}_{\chi_{\beta}A^{\nu}\lambda^{\alpha}}(-p,p)(\sigma_{\mu\rho})_{\beta}{}^{\alpha} - p^{\rho}\Gamma^{(l)}_{\chi_{\beta}A^{\mu}\lambda^{\alpha}}(p,-p)(\sigma_{\nu\rho})_{\beta}{}^{\alpha}. \tag{75}$$

As for the invariant counterterms (cf. (57)), the one-loop level is distinguished from higher loop orders in the expression (75): For $l = 1$ the photon self energy drops out, and it cannot be related to a non-local, renormalization-independent expression. It is divergent and the divergences are absorbed by the counterterm

$z_{\rm kin}(1)$. At the 1-loop level the identity (75) together with the gauge Ward identity instead determines the vertex function $\Gamma^{(1)}_{\chi_\beta A^\nu \lambda^\alpha}$:

$$\Gamma^{(1)}_{\chi_\beta A^\nu \lambda^\alpha}(-p,p) = 0. \tag{76}$$

In higher orders eq. (75) relates the photon self energy to the Green functions $\Gamma^{(l)}_{\chi_\beta A^\mu \lambda^\alpha}$. Moreover, we are able to extract a supersymmetry transformation $\bar{\delta}_{\dot\alpha}$ from the chiral Green function $\Gamma^{(l)}_{\chi_\beta A^\mu \lambda^\alpha}$ by testing the identity (63) with respect to λ^β, A^μ and χ^γ. Using the topological formula (35) and the relation

$$\Gamma_{\overline{\epsilon}^{\dot\beta}\overline{\chi}_{\dot\beta}\lambda^\beta Y_{\lambda\alpha}}(-p,p) = \delta^\alpha_\beta \ , \tag{77}$$

which follows from the linear equation (34), we find in momentum space

$$\begin{aligned} 2e^2 l\ \Gamma^{(l)}_{\chi_\gamma \lambda^\beta A^\mu}(-p,p) = {} & i\sigma^\rho_{\beta\dot\beta}\Gamma^{(l)}_{\overline{\chi}_{\dot\beta}\chi_\gamma A^\mu A^\rho}(p,-p) - p_\rho(\overline{\sigma}^{\rho\mu})^{\dot\alpha}{}_{\dot\beta}\Gamma^{(l)}_{\overline{\chi}_{\dot\beta}\chi_\gamma\lambda^\beta\overline{\lambda}^{\dot\alpha}}(-p,p) \\ & - \sum_{k=0}^{l}\Gamma^{(l-k)}_{\overline{\epsilon}^{\dot\beta}\overline{\chi}_{\dot\beta}\chi_\gamma A^\mu Y^{\alpha}_{\overline{\lambda}}}(p,-p)\Gamma^{(k)}_{\lambda^\beta\overline{\lambda}_\alpha}(-p,p). \end{aligned} \tag{78}$$

Used in eq. (75) this expression determines the photon self energy in $l \geq 2$.

Let us discuss the right-hand-side of eq. (78). On the one hand, the naive degree of power counting is not improved, but the individual parts are linearly and logarithmically divergent. On the other hand, however, all Green functions appearing with the photon field are constrained by the gauge Ward identity. In this way all local divergent contributions are uniquely determined by non-local, convergent contributions, as we will now show.

The most important term in (78) is the linearly divergent Green function $\Gamma_{\overline{\chi}\chi A^\mu A^\nu}$. From the gauge Ward identity one finds

$$p_1^\mu\Gamma_{\overline{\chi}_{\dot\beta}\chi_\gamma A^\mu A^\nu}(p_3,p_4,p_1,p_2) = p_2^\nu\Gamma_{\overline{\chi}_{\dot\beta}\chi_\gamma A^\mu A^\nu}(p_3,p_4,p_1,p_2) = 0 \ , \tag{79}$$

where temporarily non-zero momenta p_3 and p_4 have been assigned to the χ- and $\overline{\chi}$-vertices. Taking into account parity conservation we find the following covariant tensor decomposition for the symmetrized expression

$$\begin{aligned} & \frac{1}{2}\big(\Gamma_{\overline{\chi}^{\dot\beta}\chi^\gamma A^\mu A^\nu}(p_3,0,p_1,p_2) + \Gamma_{\overline{\chi}^{\dot\beta}\chi^\gamma A^\mu A^\rho}(0,p_3,p_1,p_2)\big) \\ & = i\sigma^\lambda_{\gamma\dot\beta}\big(\epsilon_{\lambda\mu\sigma\rho}p_1^\rho(\eta^\sigma{}_\nu\Sigma_1(p_1,p_2) - p_2^\sigma p_1^\nu\Sigma_2(p_1,p_2) - p_2^\sigma p_{2\nu}\Sigma_3(p_1,p_2)) \\ & \quad + \epsilon_{\lambda\nu\sigma\rho}p_2^\rho(\eta^\sigma{}_\mu\Sigma_1(p_2,p_1) - p_1^\sigma p_{2\mu}\Sigma_2(p_2,p_1) - p_1^\sigma p_{1\mu}\Sigma_3(p_2,p_1)) \\ & \quad + \epsilon_{\mu\nu\sigma\rho}p_1^\rho p_2^\sigma(p_{1\lambda}\Sigma_4(p_1,p_2) + p_{2\lambda}\Sigma_4(p_2,p_1))\big). \end{aligned} \tag{80}$$

For $p_1 \neq p_2$ transversality (79) restricts the single parts appearing in the above expression,

$$\Sigma_1(p_1,p_2) = p_1 \cdot p_2 \Sigma_2(p_1,p_2) + p_2^2 \Sigma_3(p_1,p_2) \ , \tag{81}$$

but due to analyticity (81) has to hold also for $p_1 = p_2$. The linearly divergent part Σ_1 in $\Gamma_{\overline{\chi}_{\dot\beta}\chi^\gamma A^\mu A^\nu}(0,0,-p,p)$ is therefore completely determined by the non-local parts Σ_2 and Σ_3.

In a similar way we find

$$\Gamma_{\overline{\epsilon}^{\dot\beta}\overline{\chi}_{\dot\beta}\chi^\gamma A^\mu Y^\alpha_{\overline{\lambda}}}(p,-p) = i\sigma^\rho_{\gamma\dot\alpha}(\eta_{\rho\mu}p^2 - p_\rho p_\mu)\Sigma_5(p^2) \ . \tag{82}$$

Again the local part is completely determined by the non-local part. Finally we decompose also the logarithmically divergent Green function $\Gamma_{\overline{\chi}_{\dot\beta}\chi_\gamma\lambda^\beta\overline{\lambda}^\alpha}(-p,p)$:

$$\begin{aligned}\Gamma_{\overline{\chi}^{\dot\beta}\chi^\gamma\lambda^\beta\overline{\lambda}^\alpha}(-p,p) &= \epsilon_{\gamma\beta}\epsilon_{\dot\alpha\dot\beta}\Sigma_6(p^2) \\ &\quad + \sigma^\lambda_{\gamma\dot\alpha}\sigma^\rho_{\beta\dot\beta}p_\rho p_\lambda \Sigma_7(p^2) + \sigma^\lambda_{\beta\dot\alpha}\sigma^\rho_{\gamma\dot\beta}p_\rho p_\lambda \Sigma_8(p^2).\end{aligned} \tag{83}$$

Inserting (83) into (78) the first term drops out, and only the non-local parts $\Sigma_7(p^2) + \Sigma_8(p^2)$ contribute to the photon self energy.

When we insert the above expressions (80) with (81), (82) and (83) into eq. (78), we find a non-local regularization independent expression for the vertex function $\Gamma_{\chi\lambda A}$. When we finally use the relation (75) we find the desired result, in which the photon self energy for $l \geq 2$ is completely expressed by non-local Green functions:

$$\begin{aligned}l(l-1)e^4\Sigma_A^{(l)}(p^2) = p^2\big(&-4(\Sigma_2^{(l)}(p^2) - \Sigma_3^{(l)}(p^2)) + \Sigma_7^{(l)}(p^2) + \Sigma_8^{(l)}(p^2) \\ &+\Sigma_5^{(l)} + \sum_{k=1}^{l-2}\Sigma_5^{(l-k)}(p^2)\Sigma_A^{(k)}(p^2)\big)\end{aligned} \tag{84}$$

Here we have used the supersymmetry relation between photon and photino self-energy (72), which makes possible to express the self energy of the photino appearing in (78) by the photon self energy. From 3-loop order onwards the divergent 1-loop self energy of the photon appears in the above formula, but only in a product with the non-local contribution Σ_5. Such an expression does not contribute to superficial divergences of the photon self energy in $l \geq 2$ but is related to the appearance of subdivergences.

The arguments we have used here for relating the photon self energy to non-local expressions are analogous to the ones used for calculating the axial-current–photon–photon Green functions (see [9] and [23] for a recent review). There,

too, the divergent part of the triangle diagram is uniquely related to non-local expressions by gauge invariance. In fact, there is a connection between the photon self energy and the axial-vector Green functions and the Adler–Bardeen anomaly. This is revealed when the SQED action with local coupling is extended by the axial vector multiplet. Then we will see that the $\overline{\chi}\chi$-insertions contributing on the right-hand-side of (78) induce nothing but insertions of the axial current (see section 9).

6 Softly broken axial symmetry and the axial vector multiplet

Although the perturbative construction of SQED with local couplings is well-defined, it is not complete in the sense of multiplicative renormalization. The invariant counterterm $\Gamma_{\text{ct,matter}}$ (46) cannot be related to a wave function renormalization as long as the coupling is local. Therefore it is impossible to interpret all free parameters of higher orders as field and coupling renormalizations. In this section multiplicative renormalizability is restored by introducing an additional axial vector multiplet. Then the counterterm $\Gamma_{\text{ct,matter}}$ (46) can be understood as a field renormalization of the axial vector multiplet and the matter fields, and it is possible to derive the Callan–Symanzik equation.

The axial symmetry

$$\begin{aligned} &\delta^5\phi_A = -i\tilde{\omega}(x)\phi_A \ , \qquad \delta^5\bar{\phi}_A = i\tilde{\omega}(x)\bar{\phi}_A \ , \qquad \phi = \varphi, \psi, F; \quad A = L, R \\ &\delta^5 A^\mu = \delta^5\lambda = \delta^5\overline{\lambda} = \delta^5 D = 0 \end{aligned} \tag{85}$$

is softly broken by the matter mass term and gives rise to a partially conserved axial current in the classical approximation:

$$\delta^5\Gamma_{\text{SQED}} = -\int d^4x\ \tilde{\omega}(x)\partial^\mu j_\mu^{\text{axial}} - 2im\int d^4x\ \tilde{\omega}(x)(L_{\text{mass}} - \bar{L}_{\text{mass}}) \tag{86}$$

with the axial current

$$j_\mu^{\text{axial}} = i\left(\overline{\varphi}_L D_\mu\varphi_L - \varphi_L D_\mu\overline{\varphi}_L + i\psi_L\sigma_\mu\overline{\psi}_L + (_{L\to R})\right) \ . \tag{87}$$

(We take the local coupling in all covariant derivatives.)

