ON SUMS OF POWERS OF ZEROS OF POLYNOMIALS [1]

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Abstract

Due to Girard's (sometimes called Waring's) formula the sum of the $r-$th power of the zeros of every one variable polynomial of degree N, $P_N(x)$, can be given explicitly in terms of the coefficients of the monic $\tilde{P}_N(x)$ polynomial. This formula is closely related to a known $N-1$ variable generalization of Chebyshev's polynomials of the first kind, $T_r^{(N-1)}$. The generating function of these power sums (or moments) is known to involve the logarithmic derivative of the considered polynomial. This entails a simple formula for the Stieltjes transform of the distribution of zeros. Perron-Stieltjes inversion can be used to find this distribution, *e.g.* for $N \to \infty$.

Classical orthogonal polynomials are taken as examples. The results for ordinary Chebyshev $T_N(x)$ and $U_N(x)$ polynomials are presented in detail. This will correct a statement about power sums of zeros of Chebyshev's $T-$polynomials found in the literature. For the various cases (Jacobi, Laguerre, Hermite) these moment generating functions provide solutions to certain Riccati equations.

1 Introduction

Sums of powers of the zeros of every one variable polynomial $P_N(x)$ of degree N can be given in terms of its coefficients as $(N-1)$-fold nested sums. If the polynomial at hand has definite parity, *i.e.* if $P_N(-x) = (-1)^N P_N(x)$, only $\lfloor \frac{N}{2} - 1 \rfloor$ nested sums are needed. These facts follow from *Girard*'s formula [11]: The considered r^{th} power sums are one part symmetric functions (polynomials) of weight r in the zeros, and due to the fundamental theorem of symmetric functions, *e.g.*, [6] they can be uniquely expressed in terms of the elementary symmetric functions $\{\sigma_i\}_1^N$ of the zeros, i.e. (modulo signs) in terms of the coefficients of the corresponding monic polynomial $\tilde{P}_N(x)$. *Girard*'s formula which solves this problem is usually, *e.g.*, [11] given in a

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'semi-explicit' form where still partitions have to be performed. A simplification of this formula has been given in [9], [8], [10]. Here we shall present the explicit formula. The latter authors defined a certain $(N-1)$ variable generalization of *Chebyshev*'s polynomials of the first kind, $T_r^{(N-1)}(\sigma_1, ..., \sigma_N)$, by the *Girard-Waring* expression after putting σ_N, the product of the zeros, to one.

The zeros of characteristic polynomials $P_N(x)$ determine the eigenvalue spectrum of (finite-dimensional) operators. Therefore information on these zeros is welcome. In most cases these zeros are not known, and numerical or perturbative methods are employed. Often, the limit case $N \to \infty$ is of interest and the limit density of zeros $\rho(x) := \lim_{N\to\infty} \rho(N, x)$ is asked for. Here the power sums of zeros become important because, for given $P_N(x)$ with only real zeros (hence for all orthogonal polynomials with positive moment functional), the (ordinary) generating function of all (positive integer) power sums, $G(N, z)$, is related in a simple way to the *Stieltjes*-transform of the discrete distribution of zeros $\rho(N, x)$ (the counting measure of the zeros which lives on the real line). This function $G(N, z)$, which generates the generalized *Chebyshev* polynomials of the first kind, is essentially given by the logarithmic derivative of the polynomial $P_N(x)$ under consideration. If it can be considered as a function of real N, and the limit $N \to \infty$ exists, then one can obtain the limit distribution of zeros of $P_N(x)$ by *Perron-Stieltjes* inversion [15], [1], [4], [2].

In this report which is based on the more detailed work [7] (which is referred to for proofs and intermediate steps) we demonstrate the above mentioned facts for the classical orthogonal polynomials. The generating functions $G(N, z)$ are computed, and they are seen to satisfy certain *Riccati*-equations, or their equivalent linear second order differential equations. For *Chebyshev*'s polynomials of both kinds simple expressions for this generating functions of the moments of the dicrete counting measure of the zeros result. It becomes transparent why in these cases the $1/N$ expansion for the power sums or moments becomes unreliable. Finally the well-known limit distribution of zeros for the *Jacobi*-polynomials and (appropriately scaled) generalized *Laguerre*- and *Hermite*-polynomials are found from the *Stieltjes*-transform *via* *Perron-Stieltjes*-inversion.

As explained above, this method to find limit distributions of zeros works for all polynomials with real zeros whose logarithmic derivative can be considered as function of the degree number N with existing limit $N \to \infty$ (possibly rescaled).

