ON A CONFORMAL MAPPING OF GOLDEN TRIANGLES[1]

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Golden triangles appear in the pentagon when diagonals are drawn in to produce the inscribed pentagram (or 5-star). Figure 1 shows such a golden triangle $(D_1D_2D_3)$. M_1 is the centre of the circle where the original pentagon was inscribed. We shall take its radius to be $R=1$ thus e.g. $\overline{M_1D_1} = 1$. The salient feature of this triangle is that the ratio of the length of the longer side (the diagonal in the pentagon) to its smaller one (the pentagon side length) is the golden mean $\varphi := \frac{1}{2}(1+\sqrt{5})$, the solution of $\varphi^2 = \varphi + 1$ with $\varphi > 1$: $\overline{D_1D_2} / \overline{D_2D_3} = s_0/s_1 = \varphi$.

$$\tag{1}$$

This is known at least since Euclid gave a solution to the problem of constructing a pentagon. Gauss has shown that regular p-gons with a prime number of the Fermat type, i.e. $p = 2^{2^n} + 1$, $n \in N$, can be constructed with a pair of compasses and a ruler. The Phythagorean brotherhood used the pentagram as a symbol of health and as their emblem. It is also known that one can nest golden triangles in the way indicated in Fig. 1: the line $\overline{D_2D_4}$ appears in the pentagram and its length s_1 is the same as $\overline{D_2D_3}$. The new basis length $\overline{D_3D_4} = s_2$ is again $1/\varphi$ of S_1 One may carry this nesting of golden triangles further, as indicated in Figure 1 where five steps have been performed. s_1

1 Summary of a talk given at the Department of Mathematics and Statistics, University of Technology, Lae, Papua New Guinea, September 1989.