Although the axial symmetry is softly broken, it can be gauged and supersymmetrized in the same way as electric charge symmetry: We introduce the external axial multiplet $V^\mu, \tilde{\lambda}^\alpha, \overline{\tilde{\lambda}}^{\dot{\alpha}}, \tilde{D}$ with axial transformations

$$\delta^5 V^\mu = \partial^\mu\tilde{\omega}(x), \qquad \delta^5\tilde{\lambda}^\alpha = \delta^5\overline{\tilde{\lambda}}^\alpha = \delta^5\tilde{D} = 0, \tag{88}$$

and supersymmetry transformations in the Wess–Zumino gauge

$$\begin{aligned}
&\delta_\alpha V^\mu = i\sigma^\mu_{\alpha\dot\alpha}\overline{\tilde\lambda}^{\dot\alpha}, && \bar\delta_{\dot\alpha} V^\mu = -i\tilde\lambda^\alpha\sigma^\mu_{\alpha\dot\alpha},\\
&\delta_\alpha\tilde\lambda^\beta = \frac{i}{2}\sigma^{\mu\nu\ \beta}_{\ \alpha} F_{\mu\nu}(V) + \frac{i}{2}\delta^\beta_\alpha\tilde D, && \bar\delta_{\dot\alpha}\tilde\lambda_\beta = 0,\\
&\delta_\alpha\overline{\tilde\lambda}_{\dot\alpha} = 0, && \bar\delta^{\dot\alpha}\overline{\tilde\lambda}_{\dot\beta} = \frac{i}{2}\bar\sigma^{\mu\nu\ \dot\alpha}_{\ \dot\beta} F^{\mu\nu}(V) + \frac{i}{2}\delta^{\dot\alpha}_{\dot\beta}\tilde D,\\
&\delta_\alpha\tilde D = 2\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\overline{\tilde\lambda}^{\dot\alpha}, && \bar\delta_{\dot\alpha}\tilde D = 2\partial_\mu\tilde\lambda^\alpha\sigma^\mu_{\alpha\dot\alpha}.
\end{aligned} \tag{89}$$

When we extend the covariant derivatives to

$$D^\mu\phi_A = (\partial^\mu + ie(x)Q_A A^\mu + iV^\mu)\phi_A\ , \qquad A = L, R \tag{90}$$

in the matter part of the action

$$\begin{aligned}
\Gamma_{\text{matter}} = \int d^4x\ \Big(& D^\mu\overline{\varphi}_L D_\mu\varphi_L + i\psi^\alpha_L\sigma^\mu_{\alpha\dot\alpha}D_\mu\overline{\psi}^{\dot\alpha}_L + \overline{F}_L F_L\\
& + ieQ_L\sqrt{2}(\lambda\psi_L\overline{\varphi}_L - \overline{\lambda\psi}_L\varphi_L) + i\sqrt{2}(\tilde\lambda\psi_L\overline{\varphi}_L - \overline{\tilde\lambda}\overline{\psi}_L\varphi_L)\\
& + \frac{1}{2}eQ_L D\overline{\varphi}_L\varphi_L + \frac{1}{2}\tilde D\overline{\varphi}_L\varphi_L + (_{L\to R})\Big)
\end{aligned} \tag{91}$$

and in the supersymmetry transformations, then the kinetic part (22) and the matter part (91) are invariant under gauge symmetry with local coupling (12), axial symmetry (85, 88) and supersymmetry transformations.

Finally we assign shifted axial transformations to the neutral chiral multiplet $\mathbf{q} = (q, q^\alpha, q_F)$ (24) and its complex conjugate:

$$\delta^5\mathbf{q} = 2i(\mathbf{q} + (m,0,0))\tilde\omega(x)\ , \qquad \delta^5\bar{\mathbf{q}} = -2i(\bar{\mathbf{q}} + (m,0,0))\tilde\omega(x) \tag{92}$$

and replace the ordinary derivative by

$$D_\mu\mathbf{q} = (\partial_\mu - 2iV_\mu)(\mathbf{q} + (m,0,0)) \tag{93}$$

in the supersymmetry transformations of the $\mathbf{q}$ multiplet. Then the sum of the matter mass term Γ_{mass} (3) and the chiral $\mathbf{q}$-interaction term (25) become invariant under axial symmetry

$$\delta^5(\Gamma_{\text{mass}} + \Gamma_q) = 0, \tag{94}$$

and axial symmetry can be used as a defining symmetry of the model.

For quantization the axial symmetry (100) has to be included into the Slavnov–Taylor identity in the same way as the gauge transformations (see section 3.1).

The respective BRS transformations are obtained by replacing the transformation parameter $\tilde{\omega}(x)$ with the parity odd axial ghost $\tilde{c}(x)$. The BRS transformation of the axial ghost is determined by the structure constants of the algebra:

$$s\tilde{c} = 2i\epsilon\sigma\overline{\epsilon}V - i\omega\partial\tilde{c}. \tag{95}$$

The complete BRS transformations of SQED with local coupling and the axial vector multiplet are summarized in appendix C.

The Slavnov–Taylor operator of the extended action includes the BRS transformations of the axial vector multiplet ($V^i = (V^\mu, \tilde{\lambda}^\alpha, \overline{\tilde{\lambda}}^{\dot{\alpha}}, \tilde{D})$) and of the axial ghost in addition to (32):

$$\mathcal{S}(\mathcal{F}) \to \mathcal{S}(\mathcal{F}) + \int d^4x \, \Big(sV^i \frac{\delta \mathcal{F}}{\delta V^i} + s\tilde{c} \frac{\delta \mathcal{F}}{\delta \tilde{c}} \Big). \tag{96}$$

Nilpotency of the Slavnov–Taylor operator (33) and the linear equation (34) remain valid in their original form. The classical action with gauged axial symmetry satisfies the Slavnov–Taylor identity with the extended Slavnov–Taylor operator (96) by construction:

$$\mathcal{S}(\Gamma_{\rm cl}) = 0 \ . \tag{97}$$

Similarly as in eq. (37) for the Faddeev–Popov ghost $c(x)$, a linear ghost equation for the axial ghost can be derived:

$$\frac{\delta \Gamma}{\delta \tilde{c}} = i(\psi_L^\alpha Y_{\psi_L \alpha} + \varphi_L Y_{\varphi_L} - \overline{\psi}_{L\dot{\alpha}} Y^{\dot{\alpha}}_{\overline{\psi}_L} - \overline{\varphi}_L Y_{\overline{\varphi}_L}) + (_{L\to R}) \ . \tag{98}$$

Using the consistency equation

$$\frac{\delta}{\delta \tilde{c}} \mathcal{S}(\Gamma) + s_\Gamma \big(\frac{\delta}{\delta \tilde{c}} \Gamma \big) = \partial^\mu \frac{\delta}{\delta V^\mu} \Gamma \tag{99}$$

at the classical level together with (97) we can derive an axial Ward identity expressing softly broken axial symmetry:

$$\Big(\mathbf{w}^{\rm axial} - \partial^\mu \frac{\delta}{\delta V^\mu} \Big) \Gamma_{\rm cl} = -2im \Big(\frac{\delta \Gamma_{\rm cl}}{\delta q} - \frac{\delta \Gamma_{\rm cl}}{\delta \overline{q}} \Big) + \mathcal{O}(\omega) \tag{100}$$

with

$$\begin{aligned} \mathbf{w}^{\rm axial} = -i \Big(& \varphi_L \frac{\delta}{\delta \varphi_L} - Y_{\varphi_L} \frac{\delta}{\delta Y_{\varphi_L}} + \psi_L^\alpha \frac{\delta}{\delta \psi_L^\alpha} - Y^\alpha_{\psi_L} \frac{\delta}{\delta Y^\alpha_{\psi_L}} \\ & - \overline{\varphi}_L \frac{\delta}{\delta \overline{\varphi}_L} + Y_{\overline{\varphi}_L} \frac{\delta}{\delta Y_{\overline{\varphi}_L}} - \overline{\psi}^{\dot{\alpha}}_L \frac{\delta}{\delta \overline{\psi}^{\dot{\alpha}}_L} + Y^{\dot{\alpha}}_{\overline{\psi}_L} \frac{\delta}{\delta Y^{\dot{\alpha}}_{\overline{\psi}_L}} \Big) + (_{L\to R}) \\ & + 2i \Big(q \frac{\delta}{\delta q} + q^\alpha \frac{\delta}{\delta q^\alpha} + q_F \frac{\delta}{\delta q_F} - \overline{q} \frac{\delta}{\delta \overline{q}} - \overline{q}^{\dot{\alpha}} \frac{\delta}{\delta \overline{q}^{\dot{\alpha}}} - \overline{q}_F \frac{\delta}{\delta \overline{q}_F} \Big) \ . \end{aligned} \tag{101}$$

7 The Adler–Bardeen anomaly and its non-renormalization theorem

At the quantum level the axial Ward identity (100) is broken by the Adler–Bardeen anomaly [9, 10, 11]. Since the axial transformations are included in the Slavnov–Taylor identity, the anomaly will appear as an anomaly of the Slavnov–Taylor identity in the present construction. In one loop order, all breakings of the Slavnov–Taylor identity

$$\mathcal{S}(\Gamma) = \Delta + \mathcal{O}(\hbar^2) \tag{102}$$

are algebraically restricted by the nilpotency properties of the Slavnov–Taylor operator (33) and by the identity (23), which restricts the appearance of $\eta - \overline{\eta}$ to a total derivative. Therefore Δ has to satisfy the following constraints:

$$s_{\Gamma_{cl}}\Delta = 0 \qquad \text{and} \qquad \int d^4x \, \Big(\frac{\delta}{\delta\eta} - \frac{\delta}{\delta\overline{\eta}}\Big)\Delta = 0 \ . \tag{103}$$

Moreover, Δ is restricted by discrete symmetries (see appendix C) and the number of local couplings appearing in the field monomials are determined by the [3].

The anomaly is a field polynomial that appears in Δ but cannot be absorbed by a counterterm contribution. Setting the supersymmetry ghosts to zero, the anomaly appears with the ghost of axial symmetry and has the following form:

$$\Delta^{\text{anomaly}}\Big|_{\epsilon,\overline{\epsilon},\omega=0} = r^{(1)} \int d^4x \ \tilde{c}\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}(eA)F_{\rho\sigma}(eA) \ . \tag{104}$$

Using the consistency equation (99) we find that the same anomaly appears in the axial Ward identity (100) and that $r^{(1)}$ is determined by the usual triangle diagram:

$$r^{(1)} = -\frac{1}{16\pi^2} \ . \tag{105}$$

It is a crucial feature of the construction with the local coupling and axial vector field, that the axial anomaly (104) can be written as a differential operator acting on the classical action:

$$\int d^4x \ \tilde{c}\big(\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}(eA)F_{\rho\sigma}(eA) - 4\partial_\mu(\lambda\sigma^\mu\overline{\lambda})\big) = 4i \int d^4x \ \Big(\tilde{c}\Big(\frac{\delta}{\delta\eta} - \frac{\delta}{\delta\overline{\eta}}\Big)\Gamma_{\rm cl} \ . \tag{106}$$

[3] The topological formula (35) is valid also in presence of the axial vector multiplet.