2 Girard-Waring formula for the power sums of zeros of polynomials

A monic polynomial of degree N in one real variable x is written as

$$\tilde{P}_N(x) = \sum_{k=0}^{N} (-1)^k \, \sigma_k(N) \, x^{N-k} \quad = \quad \prod_{i=1}^{N} (x - x_i^{(N)}). \tag{1}$$

The fundamental theorem of algebra has been used. The second equation defines the k^{th} elementary symmetric polynomials $\sigma_k(N) \equiv \sigma_k$ of the N zeros $\{x_i^{(N)}\}_1^N$ of $P_N(x)$. $\sigma_0(N) \equiv 1$.

The normalized sum of the r^{th} power of the zeros is

$$\begin{aligned} m_r(N) &:= \frac{1}{N} \sum_{i=1}^{N} (x_i^{(N)})^r & (2) \\ &= \int_{-\infty}^{+\infty} x^r \; d\rho_N(x) & (3) \\ d\rho_N(x) &:= \frac{1}{N} \sum_{i=1}^{N} \delta(x - x_i^{(N)}) \; dx. & (4) \end{aligned}$$

The discrete counting measure for the zeros, $\rho_N(x)$, is used only for polynomials with only real zeros, in order to have this measure normalized to 1. Under this assumption the power sums are the moments of this real discrete measure. (Sometimes we shall call the power sums of zeros of *general* polynomials also moments.)

For an arbitrary polynomial *Girard*'s formula, when written explicitly, looks for $r \in \mathbf{N}$, $N = 2, 3, \ldots$ as follows:

$$\begin{aligned} m_r(N) \;&=\; \tfrac{1}{N}\, \sigma_1^r \, \left(\prod_{k=1}^{N-1} (\sum_{i_k=0}^{\lfloor b_k \rfloor} \tfrac{(-1)^{(N-k) i_k}}{(i_k)!}) \right) \; < N, r, \{i_j\}_1^{N-1} > \; \prod_{p=1}^{N-1} (\tfrac{\sigma_{p+1}}{\sigma_1^{p+1}})^{i_{N-p}} & (5) \\ &=: \; t_r^{(N)}(\sigma_1, \ldots, \sigma_N) \quad , & (6) \end{aligned}$$

where the $N-1$ sums are ordered from the left to the right with increasing k value, and the upper boundaries of these sums are the greatest integer of $b_1 := r/N$ and

$$b_k \; := \frac{r \sum_{j=1}^{k-1} \, (N+1-j) \; i_j}{N+1-k} \quad , \quad k = 2, \ldots, N-1 \; . \tag{7}$$

The combinatorial factor which appears in eq. (5) is

$$< N, r, \{i_j\}_1^{N-1} > \; := \frac{r \; (r - 1 - \sum_{j=1}^{N-1} \, (N-j) i_j)!}{(r - \sum_{j=1}^{N-1} \, (N+1-j) \; i_j)!} \; . \tag{8}$$

For $r = 0$ one has $m_0(N) \equiv 1$. $t_r^{(N)}(\sigma_1, \ldots, \sigma_N)$, which is a polynomial, has been introduced because it is related to an $N-1$ variable generalization of *Chebyshev*'s polynomial of the first kind $T_r(x)$. This connection is given after the rescaling

$$T_r^{(N-1)}(s_1, \ldots, s_{N-1}) \; = t_r^{(N)}(\sigma_1, \ldots, \sigma_N) / (\sigma_N)^{r/N} \quad , \tag{9}$$

with

$$s_k \; := \sigma_k / (\sigma_N)^{k/N} \quad , \quad k = 1, \ldots, N-1 \; . \tag{10}$$

The ordinary *Chebyshev* polynomials are now $T_r(x) \; = T_r^{(1)}(2x)$.

Corollary: Moments for $\mathbf{P_N(-x) = (-1)^N \; P_N(x)}$ polynomials

$$i) \quad m_0(N) \equiv 1 \quad , \quad N \in \mathbf{N_0} \qquad\qquad ii) \; m_{2l+1}(N) \equiv 0 \quad , \quad l \in \mathbf{N_0} \; , \; N \in \mathbf{N} \; , \tag{11}$$

$$iii) \; m_{2l}(N) \; =$$

$$\frac{2}{N}\,(-\sigma_2)^l\,\left(\prod_{k=1}^{\lfloor\frac{N}{2}\rfloor-1}\,(\sum_{i_k=0}^{\lfloor B_k\rfloor}\,(-1)^{(\lfloor\frac{N}{2}\rfloor-k)i_k}\frac{1}{(i_k)!})\right)\,<\lfloor\tfrac{N}{2}\rfloor,l,\{i_j\}_1^{\lfloor\frac{N}{2}\rfloor-1}>\,\prod_{p=2}^{\lfloor\frac{N}{2}\rfloor}(\frac{\sigma_{2p}}{\sigma_2^p})^{i_{\lfloor\frac{N}{2}\rfloor-p+1}} \tag{12}$$

$$= \frac{2}{N}\lfloor\tfrac{N}{2}\rfloor(-1)^l\ t_l^{(\lfloor\frac{N}{2}\rfloor)}(\sigma_2,\sigma_4,...,\sigma_{2\lfloor\frac{N}{2}\rfloor}) \tag{13}$$

