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Chebyshev Polynomials and Certain Quadratic Diophantine Equations

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Abstract

Classes of binary quadratic diophantine equations (including the standard types of *Pell* equations) which follow from the well-known *Cassini-Simson*-identity for *Chebyshev*'s polynomials of the second kind are derived and their solutions are discussed.

1 Introduction and Summary

Chebyshev's $\{S_n(x) := U_n(x/2)\}$ polynomials of the second kind [10] are defined by the recurrence relation

$$(R;x,n): \qquad S_n(x) \;=\; x\,S_{n-1}(x) \;-\; S_{n-2}(x), \tag{1}$$

with $S_{-1}(x) = 0$ and $S_0(x) = 1$. Sometimes $S_{-2}(x) = -1$ is also used. They satisfy the so called *Cassini-Simson* identity

$$(C-S;x,n): \qquad S_{n-1}^2(x) - 1 \;=\; S_{n-2}(x)\,S_n(x)\ , \text{ for all } n \in \mathbb{N}_0,\ x \in \mathbb{R}, \tag{2}$$

The ordinary generating function (*o.g.f.*) is $S(x;y) := \sum_{n=0}^{\infty} S_n(x)\,y^n = 1/(1-x\,y+y^2)$. *Chebyshev*'s polynomials of the first kind will also appear. They satisfy $(R;x,n)$ with offset $T_0(x) = 1$ and $T_1(x) = x$, Sometimes $T_{-1}(x) = x$ is also used. $T_n(x/2) = (S_n(x) - S_{n-2}(x))/2 = S_n(x) - x\,S_{n-1}(x)/2$ with *o.g.f.* $T(x/2;y) = \sum_{n=0}^{\infty} T_n(x/2)\,y^n = (1 - y\,x/2)\,S(x;y)$.
The $(C-S;x,n)$ identity in combination with recurrence $(R;x,n)$ gives rise to the following two types of quadratic identities:
Type A: $\alpha \in \mathbb{R} \setminus \{-1,+1\}$, $\alpha^2 + x\,\alpha + 1 \neq 0,\ x \in \mathbb{R},\ n \in \mathbb{N}_0$

$$(A;\alpha;x,n): \qquad \alpha_n^2(\alpha,x) \;-\; (x^2 - 4)\,\beta_n^2(\alpha,x) \;=\; 4\,(1+\alpha\,(\alpha + x)), \tag{3}$$

with

$$\begin{aligned} \alpha_n(\alpha,x) \;&:=\; 2\,T_{n+1}\left(\frac{x}{2}\right) \;+\; \alpha\,2\,T_n\left(\frac{x}{2}\right), \\ &=\; (2\,\alpha + x)\,S_n(x) \;-\; (2 + \alpha\,x)\,S_{n-1}(x)\ , \end{aligned} \tag{4}$$

$$\beta_n(\alpha,x) \;:=\; S_n(x) \;+\; \alpha\,S_{n-1}(x)\ . \tag{5}$$

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Type B: $n \in \mathbb{N}_0,\ x \in \mathbb{R}$

$$(B;x,n): \qquad \frac{1}{2}\,(1\,+\,\frac{x}{2})\,\gamma_n^2(x)\;+\;\frac{1}{2}\,(1\,-\,\frac{x}{2})\,\delta_n^2(x)\;=\;1, \tag{6}$$

with

$$\begin{aligned}\gamma_n(x) &:= S_n(x)\,-\,S_{n-1}(x)\\ &= \frac{2}{\sqrt{2\,+\,x}}\,T_{2\,n+1}\left(\frac{\sqrt{2\,+\,x}}{2}\right)\;=\;(-1)^n\,S_{2n}(i\,\sqrt{x-2}\,)\;,\end{aligned} \tag{7}$$

$$\begin{aligned}\delta_n(x) &:= S_n(x)\,+\,S_{n-1}(x)\;=\;S_{2n}(\sqrt{2\,+\,x})\\ &= (-1)^n\,T_{2\,n+1}(i\,\sqrt{x-2}/2)/(i\,\sqrt{x-2}/2)\;.\end{aligned} \tag{8}$$

For $\alpha\,\to\,+1$, resp. $\alpha\,\to\,-1$, $(A;\alpha;x,n)$ reduces to $(B;x,n)$ provided $x\,\neq\,-2$, resp. $\neq\,+2$. From recurrence relation $(R;x,n)$ it is clear that $S_n(x)$ is integer for all $n\in\mathbb{N}_0$ iff $x\in\mathbb{Z}$ (in fact for $n\in\mathbb{Z}$ because $S_{-n}(x)\;=\;-\,S_{n-2}(x)$). For the diophantine analysis of the two types of quadratic identities we restrict ourselves to $x\in\mathbb{N}_0$. Type B leads to four classes of such diophantine equations, each with parameter $p\in\mathbb{N}_0$.

$$(B1;p,n): \qquad (p\,+\,1)\,\gamma_n^2(2\,(2\,p\,+\,1))\;-\;p\,\delta_n^2(2\,(2\,p\,+\,1))\;=\;1\;, \tag{9}$$

$$(B2;p,n): \qquad (2\,p\,+\,1)\,\gamma_n^2(4\,p)\;-\;(2\,p\,-\,1)\,\delta_n^2(4\,p)\;=\;2\;, \tag{10}$$

$$(B3;p,n): \qquad (4\,p\,+\,3)\,\gamma_n^2(4\,p\,+\,1)\;-\;(4\,p\,-\,1)\,\delta_n^2(4\,p\,+\,1)\;=\;4\;, \tag{11}$$

$$(B4;p,n): \qquad (4\,p\,+\,5)\,\gamma_n^2(4\,p\,+\,3)\;-\;(4\,p\,+\,1)\,\delta_n^2(4\,p\,+\,3)\;=\;4\;. \tag{12}$$

In each case one could write a companion identity by using negative arguments, but due to $S_n(-x)\;=\;(-1)^n\,S_n(x)$ this would correspond to the identity with γ_n and δ_n interchanged.