Supersymmetry determines a supersymmetric extension of the Adler–Bardeen anomaly [15, 16]. In our framework, its form can be found by extending the right-hand-side of (106) to a BRS-invariant operator. As a result, the Adler–Bardeen anomaly and its supersymmetric extension can be written as a BRS-invariant operator acting on the classical action:

$$\begin{aligned}\Delta^{\text{anomaly}} &= 4ir^{(1)}\int d^4x\,\Big(\tilde{c}\Big(\frac{\delta}{\delta\eta}-\frac{\delta}{\delta\overline{\eta}}\Big)+2i(\epsilon\sigma^\mu)^{\dot\alpha}V_\mu\frac{\delta}{\delta\overline{\chi}^{\dot\alpha}}-2i(\sigma^\mu\overline{\epsilon})^\alpha V_\mu\frac{\delta}{\delta\chi^\alpha}\\ &\qquad +2\overline{\epsilon}_{\dot\alpha}\overline{\tilde{\lambda}}^{\dot\alpha}\frac{\delta}{\delta f}-2\tilde{\lambda}^\alpha\epsilon_\alpha\frac{\delta}{\delta\overline{f}}\Big)\Gamma_{\text{cl}}\\ &\equiv -r^{(1)}\delta\mathcal{S}\Gamma_{\text{cl}}\ .\end{aligned}\tag{107}$$

Since the anomaly Δ^{anomaly} can be expressed in this form, it can be absorbed into a redefined Slavnov–Taylor operator:

$$\mathcal{S}(\Gamma)+r^{(1)}\delta\mathcal{S}\Gamma=0+\mathcal{O}(\hbar^2)\ .\tag{108}$$

The new piece $\delta\mathcal{S}$ being a symmetric operator, the operator $\big(\mathcal{S}+r^{(1)}\delta\mathcal{S}\big)(\mathcal{F})$ and its linearized version $s_{\mathcal{F}}+r^{(1)}\delta\mathcal{S}$ have the same nilpotency properties as the original Slavnov–Taylor operator (see (33)). Owing to this property, algebraic renormalization can be continued to higher orders in spite of the presence of the anomaly.

In particular, the breakings of higher orders are restricted by (103) in the same way as the ones of 1-loop order. Taking into account the topological formula, the gauge anomaly of loop order l takes the general form:

$$\Delta_l^{\text{anomaly}}\Big|_{\epsilon,\overline{\epsilon},\omega=0}=r^{(l)}\int d^4x\ e^{2(l-1)}(x)\tilde{c}\,\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}(eA)F_{\rho\sigma}(eA)\ .\tag{109}$$

Apparently the 1-loop case $l=1$ is special. At this order the coefficient of the ghost in (109) is a total derivative, but with local coupling it is not a total derivative at higher orders. Hence, when we use the consistency equation (99) and integrate the axial Ward identity, the anomaly appears in the the Ward identity of global axial symmetry for $l\geq 2$

$$\mathcal{W}^{\text{axial}}\Gamma=r^{(l)}\int d^4x\ e^{2(l-1)}(x)\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}(eA)F_{\rho\sigma}(eA)+\text{soft terms}\tag{110}$$

with

$$\mathcal{W}^{\text{axial}}\equiv\int d^4x\ \mathbf{w}^{\text{axial}}(x)\ .\tag{111}$$

Only for $l = 1$ the integrand in (110) is a total derivative and the integral vanishes.

Owing to the structure of $\mathcal{W}^{\text{axial}}$, no term on the l.h.s. of (110) is composed solely of photon fields. Hence, testing with respect to two photon fields shows that the coefficients $r^{(l)}$ have to vanish identically for $l \geq 2$:

$$r^{(l)} = 0 \quad \text{if} \quad l \geq 2. \tag{112}$$

Local couplings provide therefore a very simple and elegant way for proving the non-renormalization of the axial anomaly [12]. It is clear that the derivation is not restricted to supersymmetric gauge theories but can be applied in the same way to ordinary gauge theories.

In summary we find that the Green functions of SQED with local gauge coupling and an axial vector multiplet satisfy the anomalous Slavnov–Taylor identity

$$\big(\mathcal{S} + r^{(1)}\delta\mathcal{S}\big)\Gamma = 0 \tag{113}$$

and the anomalous axial Ward identity (cf. (100)):

$$\Big(\mathbf{w}^{\text{axial}} - \partial^\nu \frac{\delta}{\delta V^\nu}\Big)\Gamma = 4ir^{(1)}\Big(\frac{\delta}{\delta\eta} - \frac{\delta}{\delta\overline{\eta}}\Big)\Gamma - 2im\Big(\frac{\delta}{\delta q} - \frac{\delta}{\delta\overline{q}}\Big)\Gamma + \mathcal{O}(\omega) \tag{114}$$

to all orders of perturbation theory.

We want to note already here that the supersymmetric extension of the Adler–Bardeen anomaly induces a modification of supersymmetry transformations for axial-current Green functions. A detailed discussion of this point and its relation to the manifestly supersymmetric gauge can be found in section 9.

8 The Callan–Symanzik equation

In this section the power of the construction using the local gauge coupling and the anomalous axial current becomes evident. With the axial multiplet all invariant counterterms can be given as invariant field operators acting on the classical action; consequently the model possesses a valid Callan–Symanzik equation. The non-renormalization theorems derived in section 3 and 4 will then appear as restrictions on the various Callan–Symanzik coefficients. Indeed, we will show that the Callan–Symanzik equation contains only two gauge-independent, physical coefficients: the 1-loop β-function of the gauge coupling and the anomalous dimension of the mass, γ. In particular the construction provides the β-function of the gauge coupling in its closed form.

8.1 The CS equation with local coupling

We start the construction of the Callan–Symanzik (CS) equation with local couplings in the classical approximation:

$$\begin{aligned}\mu_i\partial_{\mu_i}\Gamma_{\rm cl} &= -2m^2(\varphi_L\overline{\varphi}_L+\varphi_R\overline{\varphi}_R)-m(\psi_L\psi_R+\overline{\psi}_L\overline{\psi}_R) \\ &= m\int d^4x\,\Big(\frac{\delta}{\delta q}+\frac{\delta}{\delta\overline{q}}\Big)\Gamma_{\rm cl}\ .\end{aligned} \tag{115}$$

The mass differentiation $\mu_i\partial_{\mu_i}$ contains the differentiation with respect to all mass parameters of the theory (κ is a normalization point introduced for an off-shell normalization of the matter field residua):

$$\mu_i\partial_{\mu_i} = m\partial_m+\kappa\partial_\kappa\ . \tag{116}$$

Eq. (115) can be rewritten as

$$\mu_i D_{\mu_i}\Gamma_{\rm cl} = 0 \tag{117}$$

with the operator

$$\mu_i D_{\mu_i} \equiv m\partial_m+\kappa\partial_\kappa - m\int d^4x\,\Big(\frac{\delta}{\delta q}+\frac{\delta}{\delta\overline{q}}\Big)\ , \tag{118}$$

which is symmetric with respect to the anomalous Slavnov–Taylor identity (113):

$$(s_\Gamma+r^{(1)}\delta\mathcal{S})\mu_i D_{\mu_i}\Gamma = \mu_i D_{\mu_i}(\mathcal{S}+r^{(1)}\delta\mathcal{S})(\Gamma) = 0\ . \tag{119}$$

Let us briefly outline the construction of the CS equation in higher orders before entering the calculational details. In higher orders eq. (117) is broken by hard terms, the dilatational anomalies:

$$\mu_i D_{\mu_i}\Gamma = \Delta_m\cdot\Gamma\ . \tag{120}$$

Since the operator $\mu_i D_{\mu_i}$ is symmetric with respect to the symmetries of Γ, the dilatational anomalies are completely characterized by the symmetries of Γ:

$$\big(s_\Gamma+r^{(1)}\delta\mathcal{S}\big)\big(\Delta_m\cdot\Gamma\big)=0\,,\qquad \int d^4x\,\Big(\frac{\delta}{\delta\eta}-\frac{\delta}{\delta\overline{\eta}}\Big)\big(\Delta_m\cdot\Gamma\big)=0\ . \tag{121}$$

In the Callan-Symanzik equation the breakings $\Delta_m\cdot\Gamma$ are absorbed by symmetric operators acting on Γ. These operators can be combined to the Callan-Symanzik operator $\mathcal{C}=\mu_i D_{\mu_i}+\mathcal{O}(\hbar)$, which possesses the same symmetry properties as $\mu_i D_{\mu_i}$, in particular it is an $s_\Gamma+r^{(1)}\delta\mathcal{S}$-symmetric operator:

$$\big(s_\Gamma+r^{(1)}\delta\mathcal{S}\big)\mathcal{C}\Gamma = \mathcal{C}\big(\mathcal{S}+r^{(1)}\delta\mathcal{S}\big)\Gamma+\big(s_\Gamma+r^{(1)}\delta\mathcal{S}\big)\Delta_Y\ . \tag{122}$$

The expression Δ_Y is defined to be a collection of field monomials which are linear in propagating fields. As such, Δ_Y appears as a trivial insertion and does not need to be absorbed into an operator.

The construction of the Callan-Symanzik equation proceeds by induction. When the breaking Δ_m is absorbed into a symmetric Callan-Symanzik operator to loop order $l-1$, the breaking of loop order l is a local field monomial:

$$\mathcal{C}^{(l-1)}\Gamma = 0 + \Delta_Y + \mathcal{O}(\hbar^l) \quad \Rightarrow \quad \mathcal{C}^{(l-1)}\Gamma = \Delta_m^{(l)} + \Delta_Y + \mathcal{O}(\hbar^{l+1}) \ . \tag{123}$$

The local field monomial $\Delta_m^{(l)}$ is characterized by the symmetries of the classical action:

$$s_{\Gamma_{\rm cl}}\Delta_m^{(l)} = 0 \qquad \text{and} \ \int d^4x \, \Big(\frac{\delta}{\delta\eta} - \frac{\delta}{\delta\overline{\eta}}\Big)\Delta_m^{(1)} = 0 \ , \tag{124}$$

and the numbers of local couplings in $\Delta_m^{(l)}$ are constrained by the topological formula (35).

Since $\Delta_m^{(l)}$ has the same quantum numbers and the same symmetries as the invariant counterterms of loop order l (see (42)), it consists of the same type of field monomials as the invariant counterterms (57)[4]:

$$\Delta_m^{(l)} \in \big[\Gamma^{(1)}_{\rm ct,kin}\delta_{l1}, \Gamma^{(l)}_{\rm ct,matter}, \Gamma^{(l)}_{{\rm ct},\varphi}(\xi), \Gamma^{(l)}_{{\rm ct},\psi}(\xi), \Gamma^{(l)}_{{\rm ct},\psi\varphi}(\xi)\big] \ . \tag{125}$$

Suppose, we can express the invariant counterterms in (125) as $s_\Gamma + r^{(1)}\delta\mathcal{S}$-symmetric operators in the sense of eq. (122). Then $\Delta_m^{(l)}$ can be absorbed by an additional piece of loop order l in the CS operator without destroying (122). This establishes the CS equation

$$\mathcal{C}\Gamma = \Delta_Y \tag{126}$$

by induction.

