Karlsruhe
April 08 2015

A256099: A Geometrical Problem of Omar Khayyám and its Cubic

Wolfdieter L a n g [1]

In *Alten et al.* [1], pp. 190-192, a geometrical problem of Omar Khayyám (called there Umar Hayyām) is presented [2]. See also [5, 3]. The problem is to find the point P on a circle (Radius R) in the first quadrant such that the ratio of the normal $\overline{P,H} = x$ and the radius R equals the ration of the segments $\overline{H,Q} = R - h$ and $\overline{H,O} = h$, *i.e.*, (see *Figure 1*)

$$\frac{x}{R} = \frac{R-h}{h}\,, \quad \text{or } \hat{x} = \frac{1}{\hat{h}} - 1\,, \text{with } \hat{x} = \frac{x}{R} \text{ and } \hat{h} = \frac{h}{R}\,.$$

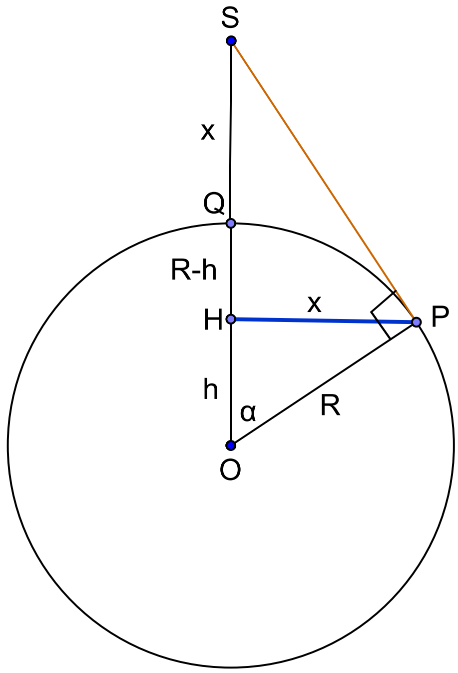

Figure 1: A geometrical problem of Omar Khayyám involving a cubic for the ratio $\tilde{x} = \frac{x}{h}$.

This leads to a cubic equation in the following way: replace $R^2 = x^2 + h^2$ in the squared equation $h\,R = R^2 - h\,x$. That is, $(x^2 + h^2)\,h^2 = (x^2 + h^2 - h\,x)^2$ which yields $0 = x\left(x^3 - 2\,h\,x^2 + 2\,h^2\,x - 2\,h^3\right)$. (With h replaced by μ the cubic factor is the one given as first equation on p. 192 of [1]).

[1] wolfdieter.lang@partner.kit.edu, http://www/kit.edu/~wl/

With $\tilde{x} := \frac{x}{h} = \frac{\hat{x}}{\hat{h}} = \tan\,\alpha$ this cubic becomes ($x \neq 0$ for the solution of the problem)

$$\tilde{x}^3 - 2\,\tilde{x}^2 + 2\,\tilde{x} - 2 = 0.$$

The discriminant of this cubic is $D = q^2 + p^3$ with $q = \frac{37}{27}$ and $p = \frac{2}{9}$. Because $D = \frac{17}{9} > 0$ this cubic has one real solution and two complex conjugated solutions.
The real solution is (thanks to Maple [4])

$$\tilde{x} = \tan\,\alpha = \frac{1}{3}\left((3\sqrt{33} + 17)^{1/3} - (3\sqrt{33} - 17)^{1/3} + 2\right).$$

The decimal expansion of the ratio $\tilde{x}$ is given in A256099: $\tilde{x} = 1.54368901....$ This corresponds to the angle $\alpha = \arctan(\tilde{x})\,\frac{180}{\pi} \approx 57.065^o$.
Because $\hat{h} = \cos\,\alpha$ and $\hat{x} = \sin\,\alpha$ the original ratio equation can also be written $\sin\,\alpha = \frac{1}{\cos\,\alpha} - 1$, or

$$\sqrt{\sin(2\,\alpha)} = 2\,\sin\left(\frac{\alpha}{2}\right).$$

This checks with $\alpha = \arctan\,\tilde{x} \approx 0.99597$. The two complex solutions are z and $\bar{z}$ (thanks to Maple [4])

$$\begin{aligned} z &= a + i\,b, \text{ with} \\ a &= -\frac{1}{6}\left((17 + 3\sqrt{33}\,)^{1/3} - (-17 + 3\sqrt{33}\,)^{1/3} + 4\right), \\ b &= \frac{1}{6}\sqrt{3}\left((17 + 3\sqrt{33}\,)^{1/3} + (-17 + 3\sqrt{33}\,)^{1/3}\right). \end{aligned}$$