$$= \frac{2}{N}\lfloor\tfrac{N}{2}\rfloor(-1)^l\ (\sigma_{2\lfloor\frac{N}{2}\rfloor})^{l/\lfloor\frac{N}{2}\rfloor}\ T_l^{(\lfloor\frac{N}{2}\rfloor-1)}(e_1,...,e_{\lfloor\frac{N}{2}\rfloor-1}) \tag{14}$$

$$\text{for } l\in\mathbf{N_0}\ ,\quad N=4,5,...,$$

where B_k, resp. $<\lfloor\frac{N}{2}\rfloor,l,\{i_j\}_1^{\lfloor\frac{N}{2}\rfloor-1}>$, resp. $t_l^{(\lfloor\frac{N}{2}\rfloor)}$, is obtained from b_k of (7), resp. $<N,r,\{i_j\}_1^{N-1}>$ of (8), resp. $t_r^{(N)}$ of (6), by replacing $N\to\lfloor\frac{N}{2}\rfloor$ and $r\to l$. The variables of $T_l^{(\lfloor\frac{N}{2}\rfloor-1)}$, which is also obtained by the same replacement from $T_r^{(N-1)}$ given in (9), are

$$e_k\equiv e_k(N)\ =\sigma_{2k}/(\sigma_{2\lfloor\frac{N}{2}\rfloor})^{k/\lfloor\frac{N}{2}\rfloor}\quad,\quad k=1,...,\lfloor\frac{N}{2}\rfloor-1\ . \tag{15}$$

In addition: $m_{2l}(1)\ =0\quad,\quad m_{2l}(2)\ =(-\sigma_2)^l\quad,\quad m_{2l}(3)\ =\frac{2}{3}(-\sigma_2)^l$.

Example:

Consider *Chebyshev*'s polynomials of the first kind, $T_N(x)$, which are classical orthogonal polynomials of the *Jacobi* type [14], [4]. The zeros are known, and the non-vanishing moments are $m_{2l}^{(T)}(N)\ =\frac{2}{N}\sum_{k=1}^{\lfloor\frac{N}{2}\rfloor}(cos(\frac{2k-1}{2N}\pi))^{2l}$. The *Corollary* gives an alternative expression for this sum.

$$\text{For } l\in\mathbf{N}\ ,\ M\in\mathbf{N}\ :\ m_{2l}^{(T)}(2M)\ =T_l^{(M-1)}(e_1(2M),...,e_{M-1}(2M))/2^{(2-\frac{1}{M})l}\ , \tag{16}$$

$$\text{with}\ \ e_k(2M)\equiv\frac{2M}{k}\ 2^{-\frac{k}{M}}\binom{2M-k-1}{k-1}\ ,\quad k=1,...,M-1, \tag{17}$$

$$m_{2l}^{(T)}(2M+1)\ =\frac{2M}{2M+1}\ (\frac{(2M+1)^{\frac{1}{M}}}{4})^l\ T_l^{(M-1)}(e_1(2M+1),...,e_{M-1}(2M+1)), \tag{18}$$

$$\text{with}\ \ e_k(2M+1)\equiv\frac{2M+1}{k}\ \frac{1}{(2M+1)^{\frac{k}{M}}}\ \binom{2M-k}{k-1},\ k=1,...,M-1. \tag{19}$$

and $m_0^{(T)}(N)\ =1\quad,\quad m_{2l}^{(T)}(1)\ =0$.

Some instances for $l\in\mathbf{N}$ are:

$$m_{2l}^{(T)}(2)\ =(\tfrac{1}{2})^l\ ,\ m_{2l}^{(T)}(4)\ =T_l(\sqrt{2})/(2\sqrt{2})^l\ ,$$
$$m_{2l}^{(T)}(6)\ =T_l^{(2)}(3\cdot2^{\frac{2}{3}},9\cdot2^{-\frac{2}{3}})/2^{\frac{5}{3}l}\ , \tag{20}$$

$$m_{2l}^{(T)}(3)\ =3^{l-1}/2_{2l-1}\quad,\quad m_{2l}^{(T)}(5)\ =(\sqrt{5})^{l-2}\ T_l(\sqrt{5}/2)/2^{2(l-1)}\ ,$$
$$m_{2l}^{(T)}(7)\ =\frac{6}{7}\ (\frac{7^{\frac{1}{3}}}{4})^l\ T_l^{(2)}(7^{\frac{2}{3}},2\cdot7^{\frac{1}{3}})\ . \tag{21}$$

From this one finds for N up to 7:

$$m_2^{(T)}(N)\ =\begin{cases}0 & \text{N=1}\\ \frac{1}{2} & \text{N=2,...}\end{cases}\ ,\ m_4^{(T)}(N)\ =\begin{cases}0 & \text{N=1}\\ 1/4 & \text{N=2}\\ 3/8 & \text{N=3,...}\end{cases}\ ,\ m_6^{(T)}(N)\ =\begin{cases}0 & \text{N=1}\\ 1/8 & \text{N=2}\\ 9/32 & \text{N=3}\\ 5/16 & \text{N=4,...}\end{cases}\ . \tag{22}$$

These values are in conflict with a statement found in [3]. There it is claimed that the moments (22) are given by the highest values, independently of N, which is false. Only the values for all $N \geq \frac{r}{2}+1$ are correctly given in [3].
The example of *Chebyshev*'s polynomials of the second type is also treated in [7].