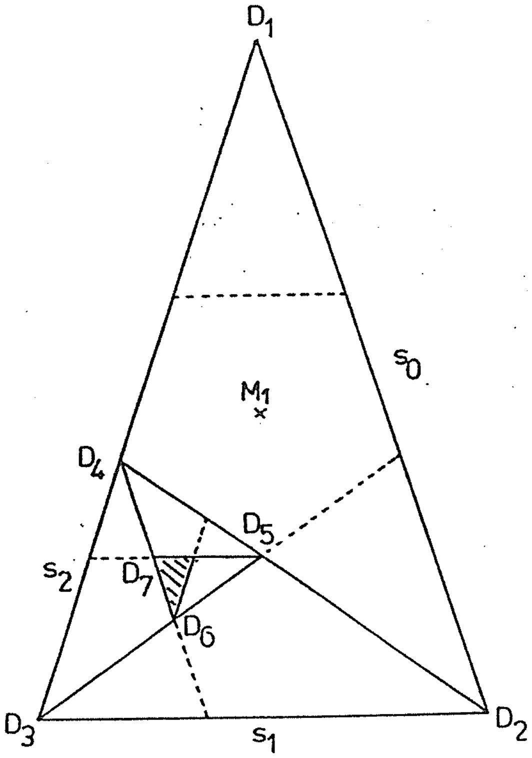

FIG.1

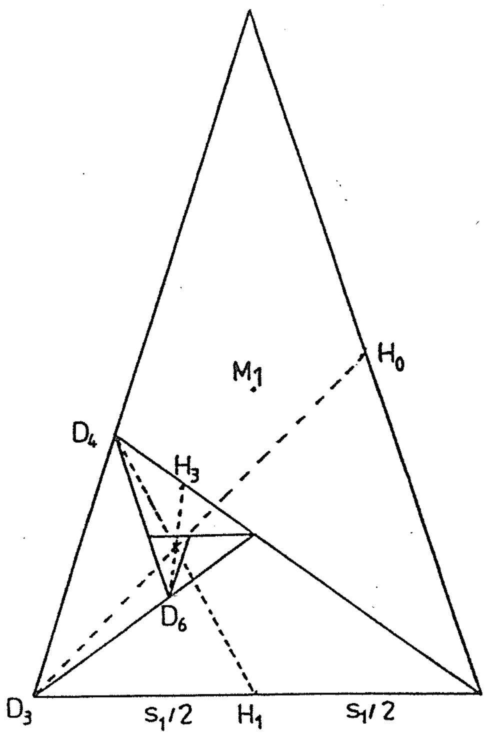

FIG.2

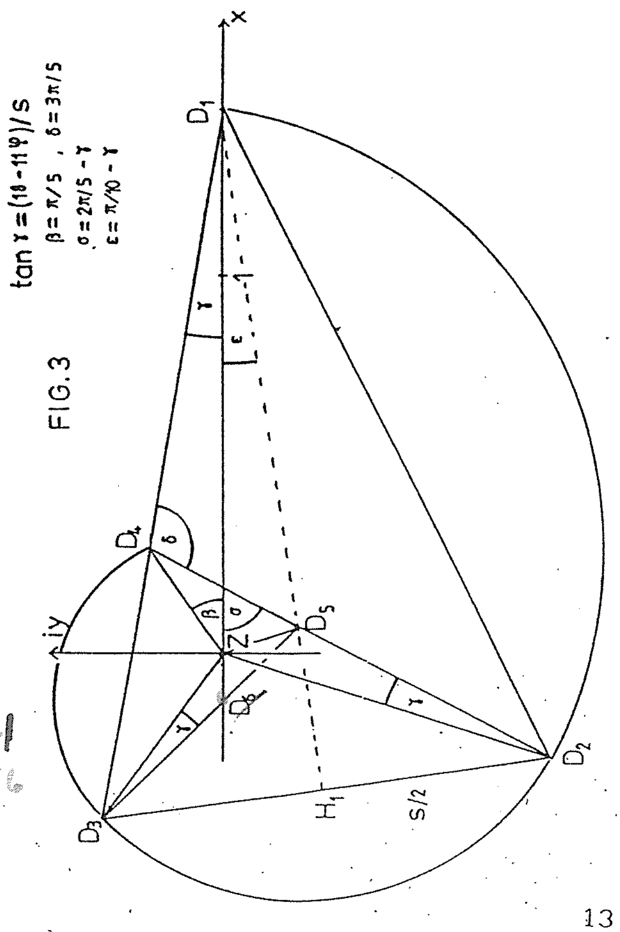

FIG.3

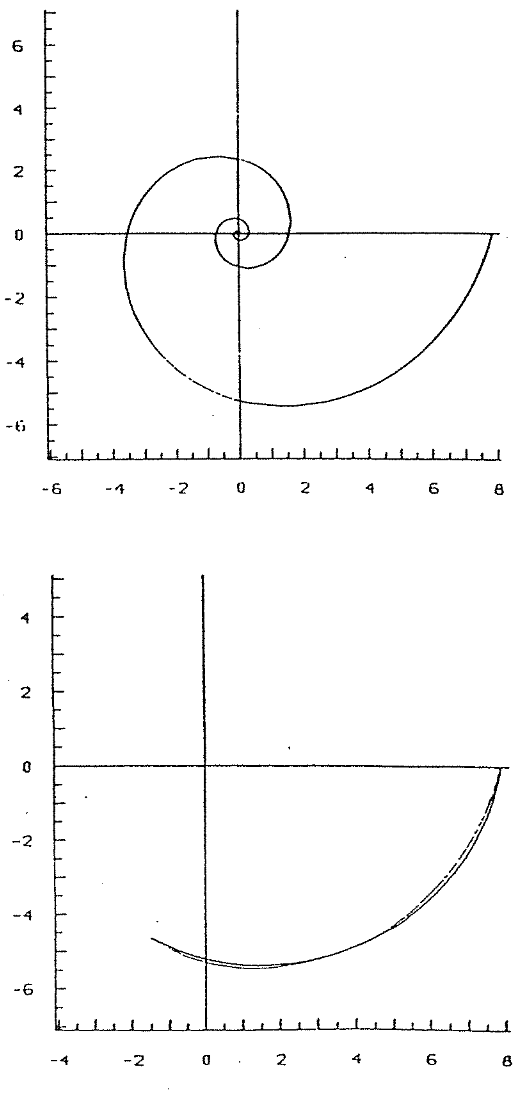

FIG. 4

The n-th golden triangle will have D_n as top vertex. The corresponding base-length will be called s_n. n may also be non-positive, indicating that this nesting process should be continued in the opposite direction. S_0 is the base of the zeroth golden triangle, which is not drawn in Figure 1. Due to the symmetry in the starting triangle one can also have a nesting mirrored at the line $\overline{D_1M_1}$. The angles in a golden triangle are twice $2\pi/5$ and, at the top, $\pi/5$. For general n one has: | s_0

$$\frac{s_n}{s_{n+1}} = \varphi\ ,\quad s_n = s_{n+1} + s_{n+2} \tag{2}$$

$$\text{with } s_1 \equiv s = \sqrt{3 - \varphi} \tag{3}$$

The pentagon length can be found, for instance, from a comparison of the trigonometric solution of $x^5 - 1 = 0$ in the complex plane with its analytic solution, using $x^5 - 1 = (x - 1)\, x^2((y^2 - 2) - y + 1)$ with $-y: = x + 1/x$. The solutions of the quadratic equation for y are φ and $\overline{\varphi} := \frac{1}{2}(1 - \sqrt{5}) = 1 - \varphi = -1/\varphi$. This comparison also shows that