According to [1] Omar Khayyám used for this problem a rectangular triangle $\triangle(O, P, S)$ such that $\overline{O, P} + \overline{P, H} = \overline{OS}$ *i.e.*, $R + x = R + \overline{Q, S}$, hence $\overline{Q, S} = x$ (see *Figure 1*). From a well-known theorem one has $h\,(R - h + x) = x^2$ or $\frac{R}{h} = \tilde{x}^2 - \tilde{x} + 1$. Squaring, with $\left(\frac{R}{h}\right)^2 = \tilde{x}^2 + 1$, leads again to $0 = \tilde{x}\,(\tilde{x}^3 - 2\,\tilde{x}^2 + 2\,\tilde{x} - 2)$; hence for this problem the above found cubic for $\tilde{x} = \tan\,\alpha$ is recovered.

Omar Khayyám used for $\hat{h} = \frac{h}{R} = 10$ and obtained the cubic for $\hat{x} = \frac{x}{R}$, namely $\hat{x} - 20\,\hat{x} + 200\,\hat{x} - 2000 = 0$.
To solve this equation Omar Khayyám used a geometric method intersecting a circle with a right angular hyperbola. For the (general) cubic for $\tilde{x}$ one uses the circle $\hat{y}^2 = (\hat{x} - \hat{h})\,(2\,\hat{h} - \hat{x})$ and the rectangular hyperbola $\hat{y} = \sqrt{2}\,\hat{h}\frac{\hat{x} - \hat{h}}{\hat{x}}$. The intersection leads to $\hat{x} = \hat{h}$ (not of interest here) and $\hat{x}^2\,(2\,\hat{h} - \hat{x}) = 2\,\hat{h}^2\,(\hat{x} - \hat{h})$ which is the cubic from above, and after division by $\hat{h}^3$ this yields the cubic for $\tilde{x} = \frac{\hat{x}}{\hat{h}}$. See *Figure 2*.

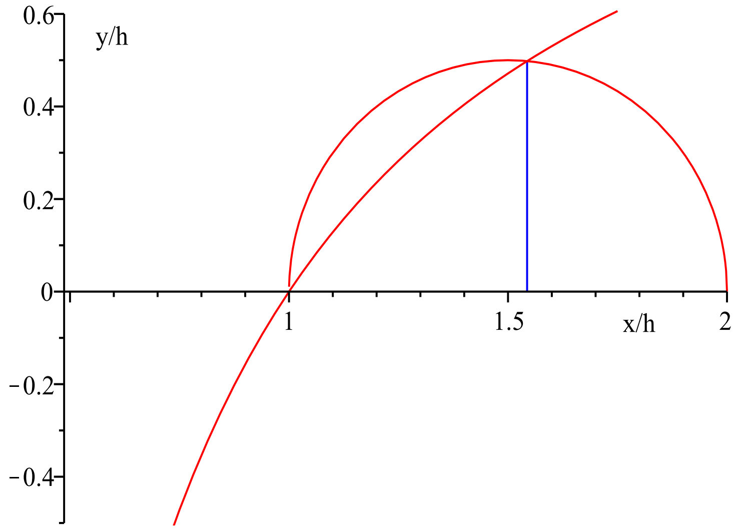

Figure 2: The geometric method of Omar Khayyám to solve the cubic for the ratio $\tilde{x} = \frac{x}{h}$.
With dimensionless coordinates (scaled with h) the circle is $y^2 = (x - 1)\,(2 - x)$ and the hyperbola is $x\,y = \sqrt{2}\,(x - 1)$.
The abscissa for the intersection point is $\tilde{x} \approx 1.54$.