3 Moment generating functions

The generating function of $p_r(N)$, the sum of the r^{th} power of N variables $\{x_i\}_1^N$, is given by the logarithmic derivative of the generating function of the elementary symmetric functions $\{\sigma_k\}$ of these variables, see *e.g.* [6]. With

$$E(z) \equiv E_N(z) := \prod_{i=1}^{N} (1 - x_i^{(N)}\ z) \ = \sum_{r=0}^{N} (-1)^r \sigma_r\ z^r \quad , \ \sigma_0 = 1 \ , \tag{23}$$

which is also given by $E(z) \ = z^N \tilde{P}_N(1/z)$, one finds from

$$(ln\ E(z))' \ = E'(z)/E(z) \ = -\frac{1}{z} \sum_{r=1}^{\infty} p_r(N)\ z^r \tag{24}$$

and *Girard's* fomula (5), (6)

$$G(N,z) \ := \sum_{r=0}^{\infty} m_r(N)\ z^r \ = \ \textstyle\sum_{r=0}^{\infty}\ t_r^{(N)}(\sigma_1, ..., \sigma_N)\ z^r \ = \ \frac{1}{N} \frac{x \frac{d}{dx} P_N(x)}{P_N(x)}\Big|_{x=1/z} \ =$$

$$= \ \frac{\sum_{r=0}^{N-1} (-1)^r (1-\frac{r}{N})\ \sigma_r(N)\ z^r}{\sum_{r=0}^{N} (-1)^r \sigma_r(N)\ z^r}. \tag{25}$$

As an application we computed the moment generating functions $G(N,z)$ for the classical orthogonal polynomials: *Jacobi* $P_N^{(\alpha,\beta)}(x)$, generalized *Laguerre* $L_N^{(\alpha)}(x)$ and *Hermite* $H_N(x)$. The derivative rules for these polynomials (*cf.* [4]) have been used. We quote the results for *Chebyshev*'s polynomials of both kinds separately because they can be written in terms of an elementary function. For details see [7]. The hypergeometric function ${}_2F_1(a,b,c;x)$ and its confluent version $F(a,c;x) \equiv {}_1F_1(a,c;x)$ appear in *table 1.*
The $(N+1)-$term recursion relation for the $t_r^{(N)}-$polynomials which can be derived from eq.(25), is:

$$\text{for } n \ = \ N, N+1, ...,: \quad \sum_{j=0}^{N} (-1)^j \sigma_j\ t_{n-j}^{(N)} \ = \ 0 \quad , \quad \sigma_0 = 1 \ , \tag{26}$$

where the N input quantities $\{t_j^{(N)}\}_0^{N-1}$ are either computed from *Girard*'s formula or from the generating function $G(N,z)$.

We also comment on the generating functions $G_<(N,z)$ for the normalized sums of negative powers of the zeros: $m_{-r}(N) \ := \frac{1}{N}\sum_{i=1}^{N} \ (x_i^{(N)})^{-r}$, $r \in \mathbf{N}$. Here we restrict the discussion to polynomials which have no vanishing zeros. With $E_N^<(z) \ := \prod_{i=1}^{N}(1 - z/x_i^{(N)}) \ = \ (-1)^N \tilde{P}_N(z)/\sigma_N$ one finds

$$G_<(N,z) \ := \ \sum_{r=1}^{\infty} \ m_{-r}(N)\ z^r \ = \ -\frac{z}{N}(ln\ E_N^<(z))' \ = \ -\frac{z}{N}\frac{\tilde{P}_N'(z)}{\tilde{P}_N(z)} \ = \ -\ G(N, 1/z)\ . \tag{27}$$