$$\sin 2\pi/5 = \frac{1}{2}\sqrt{2 + \varphi}\ ,\quad \cos 2\pi/5 = \frac{1}{2}(\varphi - 1) \tag{4}$$

$$\text{and } \sin \pi/5 = \frac{1}{2}s,\quad \cos \pi/5 = \varphi/2.$$

|s

This concludes the introduction to golden triangles. One should compute various interesting lengths, for instance, the height of the 1st triangle is $h_1 = \frac{1}{2}(2 + \varphi)$. The n-th triangle has height $h_n = (\varphi - 1)^{n-1}h_1$. the length $\overline{M_1D_5}$ is $2 - \varphi = m$.

<u>Problem</u>: Find the point of convergence of the nesting of golden triangles shown in Figure 1, which was explained above.

<u>Solution</u>: We use a conformal transformation to map the golden triangle Δ_1 to the similar one Δ_2. In fact, a Mobius transformation $W(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$, $\alpha\delta - \beta\gamma \neq 0$, will do the job, due to a lemma of function theory: Given 3 points z_1, z_2 and z_3 and 3 points w_1, w_2 and w_3 there exists exactly one Mobius transformation w with $w(z_k) = w_k$ for k = 1, 2 and 3. [w

This lemma can be proved using the Möbius invariance of the double quotient

$$Dq\ (z_1,\ z_2,\ z_3,\ z_4):\ = \frac{z_4 - z_3}{z_4 - z_1} \Big/ \frac{z_2 - z_3}{z_2 - z_1}\ .$$

Then one just has to solve

$$Dq\ (w_1,\ w_2,\ w_3,\ w) = Dq\ (z_1,\ z_2,\ z_3,\ z)$$

for $w = w(z)$ to find the unique map in question.

In our case we need the coordinates of the triangle's vertices. The origin will be taken to be M_1 (see Fig. 1) and the imaginary axis is along the line $\overline{M_1 D_1}$, i.e. the D_1 coordinate is $D_1 = i$.

$D_2 = \frac{s}{2} - i\,\frac{\varphi}{2}$, $D_3 = -\,\frac{s}{2} - i\,\frac{\varphi}{2}$ and $D_4 = -\,\frac{s_2}{2} - i\,\frac{1}{2}\,(2\varphi - 3)$ with $s_2 = (\varphi - 1)s$. The Mobius transformation mapping D_1, D_2, D_3 into D_2, D_3, D_4, respectively, turns out to be a linear transformation $(\gamma = 0)$.

$$w(z) = az + b \qquad \left(a = \frac{\alpha}{\delta},\quad b = \frac{\beta}{\delta}\right) \tag{5a}$$

with the values

$$a = -\,\frac{1}{2}\,(2 - \varphi) - \frac{s}{2}\,i,\ \pm(\varphi-1)\,e^{-3\pi/5} \tag{5b}$$

$$b = -(\varphi - 1)i.$$

Such transformations have exactly one finite fixed point (there is also the fixed point $z = \infty$)[2].

$$Z = \frac{b}{1 - a} = -\,\frac{1}{2\cdot 11}\,[(3\varphi - 1) + (10\varphi - 7)i] \equiv -(A + Bi). \tag{6}$$

[2] In order to write $\frac{1}{\alpha} \equiv \frac{1}{a + b\varphi} = A + B\varphi$ for a and b not both zero, one uses the formula:

$A = \frac{a + b}{N(\alpha)}$, $B = \frac{-\,b}{N(\alpha)}$ where $N(\alpha)\ :=\ \alpha\,\bar{\alpha} = a^2 + ab - b^2$,

$(\bar{\alpha} = a + b\bar{\varphi} = a - b(\varphi - 1)$ is the norm of the integer number α of the real quadratic arithmetic field $K(\sqrt{5})$ (c.f. the number theory book of Hardy and Wright).

* $\tilde{W}(z') = w(z'+Z) - Z$

This is in fact the point of convergence of the nesting of golden triangles, because a change of origin to Z, and use of polar coordinates results in the map

$$\tilde{W}(z') = (\varphi - 1) \exp\left(\frac{-3\pi i}{5}\right) z', * \qquad (7)$$

and this rotation (with an angle of 108^0 in the negative sense) accompanied by a shrinking (by a factor $1/\varphi = \varphi - 1$) maps the n-th golden triangle into the (n + 1) th[3].

Exercise: Show that the point of convergence Z given in (6) can be found geometrically in the way indicated in Figure 2: Find the points H_0 and H_1 where $\overline{D_1 D_2}$ and $\overline{D_3 D_2}$ are segmented into two equal pieces, respectively. The intersection point of the two straight lines $\overline{D_3 H_0}$ and $\overline{D_4 H_1}$ will be Z.