Addendum, April 24, 2015: On the second figure in Omar Khayyám's paper

In *Omar Khayyám*'s paper [2] there is a second figure, also shown in [1], Abb. 3.3.21 on p. 191 (no comments are given there). This figure is contained in the present *Figure 3*. First forget about the outer circle, the $a-$square with diagonal c and the new axes x' and y', which are not in *Khayyám's* second figure. The radius R, the straight line segments $x = \overline{O,B} = \overline{H,P}$, $h = \overline{O,H}$ and the angle $\alpha = \angle(B,P,O) = \angle(H,O,P)$ are like in *Figure 1*.

Omar Khayyám shows first, based on *Euclid*, that the area of the two rectangles $A_1 = A(P,O',T,L)$ and $A_2 = A(B,O',Q,O)$ are identical. This is clear, because $(R + x)\,(R - h) = R\,x$, due to the equal ratios given in the first equation above, written as $R\,(R - h) = x\,h$. This means tat $A(H,Q,T,L) = A(H,O,B,P)$. See the shaded areas in *Figure 3*.

Then he considers one branch of a rectangular hyperbola, called here Hy, which passes through the origin O and has asymptotic lines given by the continuation of $\overline{B,O'}$ and $\overline{O',T}$. He invokes propositions of *Apollonius*' *Conics* to show that this hyperbola has to pass also through point L which is on the line segment $\overline{TA}$. Therefore, the construction of this hyperbola Hy would solve the originally posed problem on the ratios, because the ordinate of L and H is h and $\overline{H,P} = x$. He says (in the translation [2]) " carrying out this method to the end is difficult and needs a few introductions from the *conic sections*. We do not complete this in the geometric way in order that those who know conics can, if they wish , finish it later, [...]". This is what we want do now.

The idea is to introduce a new rectangular coordinate system, adapted to the asymptotics of the hyperbola Hy, namely the new origin O' and the axis (x', y'), such that the angle $\delta = \angle(B,O',S) = \frac{\pi}{4}$.

Then the equation of *Hy* is $x'^2 - y'^2 = a^2$, and the branch with $x' < 0$ is considered. If one introduces the angle $\omega = \angle(O, O', B)$, which is not marked in *Figure 3*, then in this new system the coordinates of the old origin O are $\sqrt{R^2 + x^2}\left(\cos\left(\frac{\pi}{4} - \omega\right), -\sin\left(\frac{\pi}{4} - \omega\right)\right) = \frac{1}{\sqrt{2}}\,(R + x, -(R - x))$. This follows from the trigonometric addition theorem, using cos and sin as function of tan, and inserting $\tan\omega = \frac{x}{R}$. Because O lies on *Hy* we find $a^2 = 2\,R\,x$. The geometric construction of a as geometric mean is then done by the *Thales* circle with origin M (in the old coordinate system $\left(\frac{x}{2}, 0\right)$) and radius $R + \frac{x}{2}$. The vertical at B then intersects this circle and provides a. Thus we find the vertex S of *Hy* from $\overline{O', S} = a$. The coordinates of S in the (x', y') system are then $(-a, 0)$ and in the old system $\left(-\left(\frac{a}{\sqrt{2}} - x\right), R - \frac{a}{\sqrt{2}}\right)$. Then it is clear that L lies on *Hy* because with the angle $\sigma = \angle(L, O', T)$ one has $\tan\sigma = \frac{R - h}{R + x}$ and $x'_L = \cos\left(\frac{\pi}{2} - \sigma\right)\sqrt{(R + x)^2 + (R - h)^2}$ and $y'_L = \sin\left(\frac{\pi}{2} - \sigma\right)\sqrt{(R + x)^2 + (R - h)^2}$. That is, again with the addition theorem, and cos and sin as functions of tan σ, $L = \frac{1}{\sqrt{2}}\sqrt{(R + x)^2 + (R - h)^2}\,(-(\cos\sigma + \sin\sigma), \cos\sigma - \sin\sigma) = \frac{1}{\sqrt{2}}\,(-(2\,R + x - h), x + h)$. This checks with $2\,R\,x = a^2 = x'^2_L - y'^2_L = 2\cos\sigma\sin\sigma\,((R + x)^2 + (R - h)^2)$, which becomes indeed $2\tan\sigma\,(R + x)^2 = 2\,(R - h)\,(R + x)$, and this has been shown above to be $2\,R\,x$ due to the original x and h relation.