Beware of expanding the identity $G_<(N,z) \ + \ G(N,1/z) \ \equiv \ 0$ in z *and* $1/z$ (and concluding falsely that all power sums have to vanish). The sums of negative powers of the $P_N(x)$ zeros can then be found by differentiation. The connection between *Faà di Bruno*'s formula and the *Bell*-polynomials [13] and the identification of the generalized *Chebyshev* polynomials $t_r^{(N)}$ as special *Bell*-polynomials leads, with the abbreviation $g(x) \ \equiv \tilde{P}_N(x)$, for $m \in \mathbf{N}$ first to

$$\frac{d^m}{dx^m}\ ln\ g(x) \ = \\ -(m-1)!\ N\ t^{(N)}(-g^{(1)}(x)/(1!g(x)),\ +g^{(2)}(x)/(2!g(x)),\ ...,(-1)^m g^{(m)}(x)/(m!g(x)))\ , \quad (28)$$

with $g^{(k)} \equiv \frac{d^k}{dx^k} g(x)$. After putting $x = 0$, using $(-1)^k g^{(k)}(0)/(k!g(0)) \ = \ \sigma_{N-k}/\sigma_N$, one finally finds

$$m_{-r}(N) = \begin{cases} t_r^{(N)}(\sigma_{N-1}/\sigma_N,\ \sigma_{N-2}/\sigma_N,\ ...,\ \sigma_{N-r}/\sigma_N, 0, ..., 0) & \text{for } \ 0 \ \leq \ r \ \leq N \\ t_r^{(N)}(\sigma_{N-1}/\sigma_N,\ \sigma_{N-2}/\sigma_N,\ ...,\ (\sigma_0 = 1)/\sigma_N) & \text{for } r \ \geq \ N\ , \end{cases} \quad (29)$$

where in the upper alternative $N - r$ zeros have to be inserted.

Table 1 : Generating functions for power sums of zeros of classical orthogonal polynomials

$\mathbf{P_N(x)}$	$\mathbf{G(N,z)}$
$P_N^{(\alpha,\beta)}$	$G^{(\alpha,\beta)}(N,z) \ = \ \frac{N+\alpha+\beta+1}{2(\alpha+1)} \frac{1}{z} \frac{{}_2F_1(-N+1,\ N+\alpha+\beta+2,\ \alpha+2;\ (z-1)/2z)}{{}_2F_1(-N,\ N+\alpha+\beta+1,\ \alpha+1;\ (z-1)/2z)}$
T_N	$G^{(T)}(N,z) \ = \ \frac{1}{\sqrt{1-z^2}} tanh(N\ ln \frac{z}{1-\sqrt{1-z^2}})$
U_N	$G^{(U)}(N,z) \ = \ \frac{1}{\sqrt{1-z^2}} \Big\{ (1+\frac{1}{N})\ tanh\Big((N+1)\ ln \frac{z}{1-\sqrt{1-z^2}}\Big) - \frac{1}{N} \frac{1}{\sqrt{1-z^2}} \Big\}$
$L_N^{(\alpha)}$	$G^{(\alpha)}(N,z) \ = \ 1 - \frac{F(-N+1,\alpha+1;1/z)}{F(-N,\alpha+1;1/z)}$
H_N	$G^{(H)}(N,z) \ = \ \frac{1}{z} \frac{F(-\frac{N-1}{2},\ \frac{1}{2};\ 1/z^2)/\Gamma(\frac{2-N}{2})\ -\frac{2}{z}\ F(\frac{2-N}{2},\ \frac{3}{2};\ 1/z^2)/\Gamma(\frac{1-N}{2})}{F(-\frac{N}{2},\frac{1}{2};1/z^2)/\Gamma(\frac{1-N}{2})\ -\frac{2}{z}\ F(\frac{1-N}{2},\ \frac{3}{2};\ 1/z^2)/\Gamma(-\frac{N}{2})}$

One can verify (*cf.* [7]) that for orthogonal polynomials the generating functions $G(N,z)$ are special solutions of certain *Riccati*-equations which we list in the following *table 2.*

Table 2 : Riccati-equations satisfied by moment generating functions of classical orthogonal polynomials

$\mathbf{P_N(x)}$	Riccati-eq. for $\mathbf{G(N,z)}$ for every $\mathbf{N}$
$P_N^{(\alpha,\beta)}$	$\frac{(1-z^2)z}{N}\frac{\partial}{\partial z}G^{(\alpha,\beta)}(N,z) = (z^2+(\alpha-\beta)z+\alpha+\beta+1)\frac{1}{N}\, G^{(\alpha,\beta)}(N,z)$ $+(1-z^2)\,(G^{(\alpha,\beta)}(N,z))^2 - (1+\frac{1+\alpha+\beta}{N})$
T_N	$\frac{(1-z^2)z}{N}\frac{\partial}{\partial z}G^{(T)}(N,z) = \frac{z^2}{N}\, G^{(T)}(N,z) + (1-z^2)(G^{(T)}(N,z))^2 - 1$
U_N	$\frac{(1-z^2)z}{N}\frac{\partial}{\partial z}G^{(U)}(N,z) = \frac{2+z^2}{N}\, G^{(U)}(N,z) + (1-z^2)(G^{(U)}(N,z))^2 - \frac{N+2}{N}$
$L_N^{(\alpha)}$	$z^2\,\frac{\partial}{\partial z}\, G^{(\alpha)}(N,z) = Nz(G^{(\alpha)}(N,z))^2 + (\alpha z-1)\, G^{(\alpha)}(N,z) + 1$
$Ls_N^{(\alpha)}$	$z^2\frac{1}{N}\,\frac{\partial}{\partial z}\, Q^{(\alpha)}(N,z) = z(Q^{(\alpha)}(N,z))^2 + (\alpha z/N-1)\, Q^{(\alpha)}(N,z) + 1$
H_N	$\frac{z^3}{2}\,\frac{\partial}{\partial z}\, G^{(H)}(N,z) = \frac{z^2}{2}N(G^{(H)}(N,z))^2 - (1+\frac{z^2}{2})G^{(H)}(N,z) + 1$
Hs_N	$\frac{z^3}{2N}\,\frac{\partial}{\partial z}\, Q^{(H)}(N,z) = \frac{z^2}{2}(Q^{(H)}(N,z))^2 - (1+\frac{z^2}{2N})Q^{(H)}(N,z) + 1$