We close this article by pointing out that one can find two spirals which connect all vertices of the nested golden triangles. The first spiral, called the golden spiral, henceforth, is obtained by patching together $\frac{3\pi}{5} \cong 108^0$ pieces of circles centered around D_{n+3} with radius $r_n := s_n = (\varphi - 1)^{n-1} s$, for $n \in Z$. 3 patches of such a golden spiral are shown in Figure 3, where a new rotated coordinate system in the complex plane has been used. The arc $D_1 D_2$ has center D_4 and radius $\overline{D_4 D_1} = s_1 = s$

We quote for the interested reader the lengths of the 'spokes'

$$\rho_n := \overline{Z D_n} = (\varphi - 1)^{n-1} \rho_1 = (-1)^{n-1}(f_n - f_{n-1}\varphi)\rho_1 \qquad (7a)$$

$$\sqrt{11}\ \rho_1 = \sqrt{8 + 9\varphi} = \sqrt{5 + 7\varphi}\ s, \qquad (b)$$

where f_n is the n-th Fibonacci number obtained via $f_n = f_{n-1} + f_{n-2}$ with initial values $f_0 = 0$ and $f_1 = 1$. In fact, one finds

[3] Such a linear transformation which is neither a pure rotation nor a pure scaling is called a loxodromic transformation, because it maps also logarithmic spirals into logarithmic spirals. A logarithmic spiral can be written in polar coordinates as $\rho(\phi) = \rho_0 e^{k\phi}$

$$\sqrt{11}\ \rho_n/s = \sqrt{(5f_{2n-1} - 7f_{2n-2}) + (7f_{2n-1} - 12f_{2n-2})\varphi}. \tag{8}$$

It is interesting to note that

$$\rho_4 = 2 \sin \gamma \tag{9}$$

where γ is the angle between the lines $\overline{ZD}_n$ and $\overline{D_{n+3}D_n}$ (c.f. Fig.3, where also $\tan\gamma$ is given). The analytic form for the n-th patch connecting D_n with D_{n+1} is (in polar coordinates with origin Z and $-\Phi = \frac{3\pi}{5}(n - 1) + \hat{\Phi}$)

$$\rho_{(n)}(\hat{\Phi}) = \rho_{n+3}\left(\cos(\frac{\pi}{5} + \hat{\Phi}) + \sqrt{\cos^2(\frac{\pi}{5} + \hat{\Phi}) + 3 + 5\varphi}\right) \tag{10}$$

with $\hat{\Phi} \in [0, 3\pi/5)$. One can check that $\rho_{(n)}(0) = \rho_n$ and that $\rho_{(n)}(3\pi/5) = \rho_{n+1}$.

One can connect the points D_n also by the logarithmic spiral:

$$\rho_{LS}(\Phi) = \rho_1 \exp\ (+5/(3\pi)\ \mathcal{L}n\varphi\ \Phi) \tag{11}$$

written in polar coordinates with origin at Z. The characteristic angle between the line $\overline{ZP}$ and the tangent at P for any point P on the spiral is given by

$$\cot \alpha_{LS} = 5/3\pi\ \mathcal{L}n\ \varphi \tag{12}$$

or $\alpha_{LS} \cong 75, 68$. This angle is not a constant for the golden spiral. It is maximal at the point D_n where (for γ see (9)).

$$\alpha_{GS}(D_n) = \pi/2 - \gamma \approx 80.25^0 \tag{13}$$

and drops at $\hat{\phi}_{min} = 3\pi/10$ to its minimal value of $\approx 73.27^0$ according to

$$\cos \alpha_{GS}(\hat{\Phi}) = \sin \gamma\ (\hat{\Phi}) = a(\hat{\Phi})\sin \gamma, \tag{14a}$$

$$a(\hat{\phi}) = (1 + \frac{\varphi}{5} \tan \hat{\Phi})\cos\hat{\Phi} = \frac{2}{s} \rho_4 \sin(\frac{\pi}{5} + \hat{\Phi}) \tag{14b}$$

A comparison of the first patch of these two spirals is shown in the lower plot of Figure 4, where a certain value of ρ_1 has been used as input. The fainter line is the logarithmic spiral piece. The other line is a 108^0 - part of a circle around D_4 with radius s. The upper plot of Figure 4 shows the logarithmic spiral (11) for $\Phi \geq 0$.

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After the seminar talk, Basil Landau showed me the reference books: "A bibliography of recreational mathematics", Vols. 1 and 2 by William L. Schaaf, in which I found the Coxeter reference.

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