Hence, the derivation of the CS equation and the non-renormalization theorems is reduced to the algebraic problem of extending the counterterms (125) to $s_\Gamma + r^{(1)}\delta\mathcal{S}$-symmetric operators.

[4]The axial multiplet does not give rise to new counterterms except for contributions depending solely on external fields and are entirely local. They are not relevant in the present context and are omitted from all equations.

8.2 Construction

For extending the counterterms (125) to $s_\Gamma + r^{(1)}\delta\mathcal{S}$-symmetric operators, we will proceed in two steps. First, the counterterms are written as s_Γ-symmetric operators, and second they are extended to $s_\Gamma + r^{(1)}\delta\mathcal{S}$-symmetric ones.

The three field redefinitions $\Gamma^{(l)}_{\mathrm{ct},\varphi}(\xi), \Gamma^{(l)}_{\mathrm{ct},\psi}(\xi)$ and $\Gamma^{(l)}_{\mathrm{ct},\psi\varphi}(\xi)$ in (125) have already been constructed as s_Γ-symmetric operators in (54), (55) and (56):

$$\Gamma^{(l)}_{\mathrm{ct},\phi}(\xi) = \mathcal{N}^{(l)}_\phi \Gamma_{\mathrm{cl}} + \Delta^{(l)}_{Y,\phi} \qquad \text{with} \quad \phi = \varphi, \psi, \psi\varphi \ . \tag{127}$$

A straightforward calculation shows that the operators $\mathcal{N}^{(l)}_\varphi$ and $\mathcal{N}^{(l)}_\psi$ are $s_\Gamma + r^{(1)}\delta\mathcal{S}$-invariant operators at all loop orders l. The local invariant $\Gamma^{(l)}_{ct,\psi\varphi}$ can be extended to an $s_\Gamma + r^{(1)}\delta\mathcal{S}$-invariant in a trivial way by constructing it as an $s_\Gamma + r^{(1)}\delta\mathcal{S}$-variation of the corresponding local field monomial:

$$\begin{aligned} &(s_\Gamma + r^{(1)}\delta\mathcal{S}) \int d^4x \ \sqrt{2} f^{(l)}_{\psi\varphi}(\tilde{\xi}) \big(\chi^{E^{(2l)}} \varphi_L Y_{\psi_L} + \overline{\chi}^{E^{(2l)}} \overline{\varphi}_L Y_{\overline{\psi}_L} + (_{L\to R}) \big) \\ &= \mathcal{N}^{(l)}_{\psi\varphi}\Gamma + \Delta^{(l)}_{Y,\psi\varphi} + r^{(1)}\delta\Delta^{(l+1)}_{Y,\psi\varphi} \end{aligned} \tag{128}$$

where

$$\delta\Delta^{(l+1)}_{Y,\psi\varphi} = -8l \int d^4x \ \sqrt{2} e^{2(l+1)} f^{(l)}_{\psi\varphi}(\tilde{\xi}) (Y_{\psi_L}\sigma^\mu\overline{\epsilon} + \epsilon\sigma^\mu Y_{\overline{\psi}_l} + (_{L\to R}))V_\mu \ . \tag{129}$$

Therefore $\mathcal{N}_\varphi, \mathcal{N}_\psi$ and $\mathcal{N}_{\psi\varphi}$ are symmetric operators describing the gauge-dependent field renormalizations in the CS equation.

The operator which expresses the field monomial $\Gamma_{\mathrm{ct,kin}}$ in (125) is a simple extension of the expression with constant coupling (51):

$$\begin{aligned} \Gamma^{(1)}_{\mathrm{ct,kin}} = &-\frac{1}{2}\int d^4x \ e^2 \Big(e\frac{\delta}{\delta e} - A^\mu \frac{\delta}{\delta A^\mu} - \lambda^\alpha \frac{\delta}{\delta\lambda^\alpha} - \overline{\lambda}^{\dot\alpha}\frac{\delta}{\delta\overline{\lambda}^{\dot\alpha}} \\ &\qquad + Y^\alpha_\lambda \frac{\delta}{\delta Y^\alpha_\lambda} + Y^{\dot\alpha}_{\overline{\lambda}} \frac{\delta}{\delta Y^{\dot\alpha}_{\overline{\lambda}}} - c\frac{\delta}{\delta c} \\ &\qquad + B\frac{\delta}{\delta B} + \overline{c}\frac{\delta}{\delta\overline{c}} - 2(\xi(x)+\xi)\frac{\delta}{\delta\xi} - 2\chi_\xi\frac{\delta}{\delta\chi_\xi}\Big) \ \Gamma_{\mathrm{cl}} \\ &-\frac{1}{2}\int d^4x \ s(e^2)\big(\lambda^\alpha Y_{\lambda\alpha} + \lambda_{\dot\alpha} Y^{\dot\alpha}_{\overline{\lambda}}\big) \\ \equiv &-\frac{1}{2}\mathcal{D}_{\mathrm{kin}}\Gamma_{\mathrm{cl}} + \int d^4x \ s(e^2)\big(\lambda^\alpha Y_{\lambda\alpha} + \lambda_{\dot\alpha} Y^{\dot\alpha}_{\overline{\lambda}}\big) \ . \end{aligned} \tag{130}$$

As can be immediately verified, $\mathcal{D}_{\rm kin}$ is a $s_\Gamma + r^{(1)}\delta\mathcal{S}$-symmetric operator.

Unlike $\Gamma_{\rm ct,kin}$, the operator expression of $\Gamma_{\rm ct,matter}$ cannot be obtained as a simple extension of the operator with constant coupling (52). As already mentioned in section 6, at this point the axial vector multiplet enters the construction. Indeed, $\Gamma_{\rm ct,matter}$ can be decomposed into field redefinitions of the matter fields (54), (55), (56) and a new $s_{\Gamma_{\rm cl}}$-invariant counterterm $\Gamma^{(l)}_{{\rm ct},Vv}$:

$$\begin{aligned}\Gamma^{(l)}_{\rm ct,matter} = \frac{1}{2}\Big(&\Gamma^{(l)}_{{\rm ct},\varphi}(f_\varphi(\xi)=1) + \Gamma^{(l)}_{{\rm ct},\psi}(f_\psi(\xi)=1) + \Gamma^{(l)}_{{\rm ct},\psi\varphi}(f_{\varphi\psi}(\xi)=1)\\ &+ \Gamma^{(l)}_{{\rm ct},Vv}\Big)\ .\end{aligned} \tag{131}$$

This new invariant counterterm describes field redefinitions of the axial vector multiplet into the components of the local supercoupling E^{2l} and field redefinitions of the $\mathbf{q}$ and $\overline{\mathbf{q}}$-multiplets:

$$\begin{aligned}\Gamma^{(l)}_{{\rm ct},Vv} = \int d^4x\ \Big(&v^{(E^{2l})\mu}\frac{\delta}{\delta V^\mu} + \lambda^{(E^{2l})\alpha}\frac{\delta}{\delta\tilde\lambda^\alpha} + \overline{\tilde\lambda}^{(E^{2l})\dot\alpha}\frac{\delta}{\delta\overline{\tilde\lambda}^{\dot\alpha}}\\ &+ d^{(E^{2l})}\frac{\delta}{\delta\tilde D} - i\big(\epsilon^\alpha\chi^{(E^{2l})}_\alpha - \overline{\chi}^{(E^{2l})}_{\dot\alpha}\overline{\epsilon}^{\dot\alpha}\big)\frac{\delta}{\delta\tilde c}\\ &- 2e^{2l}(q+m)\frac{\delta}{\delta q} - 2\big(e^{2l}q^\alpha + 2\chi^{(E^{2l})\alpha}(q+m)\big)\frac{\delta}{\delta q^\alpha}\\ &- 2\big(e^{2l}q_F + 2f^{(E^{2l})}(q+m) - \chi^{(E^{2l})\alpha}q_\alpha\big)\frac{\delta}{\delta q_F}\\ &- 2e^{2l}(\overline{q}+m)\frac{\delta}{\delta\overline{q}} - 2\big(e^{2l}\overline{q}^{\dot\alpha} + 2\overline{\chi}^{(E^{2l})\dot\alpha}(\overline{q}+m)\big)\frac{\delta}{\delta\overline{q}^{\dot\alpha}}\\ &- 2\big(e^{2l}\overline{q}_F + 2\overline{f}^{(E^{2l})}(\overline{q}+m) - \overline{\chi}^{(E^{2l})}_{\dot\alpha}\overline{q}^{\dot\alpha}\big)\frac{\delta}{\delta\overline{q}_F}\Big)\Gamma_{\rm cl}\\ \equiv\ &\mathcal{D}^{(l)}_{Vv}\Gamma_{\rm cl}\ .\end{aligned} \tag{132}$$

In eq. (132) the components of the multiplet E^{2l} are defined by the following expansion:

$$\begin{aligned}E^{2l}(x,\theta,\overline\theta) =\ &\big(\boldsymbol{\eta}(x,\theta,\overline\theta) + \overline{\boldsymbol{\eta}}(x,\theta,\overline\theta)\big)^{-l}\\ \equiv\ &e^{2l}(x) + \theta^\alpha\chi^{(E^{2l})}_\alpha + \overline{\chi}^{(E^{2l})}_{\dot\alpha}\overline\theta^{\dot\alpha} + \theta^2 f^{(E^{2l})} + \overline\theta^2\overline{f}^{(E^{2l})} + \theta\sigma^\mu\overline\theta v^{(E^{2l})}_\mu\\ &+ i\theta^2(\overline\lambda^{(E^{2l})} + \frac{1}{2}\partial_\mu\chi\sigma^\mu)\overline\theta - i\overline\theta^2\theta(\lambda^{(E^{2l})} + \frac{1}{2}\sigma^\mu\partial_\mu\overline\chi^{(E^{2l})})\\ &+ \frac{1}{4}\theta^2\overline\theta^2(d^{(E^{2l})} - \Box e^{2l})\end{aligned} \tag{133}$$

with the explicit form of the lowest components:

$$\chi_\alpha^{(E^{2l})} = -l e^{2(l+1)}\chi_\alpha \ , \qquad \overline{\chi}_{\dot\alpha}^{(E^{2l})} = -l e^{2(l+1)}\overline{\chi}_{\dot\alpha} \ ,$$
$$v_\mu^{(E^{2l})} = i\ l e^{2(l+1)}\partial_\mu(\eta - \overline{\eta}) + \frac{1}{2} l(l+1) e^{2(l+2)}(\chi\sigma_\mu\overline{\chi}). \tag{134}$$

The explicit form of the higher components is not relevant for the further construction. It is sufficient to know their supersymmetry transformations, which are uniquely determined by the expansion in superspace (see (163) in appendix A).