We have also listed the *Riccati*-equations satisfied by the moment generating functions of scaled generalized *Laguerre*- and *Hermite*-polynomials. The definitions are:

$$Ls_N^{(\alpha)}(x) := L_N^{(\alpha)}(Nx) \quad , \quad Q^{(\alpha)}(N,z) := G^{(\alpha)}(N,z/N) \quad , \tag{30}$$
$$Hs_N(x) := H_N(\sqrt{N}x) \quad , \quad Q^{(H)}(N,z) := G^{(H)}(N,z/\sqrt{N}) \quad . \tag{31}$$

The associated second order linear differential eqs. are listed in *table 3*. $G(N,z)$, resp. $Q^{(\alpha)}(N,z) := G^{(\alpha)}(N,z/N)$ and $Q^{(H)}(N,z) := G^{(H)}(N,z/\sqrt{N})$, are replaced by $-\frac{z}{N}\,\frac{\partial}{\partial z}(ln\,\mathcal{U}(N,z))$, with correspondingly labelled $\mathcal{U}(N,z)$.

Table 3: Second order linear differential equation satisfied by $\mathcal{U}(\mathbf{N},\mathbf{z})$ for given $\mathbf{N}$ for (scaled) classical orthogonal polynomials

$\mathbf{P_N(x)}$	$\mathbf{2}^{\mathrm{nd}}$ order LDE for $\mathcal{U}(\mathbf{N},\mathbf{z})$ for every $\mathbf{N}$
$P_N^{(\alpha,\beta)}$	$\left(z^2(1-z^2)\frac{\partial^2}{\partial z^2} - z(2z^2+(\alpha-\beta)z+\alpha+\beta)\frac{\partial}{\partial z} - N(N+1+\alpha+\beta)\right)\mathcal{U}^{(\alpha,\beta)}(N,z) = 0$
T_N	$\left(z^2(1-z^2)\frac{\partial^2}{\partial z^2} + z(1-2z^2)\frac{\partial}{\partial z} - N^2\right)\mathcal{U}^{(T)}(N,z) = 0$
U_N	$\left(z^2(1-z^2)\frac{\partial^2}{\partial z^2} - z(1+2z^2)\frac{\partial}{\partial z} - N(N+2)\right)\mathcal{U}^{(U)}(N,z) = 0$
$Ls_N^{(\alpha)}$	$\left(z^3\frac{\partial^2}{\partial z^2} + z(N+(1-\alpha)z)\frac{\partial}{\partial z} + N^2\right)\mathcal{U}^{(\alpha)}(N,z) = 0$
Hs_N	$\left(z^4\frac{\partial^2}{\partial z^2} + 2z(z^2+N)\frac{\partial}{\partial z} + 2N^2\right)\mathcal{U}^{(H)}(N,z) = 0$

The above given *Riccati*-equations can also be found starting with *Case*'s eqs. which have been derived for the sums of powers of (simple) zeros of polynomials satisfying certain differential equations [3]. *Table 4* shows these eqs. for the classical orthogonal polynomials. In the generalized *Laguerre* and the *Hermite* case scaled moments are used in accordance with the above defined scaled polynomials $Ls_N^{(\alpha)}(x)$ and $Hs_N(x)$:

$$q_r^{(H)}(N) = m_r^{(H)}(N)/N^{r/2} \quad , \quad q_r^{(\alpha)}(N) = m_r^{(\alpha)}(N)/N^r \ . \tag{32}$$