The distance between the focus F of *Hy* and O' is then $c = \sqrt{2}\,a$ which is constructed as the diagonal in the $a-$square shown in *Figure 3*.

We give some approximate values for various quantities. $\hat{x} = \frac{x}{R} = \tan\omega \approx 0.839$, corresponding to $\omega \approx 40.01^o$, $\hat{h} = \frac{h}{R} \approx 0.544$, $\tan\sigma = \frac{1 - \hat{h}}{1 + \hat{x}} \approx 0.248$, corresponding to 13.93^o. $\hat{a} = \frac{a}{R} = \sqrt{2\,\hat{x}} \approx 1.296$. $\hat{c} = \frac{c}{R} = \sqrt{2}\,\hat{a} \approx 1.832$. $\widehat{x'_O} = \frac{x'_O}{R} = \frac{1}{\sqrt{2}}\,(1 + \hat{x}) \approx 1.300$, $-\widehat{y'_O} = -\frac{y'_O}{R} = \frac{1}{\sqrt{2}}\,(1 - \hat{x}) \approx 0.114$. $\widehat{x'_L} = \frac{x'_L}{R} = \frac{1}{\sqrt{2}}\,(2 + \hat{x} - \hat{h}) \approx -1.623$, $\widehat{y'_L} = \frac{y'_L}{R} = \frac{1}{\sqrt{2}}\,(\hat{x} + \hat{h}) \approx 3.912$.

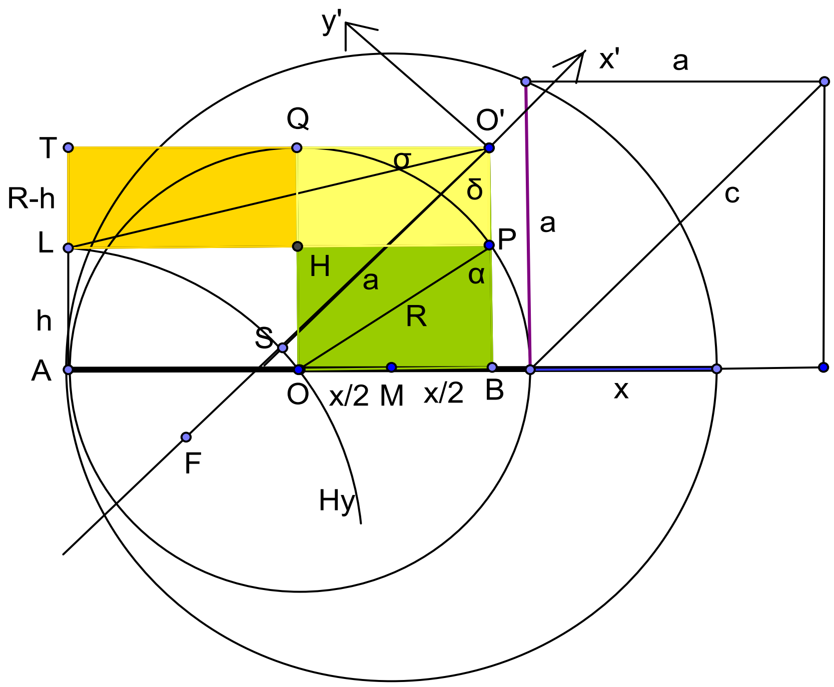

Figure 3: Omar Khayyám's second approach to his problem.

References

[1] H.-W. Alten et al., *4000 Jahre Algebra*, 2. Auflage, Springer, 2014.

[2] O Khayyam, A paper of Omar Khayyam, Scripta Math. 26 (1963), 323-337.

[3] The MacTutor History of Mathematics archive, http://www-history.mcs.st-and.ac.uk/Biographies/Kha
, Omer Khayyám.

[4] Maple^{TM}, http://www.maplesoft.com , Maplesoft.

[5] Wikipedia, https://en.wikipedia.org/wiki/Omar_Khayy%C3%A1m , Omer Khayyám.

AMS MSC numbers: 01A30, 11D25.

Keywords: Khayyám, cubic.

Concerned with OEIS sequences A256099.