The operator $\mathcal{D}_{Vv}$ is s_Γ-invariant, but it does not commute with the anomalous Slavnov–Taylor operator. A lengthy but straightforward calculation shows that it can be extended to an $s_\Gamma + r^{(1)}\delta\mathcal{S}$-invariant operator by including redefinitions of the local coupling and an anomalous dimension for the axial vector multiplet into the operator. Their coefficients are uniquely determined, and one finds the following $s_\Gamma + r^{(1)}\delta\mathcal{S}$-invariant operator:

$$\mathcal{D}_{Vv}^{\rm sym} \equiv \mathcal{D}_{Vv}^{(l)} - r^{(1)}\big(4\mathcal{D}_e^{(l+1)} + 8l(\mathcal{N}_V^{(l+1)} - 8(l+1)r^{(1)}\delta\mathcal{N}_V^{(l+2)})\big). \tag{135}$$

In this expression $\mathcal{N}_V^{(l+1)}$ contributes to an anomalous dimension of the axial vector field and its superpartners:

$$\begin{aligned}\mathcal{N}_V^{(l)} = \int d^4x\ \Big(& e^{2l}\Big(V^\mu\frac{\delta}{\delta V^\mu} + \tilde\lambda^\alpha\frac{\delta}{\delta\tilde\lambda^\alpha} + \overline{\tilde\lambda}^{\dot\alpha}\frac{\delta}{\delta\overline{\tilde\lambda}^{\dot\alpha}} + \tilde D\frac{\delta}{\delta\tilde D}\Big) \\ & -\frac{i}{2}V_\mu(\sigma^\mu\overline{\chi}^{(E^{2l})})^\alpha\frac{\delta}{\delta\tilde\lambda^\alpha} + \frac{i}{2}V_\mu(\chi^{(E^{2l})}\sigma^\mu)^{\dot\alpha}\frac{\delta}{\delta\overline{\tilde\lambda}^{\dot\alpha}} \\ & +2\Big(V^\mu v_\mu^{(E^{2l})} + i\tilde\lambda^\alpha\chi_\alpha^{(E^{2l})} - i\overline{\chi}_{\dot\alpha}^{(E^{2l})}\overline{\tilde\lambda}^{\dot\alpha}\Big)\frac{\delta}{\delta\tilde D}\Big),\end{aligned} \tag{136}$$

and

$$\delta\mathcal{N}_V^{(l)} = \int d^4x\ e^{2l}V^\mu V_\mu\frac{\delta}{\delta\tilde D}\ ; \tag{137}$$

the operator $\mathcal{D}_e$ describes a redefinition of the coupling $e(x)$ and its superpartners:

$$\begin{aligned}\mathcal{D}_e^{(l+1)} = \int d^4x \, \Big(& e^{2l+3}\frac{\delta}{\delta e} - 2\big(\chi^{(E^{2l})\alpha}\frac{\delta}{\delta\chi^\alpha} + \overline{\chi}^{(E^{2l})\dot\alpha}\frac{\delta}{\delta\overline{\chi}^{\dot\alpha}}\big) \\ & - 2\big(f^{(E^{2l})}\frac{\delta}{\delta f} + \overline{f}^{(E^{2l})}\frac{\delta}{\delta\overline{f}}\big)\Big) \\ & - e^{2(l+1)}\Big(A^\mu\frac{\delta}{\delta A^\mu} + \lambda^\alpha\frac{\delta}{\delta\lambda^\alpha} + \overline{\lambda}^{\dot\alpha}\frac{\delta}{\delta\overline{\lambda}^{\dot\alpha}} \\ & - Y_\lambda^\alpha\frac{\delta}{\delta Y_\lambda^\alpha} - Y_{\overline{\lambda}}^{\dot\alpha}\frac{\delta}{\delta Y_{\overline{\lambda}}^{\dot\alpha}} + c\frac{\delta}{\delta c} \\ & - B\frac{\delta}{\delta B} - \overline{c}\frac{\delta}{\delta\overline{c}} + 2(\xi(x)+\xi)\frac{\delta}{\delta\xi} + 2\chi_\xi\frac{\delta}{\delta\chi_\xi}\Big) \quad (138)\end{aligned}$$

Thus, we have expressed all five invariant basis elements of $\Delta_m^{(l)}$ (125) in terms of the $s_\Gamma + r^{(1)}\delta\mathcal{S}$ symmetric operators $\mathcal{D}_{\rm kin}$ (130), $\mathcal{D}_{Vv}^{\rm sym}$ (135) and $\mathcal{N}_\varphi, \mathcal{N}_\psi, \mathcal{N}_{\psi\varphi}$ (127). Therefore the dilatational anomalies can be absorbed into the following Callan-Symanzik operator

$$\begin{aligned}\mathcal{C} = \mu_i D_{\mu_i} + \hat\beta_e^{(1)}\mathcal{D}_{\rm kin} + \sum_l \big(& 4r^{(1)}\hat\gamma^{(l)}\mathcal{D}_e^{(l+1)} - \hat\gamma^{(l)}\mathcal{D}_{Vv}^{(l)} + 8r^{(1)}\hat\gamma^{(l)}(l\mathcal{N}_V^{(l+1)} \\ & - 8r^{(1)}l(l+1)\delta\mathcal{N}_V^{(l+2)}) - \hat\gamma_\varphi^{(l)}\mathcal{N}_\varphi^{(l)} - \hat\gamma_\psi^{(l)}\mathcal{N}_\psi^{(l)} - \hat\gamma_{\psi\varphi}^{(l)}\mathcal{N}_{\psi\varphi}^{(l)}\big) \quad (139)\end{aligned}$$

and the algebraic construction yields the CS equation of SQED with local coupling and gauged axial symmetry

$$\mathcal{C}\Gamma = \Delta_Y \ . \quad (140)$$

The two main properties are the restriction of $\mathcal{D}_{\rm kin}$ to one-loop and the connection between $\mathcal{D}_e$, $\mathcal{D}_{Vv}$ and $\mathcal{N}_V$ via the anomaly coefficient $r^{(1)}$. Thus there are only two gauge-independent, physical coefficients in the CS equation of SQED with local coupling and gauged axial symmetry, namely the one-loop β-function $\hat\beta_e^{(1)}$ and the anomalous dimension $\hat\gamma^{(l)}$.

8.3 The limit to constant coupling

In order to clarify the significance of the CS coefficients in (140) we turn to the limit of constant coupling and constant gauge parameter. In this limit all higher components in the θ-expansion of the supercoupling E^{2l} vanish. Therefore

the only parts which are left from the operator $\mathcal{D}^{\rm sym}_{V_v}$ (135) are contributions to the anomalous dimension of q-fields and of the axial vector multiplet, and a contribution to the gauge β-function. The latter arises from the operator $\mathcal{D}_e^{(l+1)}$ (138), and together with the one-loop operator $\mathcal{D}_{\rm kin}$ (130) it determines the β-function β_e of SQED:

$$\lim_{E\to e}(\hat\beta_e^{(1)}\mathcal{D}_{\rm kin}+4r^{(1)}\sum_l\hat\gamma^{(l)}\mathcal{D}_e^{(l+1)})\Gamma=e^3(\hat\beta_e^{(1)}+4r^{(1)}\gamma)\partial_e\Gamma+\cdots \tag{141}$$

and therefore

$$\beta_e=e^2(\hat\beta_e^{(1)}+4r^{(1)}\gamma)\quad\text{with}\quad\gamma=\sum_l e^{2l}\hat\gamma^{(l)}\ . \tag{142}$$

Inserting the values for the one-loop β-function and for the anomaly coefficient (105),

$$\hat\beta_e^{(1)}=\frac{1}{8\pi^2}\ ,\qquad r^{(1)}=-\frac{1}{16\pi^2}, \tag{143}$$

we find the closed expression for the β-function

$$\beta_e=\hat\beta_e^{(1)}e^2(1-2\gamma). \tag{144}$$

In the limit to constant coupling the functional Γ in (140) is the generating functional of 1PI Green functions of ordinary SQED extended by the anomalous axial symmetry. If the axial vector field and its superpartners are set to zero in addition it coincides with $\Gamma^{\rm SQED}$. Therefore we find in the limit to constant coupling from (140) the CS equation of SQED with gauged axial symmetry, whose coefficients are restricted by the multiplet structure of the SQED action:

$$\begin{aligned}&\Big(\mu_i\partial_{\mu_i}+e^2(\hat\beta_e^{(1)}+4r^{(1)}\gamma)(e\partial_e-N_A-N_\lambda+N_{Y_\lambda}-N_c+N_B+N_{\overline{c}}-2\xi\partial_\xi)\\&+4r^{(1)}e^3\partial_e\gamma(N_V+N_{\tilde\lambda}+N_{\tilde D})+2\gamma(N_q+N_{q^\alpha}+N_{q_F})\\&-\gamma_\varphi(N_{\varphi_L}+N_{\varphi_R}-N_{Y_{\varphi_L}}-N_{Y_{\varphi_R}})-\gamma_\psi(N_{\psi_L}+N_{\psi_R}-N_{Y_{\psi_L}}-N_{Y_{\psi_R}})\Big)\,\Gamma\\&=(1-2\gamma)\int d^4x\ m\big(\frac{\delta}{\delta q}+\frac{\delta}{\delta\overline{q}}\big)\,\Gamma+\Delta_{Y,\psi\varphi}\ .\end{aligned} \tag{145}$$

In the CS equation the non-renormalization theorems are identified as the various restrictions on the CS coefficients: The non-renormalization of chiral vertices (cf. (57) and section 5.1) is expressed by the common anomalous dimension γ for the mass and the q-field. However, non-renormalization in the strict sense

is covered up by the gauge dependent anomalous dimensions γ_ψ and γ_φ of the matter fields:

$$\gamma_\varphi = \sum_l e^{2l} f_\varphi^{(l)}(\xi)\hat{\gamma}_\varphi^{(l)} \ , \qquad \gamma_\psi = \sum_l e^{2l} f_\psi^{(l)}(\xi)\hat{\gamma}_\psi^{(l)} \ . \tag{146}$$

Only in the supersymmetric gauge they turn out to be gauge independent and to coincide with the anomalous dimension of the mass, making there the non-renormalization theorems of chiral vertices manifest.

The closed form of the gauge β-function (142) is a consequence of the non-renormalization of the photon self energy in loop orders $l \geq 2$ (cf. (57) and section 5.2). An additional interesting result of the present construction is the anomalous dimension of the axial vector multiplet appearing in (145),

$$\gamma_V = -8r^{(1)} \sum_l l\hat{\gamma}^{(l)} e^{2(l+1)} = -4r^{(1)} e^3 \partial_e \gamma. \tag{147}$$

It illustrates the deep interplay between local couplings and gauged axial symmetry.

With the use of local couplings and an axial vector field we have derived all improved properties of the CS equation in the Wess–Zumino gauge. Previously, the corresponding expressions have only been accessible in the manifestly supersymmetric gauge and by the construction of the supercurrent. Even there the explicit terms for the anomalous dimension of the axial current are available only with quite some technical effort. For this reason, we are convinced that the technique of local couplings can also enlighten some further unproven renormalization properties in more complex theories.