The input moments are:
Jacobi: $m_0(N) \equiv 1$, $m_1(N) := (\beta-\alpha)/(2N+\alpha+\beta)$.
For T_N : $\alpha = \beta = -1/2$ and for U_N : $\alpha = \beta = +1/2$.
generalized *Laguerre:* $q_0^{(\alpha)}(N) \equiv 1$.
Hermite: $q_0^{(H)}(N) \equiv 1$, $q_1^{(H)}(N) \equiv 0$.
From the *Riccati*-eqs. which the generating functions of the power sums satisfy, one infers the $N \to \infty$ limits (the leading term of the $1/N$ expansion) which are given, together with the (scaled) moments, in *table 5*. $G_0(z) := \lim_{N\to\infty} G^{(\alpha,\beta)}(N,z)$ in the *Jacobi*-case. In the generalized *Laguerre*-case $Q_0(z) := \lim_{N\to\infty} Q^{(\alpha)}(N,z)$, and in the *Hermite*-case $Q_0^{(H)}(z) := Q^{(H)}(N,z)$. $c(z)$ is the generating function of *Catalan*'s numbers $\{C_r\}$. Universality is manifest due to the independence of the parameters α, β, resp. α, in the *Jacobi*-, resp. generalized *Laguerre*-case. These asymptotic moments have also been considered in [12] (although the appearance of *Catalan*'s numbers has not been mentioned).

Table 4: Case's equations for moments of zeros of classical orthogonal polynomials

$\mathbf{P_N(x)}$	**Case-eqs. for $\{\mathbf{m_r(N)}\}$, resp. $\{\mathbf{q_r(N)}\}$, for given N**
$P_N^{(\alpha,\beta)}$	$(\alpha+\beta+2N-r)\ m_{r+1}^{(\alpha,\beta)}(N)\ = (\beta-\alpha)\ m_r^{(\alpha,\beta)}(N) - r\ m_{r-1}^{(\alpha,\beta)}(N) +$ $+\ N\sum_{s=0}^{r-1}(m_{r-1-s}^{(\alpha,\beta)}(N)\ m_s^{(\alpha,\beta)}(N)\ - m_{r-s}^{(\alpha,\beta)}(N)\ m_{s+1}^{(\alpha,\beta)}(N))$
T_N	$(2N-(r+1))\ m_{r+1}^{(T)}(N)\ = -r\ m_{r-1}^{(T)}(N) +$ $+\ N\sum_{s=0}^{r-1}(m_{r-1-s}^{(T)}(N)\ m_s^{(T)}(N)\ - m_{r-s}^{(T)}(N)\ m_{s+1}^{(T)}(N))$
U_N	$(2N-(r-1))\ m_{r+1}^{(U)}(N)\ = -r\ m_{r-1}^{(U)}(N) +$ $+\ N\sum_{s=0}^{r-1}(m_{r-1-s}^{(U)}(N)\ m_s^{(U)}(N)\ - m_{r-s}^{(U)}(N)\ m_{s+1}^{(U)}(N))$
$Ls_N^{(\alpha)}$	$q_{r+1}^{(\alpha)}(N)\ = \frac{\alpha-r}{N}\ q_r^{(\alpha)}(N)\ + \sum_{s=0}^{r}\ q_{r-s}^{(\alpha)}(N)\ q_s^{(\alpha)}(N)$
Hs_N	$q_{r+2}^{(H)}(N)\ = -\frac{r+1}{2N}\ q_r^{(H)}(N)\ + \frac{1}{2}\sum_{s=0}^{r} q_{r-s}^{(H)}(N)\ q_s^{(H)}(N)$

Table 5: (Scaled) moment generating functions in the limit $\mathbf{N\to\infty}$

$\mathbf{P_N(x)}$	$\mathbf{G_0(z)}$, **resp.** $\mathbf{Q_0(z)}$	**limit moments**
$P_N^{(\alpha,\beta)}$	$G_0(z)\ = 1/\sqrt{1-z^2}$	$m_{0,2l}\ = \binom{2l}{l}/2^{2l}$
$Ls_N^{(\alpha)}$	$Q_0(z)\ = \frac{1}{2z}(1-\sqrt{1-4z})\ =: c(z)$	$q_{0,r}\ = C_r\ := \binom{2r}{r}/(r+1)$
Hs_N	$Q_0^{(H)}(z)\ = c(z^2/2)\ = \frac{1}{z^2}(1-\sqrt{1-2z^2})$	$q_{0,2l}^{(H)}\ = C_l/2^l$

It is clear from the explicit expressions for the generating functions $G(N,z)$ in both *Chebyshev* instances that a $1/N$ expansion is doomed to fail. Such an expansion produces for the T_N-polynomials the asymptotic moment generating function $G_0(z)$ given for the *Jacobi*-case in *table* 5. This happens to be a solution of the T_N-*Riccati*-eq. shown in *table 2* for all N. However, it is not the correct solution $G^{(T)}(N,z)$ given in *table 1*. This explains the mistake which occured in [3]. It is possible to recover $G^{(T)}(N,z)$ from the *special* solution $G_0(z)$ of the T_N-*Riccati*-equation. In the *Chebyshev* U_N case the sum of the first two $1/N$-expansion coefficients furnishes a *special* solution of the U_N-Riccati-eq. (see [7]), which is also not the correct solution $G^{(U)}(N,z)$ given in *table 1*.

4 Distribution of zeros and Perron-Stieltjes inversion

For polynomials with only real zeros the (ordinary) generating function $G(N,z)$ of the sums of powers of zeros (or moments) is related to the *Stieltjes*-transform $\chi(N,z)$ of the discrete real measure (4)in a simple way. The *Stieltjes*-transform of $\rho_N(x)$ is defined by

$$\chi(N,z) \;:=\; \int_{-\infty}^{+\infty} \frac{1}{z-x}\, d\rho_N(x) \quad , \quad z \notin supp(d\rho_N)\ . \tag{33}$$

Because (in the sense of formal power series) one has with the counting measure (4)

$$G(N,z) \;=\sum_{r=0}^{\infty} m_r(N)\ z^r \;=\frac{1}{N}\sum_{i=1}^{N}\frac{1}{1-z\ x_i^{(N)}} \;=\int_{-\infty}^{\infty}\frac{1}{1-z\ x}\, d\rho_N(x)\ , \tag{34}$$

one finds the simple relation

$$\chi(N,z) \;=\frac{1}{z}\ G(N,\frac{1}{z}) \quad . \tag{35}$$

It is possible to recover the discrete measure from $\chi(N,z)$, *i.e* $G(N,z)$, with the aid of the *Perron-Stieltjes* inversion formula [15],[1],[4],[2]

$$\rho_N(t_2) \;-\; \rho_N(t_1) \;=-\frac{1}{\pi}\lim_{\eta\to+0}\int_{t_1}^{t_2} dt\ Im\ \chi(N,t+i\eta) \quad , \tag{36}$$

with $\rho_N(t_k) \;:=\frac{1}{2}(\rho_N(t_k+0) \;+\rho_N(t_k-0))$ for $k=1,2$. This inversion is valid for real measures of bounded variation, provided $\bar{\chi}_N=\chi_N(\bar{z})$ for $z\in\mathbf{C}$, and χ_N is analytic off the real axis. As long as N is finite, $\chi_N(z)$ has poles only at the zeros of P_N. The inversion formula then yields for ρ_N jumps at these real zeros $x_i^{(N)}$ with height $m_i^{(N)}/N$ where m_i counts the multiplicity of the zero $x_i^{(N)}$. ((36) shows that poles in χ lead to jumps in ρ. The converse statement is obvious from (33).) In this way one recovers the purely discontinuous measure (4).

The inversion formula becomes especially interesting if the limit $\chi(z) \;:=\lim_{N\to\infty}\chi(N,z)$ exists. Then one can find the limit measure or distribution of zeros. In our examples of classical (scaled) orthogonal polynomials the limit moment generating functions exist (see *table 5*), and hence one can find by *Perron-Stieltjes* inversion of $\chi(z):=\frac{1}{z}\ G_0(\frac{1}{z})$ (resp. $\frac{1}{z}\ Q_0(\frac{1}{z})$) the distribution of zeros in the limit $N\to\infty$. These are listed in *table 6*. In all cases the conventional definition of the square-root in $\mathbf{C}$ has been used. These limit distributions (with other normalization) have been considered in [5].

Table 6: Limit distributions of zeros of (scaled) orthogonal polynomials

$\mathbf{P_N(x)}$	$\rho(x) \; = \; \lim_{N\to\infty} \rho_N(x)$
$P_N^{(\alpha,\beta)}$	$\rho^{(\alpha,\beta)}(x) \; = \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}}$, for $\|x\| \; < \; 1$ and 0 otherwise
$Ls_N^{(\alpha)}$	$\rho^{(Ls^{(\alpha)})}(x) \; = \frac{1}{2\pi} \sqrt{\frac{4}{x} - 1}$, for $0 < x \leq 4$, and 0 otherwise
Hs_N	$\rho^{(Hs)}(x) \; = \frac{1}{\pi} \sqrt{2 - x^2}$, for $\|x\| < \sqrt{2}$ and 0 otherwise

5 Conclusion

Formula (25) which gives the (ordinary) generating function $G(N, z)$ for the sums of (positive) powers of zeros of any one-variable polynomial (1) can be simplified whenever the (logarithmic) derivative rule for the polynomial under consideration is known. If the polynomial satisfies a (second) order differential equation (like in the assumption of the work [3]) one expects $G(N, z)$ to satisfy a *Riccati*-type differential equation. For polynomials with only real zeros (*e.g.* orthogonal polynomials with positive moment functionals) one can use *Perron-Stieltjes* inversion to find the distribution of zeros in the limit $N \to \infty$, provided $\lim_{N\to\infty} \; G(N, 1/z)/z$ exists.

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