9 Supersymmetry breaking of the axial current Green functions and the Konishi anomaly

In section 5.2 we have shown that the photon self energy in $l \geq 2$ is completely determined by non-local Green functions (see (75) and (78)). Although no axial-current Green functions are involved, the expressions we have obtained resemble the axial anomaly, because they are divergent and ask for regularization, but their divergent part is completely determined by the non-local part via gauge invariance. On the other hand, the axial anomaly has played an important role in the algebraic construction of the CS equation. Using this construction the gauge β-function (142) has been shown to be completely determined by the anomalous

dimension γ and the coefficient of the axial anomaly. In this section, an underlying connection between these results is revealed by deriving explicit relations between the photon self energy and axial-current Green functions.

As the first relation, the high-energy logarithms of the 2-loop photon self energy are completely expressed by one-loop axial-current Green functions. By (75), the 2-loop photon self energy is completely determined by vertex functions of the form $\Gamma_{\chi\lambda A^\mu}$, and these vertex functions are in turn determined by (78),

$$\begin{aligned} 2e^2 l\Gamma^{(l)}_{\chi_\gamma\lambda^\beta A^\mu}(-p,p) = {} & i\sigma^\rho_{\beta\dot\beta}\Gamma^{(l)}_{\overline{\chi}_{\dot\beta}\chi_\gamma A^\mu A^\rho}(p,-p) - p_\rho(\overline{\sigma}^{\rho\mu})^{\dot\alpha}{}_{\dot\beta}\Gamma^{(l)}_{\overline{\chi}_{\dot\beta}\chi_\gamma\lambda^\beta\overline{\lambda}^\alpha}(-p,p) \\ & - \sum_{k=0}^{l}\Gamma^{(l-k)}_{\bar\epsilon^\beta\overline{\chi}_{\dot\beta}\chi_\gamma A^\mu Y^\alpha_{\overline{\lambda}}}(p,-p)\Gamma^{(k)}_{\lambda^\beta\overline{\lambda}_\alpha}(-p,p), \end{aligned} \tag{148}$$

in terms of vertex functions with a $\overline{\chi}\chi$-insertion. All such Green functions vanish at the 0- and 1-loop level. In order to find the high-energy behaviour, we apply the CS equation (140) on the right-hand-side of (148). At 2-loop order the only non-vanishing term comes from the operator $\mathcal{D}^{(1)}_{vV} = \int v^{(E^2)\mu}\delta_{V^\mu} + \cdots$ (132). This operator is the crucial element for relating the Green functions $\Gamma_{\overline{\chi}^\beta\chi^\gamma\ldots}$ to Green functions with an axial current insertion at zero momentum, for it describes a field renormalization of the axial vector field into the vector component $v^{(E^{2l})}_\mu$ of the local supercoupling (134), which contains the combination $\chi\sigma\overline{\chi}$. Explicitly we find in two-loop order:

$$m\partial_m 4e^2\Gamma^{(2)}_{\chi^\gamma\lambda^\beta A^\mu} = e^6\hat\gamma^{(1)}\sigma^\rho_{\gamma\dot\beta}\widetilde{\Gamma}^{(1)}_{\rho\mu}{}^{\dot\beta}{}_\beta(p,-p) + \text{soft terms} \tag{149}$$

with

$$\begin{aligned} \widetilde{\Gamma}_{\rho\mu\beta\dot\beta}(p,-p) = {} & -i\sigma^\lambda_{\beta\dot\beta}\Gamma_{V^\rho A^\mu A^\lambda}(0,p,-p) + p_\lambda(\overline{\sigma}^{\lambda\mu})^{\dot\alpha}{}_{\dot\beta}\Gamma_{V^\rho\lambda^\beta\overline{\lambda}^\alpha}(0,-p,p) \\ & + \Gamma_{\bar\epsilon^\beta V^\rho A^\mu Y^\alpha_{\overline{\lambda}}}(0,p,-p)\Gamma_{\lambda^\beta\overline{\lambda}_\alpha}(-p,p)\ . \end{aligned} \tag{150}$$

Via eq. (75), the identity (149) expresses the high-energy behaviour of the photon self energy explicitely in terms of lower order Green functions with axial-current insertions $\widetilde{\Gamma}_{\rho\mu\dot\beta\beta}(p,-p)$. Therefore, all 2-loop high-energy logarithms of the photon self energy are due to subdivergences related to one-loop triangle diagrams.

It is possible to derive a complementary relation for the photon self energy, where the axial Green functions in the combination of eq. (150) are entirely expressed by the photon self energy. From

$$\frac{\partial}{\partial\bar\epsilon^{\dot\beta}}\frac{\delta}{\delta V^\rho(z)}\frac{\delta}{\delta A^\mu(x)}\frac{\delta}{\delta\lambda^\beta(y)}(\mathcal{S} + r^{(1)}\delta\mathcal{S})(\Gamma) = 0 \tag{151}$$

one gets the relation

$$\widetilde{\Gamma}_{\rho\mu\dot{\beta}\beta}(p,-p) = 8r^{(1)}\sigma^{\rho}_{\alpha\dot{\beta}}\Gamma_{\chi_\alpha A^\mu \lambda^\beta}(p,-p) \tag{152}$$

between the axial Green functions in $\widetilde{\Gamma}^{(l)}_{\rho\mu\dot{\beta}\beta}(p,-p)$ and the vertex function $\Gamma_{\chi\lambda A^\mu}$. By using the relation (75) to eliminate $\Gamma_{\chi A^\mu\lambda}$, the function $\widetilde{\Gamma}_{\rho\mu\dot{\beta}\beta}(p,-p)$ can be related to the photon self energy:

$$p^\lambda(\overline{\sigma}^\rho\sigma_{\mu\lambda})^{\dot{\beta}\beta}\widetilde{\Gamma}^{(l)}_{\rho\nu\beta\dot{\beta}}(-p,p) - p^\lambda(\overline{\sigma}^\rho\sigma_{\nu\lambda})^{\dot{\beta}\beta}\widetilde{\Gamma}^{(l)}_{\rho\mu\beta\dot{\beta}}(p,-p) = 64(l-2)r^{(1)}\Gamma^{(l-1)}_{A^\mu A^\nu}. \tag{153}$$

This is the announced relation. It can be solved for the photon self energy in terms of axial-current Green functions at all orders, except for one-loop order. This exception corresponds to the only independent divergent contribution to the photon self energy in one loop order.

The two relations we have derived can be combined to rearrange the CS equation (149) into an ordinary scaling equation for the vertex function $\Gamma_{\chi\lambda A}$ or, using (75), for the photon self energy. Inserting (153) into (149) and using the relation (75) we find the CS equation of the photon self energy in 2-loop order, which is in agreement with the algebraic construction of section 8 (cf. in particular eq. (145)):

$$m\partial_m\Gamma^{(2)}_{A^\mu A^\nu} = 8e^2r^{(1)}\gamma^{(1)}\Gamma^{(0)}_{A^\mu A^\mu} + \text{soft terms}\ . \tag{154}$$

Eq. (152) relates the vertex function $\Gamma_{\chi\lambda A^\mu}$ to Green functions with axial-current insertions, and eq. (148) relates the same vertex function to the Green functions with $\chi\overline{\chi}$- insertions, multiplied with the anomaly coefficient $r^{(1)}$. Comparing both identities reveals a one-to-one correspondence of the Green functions $\Gamma^{(l)}_{V^\mu\ldots}$ and $r^{(1)}\Gamma^{(l-1)}_{\overline{\chi}^{\dot{\beta}}\chi^\gamma\ldots}$. This is an underlying connection between the axial-current Green functions and the non-local expressions defining the photon self energy.

An interpretation of the identity (152) is the invariance of axial Green functions under supersymmetry transformations. In the anomalous Slavnov–Taylor identity, the supersymmetry transformations of the axial multiplet contain contributions from the axial anomaly, and correspondingly (152) relates axial Green functions to Green functions of lower order, multiplied with the anomaly coefficient $r^{(1)}$. From the point of view of the unmodified supersymmetry transformations without the anomaly contributions, (152) can be viewed as an expression for a supersymmetry breaking for axial-current Green functions, induced by the Adler-Bardeen anomaly in the Wess–Zumino gauge.

Furthermore, the expressions (152) and (153) have an analogue in superspace. In the supersymmetric gauge the axial Ward identity is covariantly decomposed into chiral and antichiral transformations which read [5]:[5]

$$\bar{D}\bar{D}I_5\,\Gamma = (A_+\frac{\delta}{\delta A_+} + A_-\frac{\delta}{\delta A_-})\Gamma + \frac{1}{2}m[A_+A_-]_2\cdot\Gamma + r^{(1)}[W^\alpha W_\alpha]_3\cdot\Gamma,$$
$$DDI_5\,\Gamma = (\bar{A}_+\frac{\delta}{\delta \bar{A}_+} + \bar{A}_-\frac{\delta}{\delta \bar{A}_-})\Gamma + \frac{1}{2}m[\bar{A}_+\bar{A}_-]_2\cdot\Gamma + r^{(1)}[\bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}]_3\cdot\Gamma, \quad (155)$$

where W_α and $\bar{W}_{\dot\alpha}$ are the supersymmetric field strength tensors of the real superfield containing the photon and photino

$$W_\alpha = -\frac{1}{8}\bar{D}\bar{D}D_\alpha\phi, \qquad \bar{W}_{\dot\alpha} = -\frac{1}{8}DD\bar{D}_{\dot\alpha}\phi, \quad (156)$$

and I_5 is the gauge invariant multiplet of the matter Lagrangian:

$$I_5 = \frac{1}{16}[\bar{A}_+e^{e\phi}A_+ + \bar{A}_-e^{-e\phi}A_- + \mathcal{O}(\hbar)]_2\ . \quad (157)$$

Like (152) and (153), these identities represent a supersymmetric extension of the anomaly found here in the decomposition of the anomaly into a chiral and antichiral part.

Subtraction of the chiral and antichiral identities in (155) yields the axial Ward identity with its anomaly. However, if one integrates and sums the two identities, one obtains an identity for the matter insertion $\int dV\ I_5$. It contains a field redefinition and a mass insertion, but involves in particular also insertions of the gauge invariant kinetic term $\int dS\ W^\alpha W_\alpha$ and its complex conjugate. This contribution is due to the Adler–Bardeen anomaly of the axial current. (The identity of the matter term insertion is sometimes called Konishi anomaly [13].) In combination with the local version of the CS equation it yields the closed form of the β-function in the usual superspace approach without local couplings.

In the Wess–Zumino gauge the anomalous axial Ward identity (114) is left as the only equation of the covariant expressions in superspace, but the Adler–Bardeen anomaly induces a supersymmetry breaking of axial current Green functions. And the relations (152) and (153) we have derived from the anomalous Slavnov–Taylor identity have a similar physical content as the Konishi anomaly in superspace, since they relate axial current Green functions to the photon self energy. In the Wess–Zumino gauge they can therefore be considered as the analogue to the Konishi anomaly in the manifestly supersymmetric gauge.

[5] We use the conventions of Clark, Piguet and Sibold [5] with minor modifications, which concern the massive photon field and the non-renormalization of chiral vertices. We only give a short sketch of the construction and refer for details to the original paper.

10 Conclusions

We have traced back the non-renormalization theorems of chiral vertices and the generalized non-renormalization theorem of the photon self energy in SQED on a common algebraic origin, namely the nature of supersymmetric Lagrangians as the highest components of supermultiplets. This fact has been exploited using the extension of the gauge coupling to an external superfield. On the basis of symmetric counterterms, the non-renormalization theorems appear as absence of independent counterterms to the chiral vertices in $l \geq 1$ and to the kinetic term of the photon multiplet in loop order $l \geq 2$. We have related the corresponding vertex functions to non-local expressions. There is a remarkable difference between the usual and the generalized non-renormalization theorem: Chiral vertex functions are — up to gauge dependent field redefinitions — related to superficially convergent Green functions, whereas the photon self energy is related to linearly divergent Green functions. The latter expressions turn out to be independent of the renormalization procedure only if gauge invariance is taken into account. In this sense, the expressions determining the photon self energy remind of the triangle diagrams of the axial current.

Indeed, a deep connection between the local supercoupling and the axial current is revealed when we complete SQED with local gauge coupling to a multiplicatively renormalizable theory: It turns out to be necessary to introduce an axial vector multiplet whose vector component couples to the axial current. The extended action of SQED with the external superfield of the local coupling and the axial vector multiplet gives a complete description of all known renormalization properties of SQED including the Adler–Bardeen anomaly. Owing to the local gauge coupling the Adler–Bardeen anomaly can be absorbed into the Slavnov–Taylor operator and the extended model can be renormalized algebraically by using the anomalous Slavnov–Taylor identity. Even the non-renormalization theorem of the Adler–Bardeen anomaly is shown to be an almost trivial consequence of the construction with local coupling: Only the one-loop order of the anomaly is a total derivative; higher orders would contribute to the global integrated axial Ward identity, hence their coefficients vanish.

All these properties together make possible to derive the Callan–Symanzik equation for SQED including the anomalous axial current from an algebraic construction with the anomalous Slavnov–Taylor identity. The Callan–Symanzik equation depends only on two physical coefficients, the one-loop gauge β-function and the anomalous dimension of mass, γ. The higher orders of the β-function and the anomalous dimension of the anomalous axial current are related to these coefficients via the anomaly coefficient. In the course of the construction also the gauge-dependent anomalous dimensions of the matter fields are identified and disentangled from the non-renormalization properties.

In summary, usual SQED appears in the extended construction as a limiting theory of a more fundamental one, which includes all known non-renormalization theorems in its structure. It is remarkable that the soft breakings of the Giradello–Grisaru class can be introduced without further modifications. They are the lowest components of the multiplets of gauge invariant Lagrangians and are as such already included in the present construction. They can be explicitely obtained by a shift in the highest component of the corresponding external fields.

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A Conventions and Notations

2-Spinor indices and scalar products:

$$\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}, \quad \epsilon_{12} = 1, \quad \epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = \delta^{\alpha}_{\gamma}, \tag{158}$$

$$\epsilon_{\dot\alpha\dot\beta} = -\epsilon_{\dot\beta\dot\alpha}, \quad \epsilon_{\dot1\dot2} = 1, \quad \epsilon^{\dot\alpha\dot\beta}\epsilon_{\dot\beta\dot\gamma} = \delta^{\dot\alpha}_{\dot\gamma},$$

$$\psi\chi = \psi^{\alpha}\chi_{\alpha}\ , \quad \psi^{\alpha} = \epsilon^{\alpha\beta}\psi_{\beta}\ ,$$

$$\overline{\psi}\overline{\chi} = \overline{\psi}_{\dot\alpha}\overline{\chi}^{\dot\alpha}\ , \quad \overline{\psi}_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\overline{\psi}^{\dot\beta}\ . \tag{159}$$

σ matrices:

$$\sigma^{\mu}_{\alpha\dot\alpha} = (\sigma^0_{\alpha\dot\alpha}, \sigma^1_{\alpha\dot\alpha}, \sigma^2_{\alpha\dot\alpha}, \sigma^3_{\alpha\dot\alpha})\ , \tag{160}$$

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\overline{\sigma}^{\mu\dot\alpha\beta} = \epsilon^{\beta\alpha}\epsilon^{\dot\alpha\dot\beta}\sigma^{\mu}_{\alpha\dot\beta}\ .$$

$$(\sigma^{\mu\nu})_{\alpha}{}^{\beta} = \frac{i}{2}(\sigma^{\mu}\overline{\sigma}^{\nu} - \sigma^{\nu}\overline{\sigma}^{\mu})_{\alpha}{}^{\beta}\ , \quad (\overline{\sigma}^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} = \frac{i}{2}(\overline{\sigma}^{\mu}\sigma^{\nu} - \overline{\sigma}^{\nu}\sigma^{\mu})^{\dot\alpha}{}_{\dot\beta}\ .$$

Complex conjugation:

$$
\begin{aligned}
(\psi\theta)^\dagger &= \overline{\theta\psi} \ , \\
(\psi\sigma^\mu\overline{\theta})^\dagger &= \theta\sigma^\mu\overline{\psi} \ , \qquad (\psi\sigma^{\mu\nu}\theta)^\dagger = \overline{\theta}\overline{\sigma}^{\mu\nu}\overline{\psi} \ .
\end{aligned}
\tag{161}
$$

Derivatives:

$$
\begin{aligned}
&\frac{\partial}{\partial\theta^\alpha}\theta^\beta = \delta_\alpha^\beta \ , \qquad \frac{\partial}{\partial\theta_\alpha}\theta_\beta = -\delta_\beta^\alpha \ , \\
&\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}}\overline{\theta}^{\dot\beta} = \delta_{\dot\alpha}^{\dot\beta} \ , \quad \frac{\partial}{\partial\overline{\theta}_{\dot\alpha}}\overline{\theta}_{\dot\beta} = -\delta_{\dot\beta}^{\dot\alpha} \ .
\end{aligned}
\tag{162}
$$

Supersymmetry transformations in superspace

- Superfields in the real representation

$$
\begin{aligned}
\delta_\alpha\phi(x,\theta,\overline{\theta}) &= \Big(\frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}\partial_\mu\Big)\phi(x,\theta,\overline{\theta}) \ , \\
\bar\delta_{\dot\alpha}\phi(x,\theta,\overline{\theta}) &= \Big(-\frac{\partial}{\partial\theta^{\dot\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\Big)\phi(x,\theta,\overline{\theta}) \ .
\end{aligned}
\tag{163}
$$

- Superfields in the chiral representation

$$
\phi_{\rm c}(x,\theta,\overline{\theta}) = \phi(x + i\theta\sigma\overline{\theta},\theta,\overline{\theta}) \ ;
\tag{164}
$$

$$
\begin{aligned}
\delta_\alpha\phi_{\rm c}(x,\theta,\overline{\theta}) &= \frac{\partial}{\partial\theta^\alpha}\phi_{\rm c}(x,\theta,\overline{\theta}) \ , \\
\bar\delta_{\dot\alpha}\phi_{\rm c}(x,\theta,\overline{\theta}) &= \Big(-\frac{\partial}{\partial\theta^{\dot\alpha}} - 2i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\Big)\phi_{\rm c}(x,\theta,\overline{\theta}) \ .
\end{aligned}
\tag{165}
$$

- Superfields in the antichiral representation

$$
\phi_{\rm ac}(x,\theta,\overline{\theta}) = \phi(x - i\theta\sigma\overline{\theta},\theta,\overline{\theta}) \ ;
\tag{166}
$$

$$
\begin{aligned}
\delta_\alpha\phi_{\rm ac}(x,\theta,\overline{\theta}) &= \Big(\frac{\partial}{\partial\theta^\alpha} + 2i\sigma^\mu_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}\partial_\mu\Big)\phi_{\rm ac}(x,\theta,\overline{\theta}) \ , \\
\bar\delta_{\dot\alpha}\phi_{\rm ac}(x,\theta,\overline{\theta}) &= -\frac{\partial}{\partial\theta^{\dot\alpha}}\phi_{\rm ac}(x,\theta,\overline{\theta}) \ .
\end{aligned}
\tag{167}
$$

Superspace integration

$$\int dV\ \phi = \int d^4x\ \frac{\partial}{\partial\theta_\alpha}\frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}}\frac{\partial}{\partial\overline{\theta}_{\dot\alpha}}\phi\ , \tag{168}$$
$$\int dS\ A = \int d^4x\ \frac{\partial}{\partial\theta_\alpha}\frac{\partial}{\partial\theta^\alpha}A\ , \qquad \int d\bar{S}\ \bar{A} = \int d^4x\ \frac{\partial}{\partial\theta^{\dot\alpha}}\frac{\partial}{\partial\theta_{\dot\alpha}}\bar{A}\ ,$$

where ϕ is a real superfield and A and $\bar{A}$ are chiral and antichiral superfields respectively.

B Discrete symmetries

SQED with local coupling and gauged axial symmetry is invariant under the discrete symmetries parity P, charge conjugation C and R-parity. We define the transformation of the fields according to the following table:

	x^μ	A^μ	λ^α	φ_L	φ_R	ψ_L^α	ψ_R^α	c	ϵ^α	ω^ν	$\bar{c}$	B
R	x^μ	A^μ	$-i\lambda^\alpha$	$-i\varphi_L$	$-i\varphi_R$	ψ_L^α	ψ_R^α	c	$-i\epsilon^\alpha$	ω^ν	$\bar{c}$	B
C	x^μ	$-A^\mu$	$-\lambda^\alpha$	φ_R	φ_L	ψ_R^α	ψ_L^α	$-c$	ϵ^α	ω^ν	$-\bar{c}$	$-B$
CP	$(\mathcal{P}x)^\mu$	$-(\mathcal{P}A)^\mu$	$-i\overline{\lambda}_{\dot\alpha}$	$\overline{\varphi}_L$	$\overline{\varphi}_R$	$i\overline{\psi}_{L\dot\alpha}$	$i\overline{\psi}_{R\dot\alpha}$	$-c$	$-i\overline{\epsilon}_{\dot\alpha}$	$(\mathcal{P}\omega)^\nu$	$-\bar{c}$	$-B$

	e	$\eta-\overline{\eta}$	χ^α	f	q	q^α	q_F	V^μ	$\tilde{\lambda}^\alpha$	$\tilde{D}$	$\tilde{c}$
R	e	$\eta-\overline{\eta}$	$i\chi^\alpha$	$-f$	q	iq^α	$-q_F$	V^μ	$-i\tilde{\lambda}^\alpha$	$\tilde{D}$	$\tilde{c}$
C	e	$\eta-\overline{\eta}$	χ^α	f	q	q^α	q_F	V^μ	$\tilde{\lambda}^\alpha$	$\tilde{D}$	$\tilde{c}$
CP	e	$\overline{\eta}-\eta$	$i\overline{\chi}_{\dot\alpha}$	$\bar{f}$	$\overline{q}$	$i\overline{q}_{\dot\alpha}$	$\overline{q}_F$	$-(\mathcal{P}V)^\mu$	$-i\overline{\tilde{\lambda}}_{\dot\alpha}$	$\tilde{D}$	$-\tilde{c}$

Table 1: Discrete symmetries. The transformation rules for the sources Y_i can be deduced from the requirement that Γ_{ext} is invariant. The transformation rules for the complex conjugate fields are obvious except for the CP conjugation of the spinors. We define for $\chi \in \{\lambda, \psi_L, \psi_R, \epsilon\}$:

$$\chi^\alpha \overset{CP}{\to} a\overline{\chi}_{\dot\alpha} \Rightarrow \overline{\chi}_{\dot\alpha} \overset{CP}{\to} -a^*\chi^\alpha\ ,\ \chi_\alpha \overset{CP}{\to} -a\overline{\chi}^{\dot\alpha}\ ,\ \overline{\chi}^{\dot\alpha} \overset{CP}{\to} a^*\chi_\alpha\ .$$

C The BRS transformations

In this appendix we list the BRS transformations of all fields introduced in SQED with local gauge coupling and with the axial vector multiplet.

- BRS transformations of the photon multiplet

$$
\begin{aligned}
sA_\mu &= \frac{1}{e}\partial_\mu ec + i\epsilon\sigma_\mu\overline{\lambda} - i\lambda\sigma_\mu\overline{\epsilon} \\
&\quad + \frac{1}{2}e^2(\epsilon\chi + \overline{\chi\epsilon})A_\mu - i\omega^\nu\partial_\nu A_\mu \ , \\
s\lambda^\alpha &= \frac{i}{2e}(\epsilon\sigma^{\rho\sigma})^\alpha F_{\rho\sigma}(eA) - i\epsilon^\alpha\, eQ_L(|\phi_L|^2 - |\phi_R|^2) + \frac{1}{2}\epsilon^\alpha e^2(\chi\lambda - \overline{\chi}\overline{\lambda}) \\
&\quad + \frac{1}{2}e^2(\epsilon\chi + \overline{\chi\epsilon})\lambda^\alpha - i\omega^\nu\partial_\nu\lambda^\alpha \ , \\
s\overline{\lambda}_{\dot\alpha} &= \frac{-i}{2e}(\overline{\epsilon\sigma}^{\rho\sigma})_{\dot\alpha} F_{\rho\sigma}(eA) - i\overline{\epsilon}_{\dot\alpha}\, eQ_L(|\phi_L|^2 - |\phi_R|^2) + \frac{1}{2}\overline{\epsilon}_{\dot\alpha} e^2(\chi\lambda - \overline{\chi}\overline{\lambda}) \\
&\quad + \frac{1}{2}e^2(\epsilon\chi + \overline{\chi\epsilon})\overline{\lambda}_{\dot\alpha} - i\omega^\nu\partial_\nu\overline{\lambda}_{\dot\alpha} \ .
\end{aligned}
\tag{169}
$$

- BRS transformations of the axial vector multiplet

$$
\begin{aligned}
sV_\mu &= \partial_\mu\tilde{c} + i\epsilon\sigma_\mu\overline{\tilde{\lambda}} - i\tilde{\lambda}\sigma_\mu\overline{\epsilon} - i\omega^\nu\partial_\nu V_\mu \ , \\
s\tilde{\lambda}^\alpha &= \frac{i}{2}(\epsilon\sigma^{\rho\sigma})^\alpha F_{\rho\sigma}(V) + \frac{i}{2}\epsilon^\alpha\, \tilde{D} - i\omega^\nu\partial_\nu\tilde{\lambda}^\alpha \ , \\
s\overline{\tilde{\lambda}}_{\dot\alpha} &= \frac{-i}{2}(\overline{\epsilon\sigma}^{\rho\sigma})_{\dot\alpha} F_{\rho\sigma}(V) + \frac{i}{2}\overline{\epsilon}_{\dot\alpha}\tilde{D} - i\omega^\nu\partial_\nu\overline{\tilde{\lambda}}_{\dot\alpha} \ , \\
s\tilde{D} &= 2\epsilon\sigma^\mu\partial_\mu\overline{\tilde{\lambda}} + 2\partial_\mu\tilde{\lambda}\sigma^\mu\overline{\epsilon} - i\omega^\nu\partial_\nu\tilde{D} \ .
\end{aligned}
\tag{170}
$$

- BRS transformations of matter fields
  The covariant derivative is defined by eq. (90).

$$
\begin{aligned}
s\varphi_L &= -i(eQ_Lc + \tilde{c})\varphi_L + \sqrt{2}\,\epsilon\psi_L - i\omega^\nu\partial_\nu\varphi_L \ , \\
s\overline{\varphi}_L &= +i(eQ_Lc + \tilde{c})\,\overline{\varphi}_L + \sqrt{2}\,\overline{\psi}_L\overline{\epsilon} - i\omega^\nu\partial_\nu\overline{\varphi}_L \ , \\
s\psi_L^\alpha &= -i(eQ_Lc + \tilde{c})\,\psi_L^\alpha \\
&\quad -\sqrt{2}\,\epsilon^\alpha\,(\overline{q} + m)\overline{\varphi}_R - \sqrt{2}\,i(\overline{\epsilon\sigma}^\mu)^\alpha D_\mu\varphi_L - i\omega^\nu\partial_\nu\psi_L^\alpha \ , \\
s\overline{\psi}_{L\dot\alpha} &= +i(eQ_Lc + \tilde{c})\,\overline{\psi}_{L\dot\alpha} \\
&\quad +\sqrt{2}\,\overline{\epsilon}_{\dot\alpha}\,(q + m)\varphi_R + \sqrt{2}\,i(\epsilon\sigma^\mu)_{\dot\alpha} D_\mu\overline{\varphi}_L - i\omega^\nu\partial_\nu\overline{\psi}_{L\dot\alpha} \ .
\end{aligned}
$$

  and respective expressions for right-handed fields.

- The BRS transformations of ghosts

$$
\begin{aligned}
sc &= 2i\epsilon\sigma^\nu\overline{\epsilon}A_\nu + \frac{1}{2}e^2(\epsilon\chi + \overline{\chi\epsilon})c - i\omega^\nu\partial_\nu c \ , \\
s\tilde{c} &= 2i\epsilon\sigma^\nu\overline{\epsilon}V_\nu - i\omega^\nu\partial_\nu\tilde{c} \ , \\
s\epsilon^\alpha &= 0 \ , \\
s\overline{\epsilon}^{\dot\alpha} &= 0 \ , \\
s\omega^\nu &= 2\epsilon\sigma^\nu\overline{\epsilon} \ .
\end{aligned}
\tag{171}
$$

- BRS transformations of the B-field, anti-ghost and gauge parameter

$$
\begin{aligned}
sB &= 2i\epsilon\sigma^\nu\overline{\epsilon}\partial_\nu(\frac{1}{e}\overline{c}) - e^2\frac{1}{2}(\epsilon\chi + \overline{\chi\epsilon})B - i\omega^\nu\partial_\nu B \ , \\
s\overline{c} &= B - e^2\frac{1}{2}(\epsilon\chi + \overline{\chi\epsilon})\overline{c} - i\omega^\nu\partial_\nu\overline{c} \ , \\
s\chi_\xi &= 2i\frac{1}{e^2}\epsilon\sigma^\nu\overline{\epsilon}\partial_\nu(e^2\xi) + e^2(\epsilon^\alpha\chi_\alpha + \overline{\chi}_{\dot\alpha}\overline{\epsilon}^{\dot\alpha})\chi_\xi - i\omega^\nu\partial_\nu\chi_\xi \ , \\
s\xi &= \chi_\xi + e^2(\epsilon\chi + \overline{\chi\epsilon})\xi - i\omega^\nu\partial_\nu\xi \ .
\end{aligned}
\tag{172}
$$

- BRS transformations of the local coupling and its superpartners (20)

$$
\begin{aligned}
s\eta &= \epsilon^\alpha\chi_\alpha - i\omega^\nu\partial_\nu\eta \ , \\
s\overline{\eta} &= \overline{\chi}_{\dot\alpha}\overline{\epsilon}^{\dot\alpha} - i\omega^\nu\partial_\nu\overline{\eta} \ , \\
s\chi_\alpha &= 2i(\sigma^\mu\overline{\epsilon})_\alpha\partial_\mu\eta + 2\epsilon_\alpha f - i\omega^\mu\partial_\mu\chi_\alpha \ , \\
s\overline{\chi}_{\dot\alpha} &= 2i(\epsilon\sigma^\mu)_{\dot\alpha}\partial_\mu\overline{\eta} - 2\overline{\epsilon}_{\dot\alpha}\bar{f} - i\omega^\mu\partial_\mu\overline{\chi}_{\dot\alpha} \ , \\
sf &= i\partial_\mu\chi\sigma^\mu\overline{\epsilon} - i\omega^\mu\partial_\mu f \ , \\
s\bar{f} &= -i\epsilon\sigma^\mu\partial_\mu\overline{\chi} - i\omega^\mu\partial_\mu\bar{f} \ .
\end{aligned}
\tag{173}
$$

- BRS transformations of q-multiplets (24)
  The covariant derivative is defined in eq. (93)

$$
\begin{aligned}
sq &= +2i\tilde{c}(q+m) + \epsilon^\alpha q_\alpha - i\omega^\nu\partial_\nu q \ , \\
s\overline{q} &= -2i\tilde{c}(\overline{q}+m) + \overline{q}_{\dot\alpha}\overline{\epsilon}^{\dot\alpha} - i\omega^\nu\partial_\nu\overline{q} \ , \\
sq_\alpha &= +2i\tilde{c}q_\alpha + 2i(\sigma^\mu\overline{\epsilon})_\alpha D_\mu(q+m) + 2\epsilon_\alpha q_F - i\omega^\mu\partial_\mu q_\alpha \ , \\
s\overline{q}_{\dot\alpha} &= -2i\tilde{c}\overline{q}_{\dot\alpha} + 2i(\epsilon\sigma^\mu)_{\dot\alpha}D_\mu(\overline{q}+m) - 2\overline{\epsilon}_{\dot\alpha}\overline{q}_F - i\omega^\mu\partial_\mu\overline{q}_{\dot\alpha} \ , \\
sq_F &= +2i\tilde{c}q_F + iD_\mu q^\alpha\sigma^\mu_{\alpha\dot\alpha}\overline{\epsilon}^{\dot\alpha} - 4i\overline{\tilde{\lambda}}_{\dot\alpha}\overline{\epsilon}^{\dot\alpha}(q+m) - i\omega^\mu\partial_\mu q_F \ , \\
s\overline{q}_F &= -2i\tilde{c}\overline{q}_F - i\epsilon^\alpha\sigma^\mu_{\alpha\dot\alpha}D_\mu\overline{q}^{\dot\alpha} + 4i\epsilon^\alpha\tilde{\lambda}_\alpha(\overline{q}+m) - i\omega^\mu\partial_\mu\overline{q}_F \ .
\end{aligned}
\tag{174}
$$

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