For the diophantine analysis of *type A* we use $x\in\mathbb{N}_0$ and put $\alpha\;=\;p/q$ with $p\in\mathbb{Z}$ and $q\in\mathbb{N}$ ($q\neq 0$). $p\;=\;\pm q$ will also be admitted because for $\alpha\to\pm 1$ the *type B* identity is recovered.

$$(A;p,q;x,n), \qquad x\in\mathbb{N}_0,\ p\in\mathbb{Z},\ q\in\mathbb{N},\ p^2\,+\,q^2\,+\,p\,q\,x\neq 0\;:$$

$$\tilde{\alpha}_n^2(p,q;x)\;-\;(x^2\,-\,4)\,\tilde{\beta}_n^2(p,q;x)\;=\;4\,(p^2\,+\,q^2\,+\,x\,p\,q)\;, \tag{13}$$

with

$$\begin{aligned}\tilde{\alpha}_n(p,q;x) &:= q\,2\,T_{n+1}\left(\frac{x}{2}\right)\;+\;p\,2\,T_n\left(\frac{x}{2}\right)\\ &= (2\,p\,+\,q\,x)\,S_n(x)\;-\;(2\,q\,+\,p\,x)\,S_{n-1}(x)\;,\end{aligned} \tag{14}$$

$$\tilde{\beta}_n(p,q;x) := q\,S_n(x)\;+\;p\,S_{n-1}(x)\;. \tag{15}$$

This can be split into the even and odd x case.

$$(A1;p,q;k,n), \qquad p\in\mathbb{Z},\ q\in\mathbb{N},\ p^2\,+\,q^2\,+\,2\,p\,q\,k\neq 0, k\in\mathbb{N}_0,\ n\in\mathbb{N}_0\;:$$

$$\hat{\alpha}_n^2(p,q;k)\;-\;(k^2\,-\,1)\,\hat{\beta}_n^2(p,q;k)\;=\;(p^2\,+\,q^2\,+\,2\,k\,p\,q)\;, \tag{16}$$

with

$$\begin{aligned}\hat{\alpha}_n(p,q;k) &:= \tilde{\alpha}_n(p,q;2\,k)/2\;=\;q\,T_{n+1}(k)\;+\;p\,T_n(k)\\ &= (p\,+\,q\,k)\,S_n(2\,k)\;-\;(q\,+\,p\,k)\,S_{n-1}(2\,k)\;,\end{aligned} \tag{17}$$

$$\hat{\beta}_n(p,q;k) := \tilde{\beta}_n(p,q;2\,k)\;=\;q\,S_n(2\,k)\;+\;p\,S_{n-1}(2\,k)\;, \tag{18}$$

and

$$
\begin{aligned}
&(A2;p,q;k,n), && p\in\mathbb{Z},\ q\in\mathbb{N},\ p^2\,+\,q^2\,+\,(2\,k\,+\,1)\,p\,q\neq 0,\ k\in\mathbb{N}_0,\ n\in\mathbb{N}_0\ : \\
&\bar{\alpha}_n^2(p,q;k) \ - && (2\,k\,+\,3)\,(2\,k\,-\,1)\,\bar{\beta}_n^2(p,q;k) \ = \ 4\,(p^2\,+\,q^2\,+\,(2\,k\,+\,1)\,p\,q)\ ,
\end{aligned} \tag{19}
$$

with

$$
\begin{aligned}
\bar{\alpha}_n(p,q;k) \ &:= \ \tilde{\alpha}_n(p,q;2\,k\,+\,1) \ = \ q\,2\,T_{n+1}\left(\frac{2\,k\,+\,1}{2}\right) \ + \ p\,2\,T_n\left(\frac{2\,k\,+\,1}{2}\right) \\
&= \ (2\,p\,+\,(2\,k\,+\,1)\,q)\,S_n(2\,k\,+\,1) \ - \ (2\,q\,+\,(2\,k\,+\,1)\,p)\,S_{n-1}(2\,k\,+\,1),
\end{aligned} \tag{20}
$$

$$
\bar{\beta}_n(p,q;k) \ := \ \tilde{\beta}_n(p,q;2\,k\,+\,1) \ = \ q\,S_n(2\,k\,+\,1) \ + \ p\,S_{n-1}(2\,k\,+\,1)\ . \tag{21}
$$

Some special cases

a) +1 Pell equation:

$$
x^2 \ - \ (k^2\,-\,1)\,y^2 \ = \ +1, \quad k \ = \ 2,3,\ldots \tag{22}
$$

$(A1;0,q;k,n) \ \equiv \ (A1;k,n)\ ,\ n\in\mathbb{N}_0$:

$$
\begin{aligned}
x \ \equiv \ x_n(k) \ &= \ \hat{\alpha}_n(0,q;k)/q \ = \ T_{n+1}(k), \\
&= \ k\,S_n(2\,k) \ - \ S_{n-1}(2\,k),
\end{aligned} \tag{23}
$$

$$
y \ \equiv \ y_n(k) \ = \ \hat{\beta}_n(0,q;k)/q \ = \ S_n(2\,k)\ . \tag{24}
$$

Note 1: The same eq. is obtained from type $(B1;p=k^2\,-\,1,n)$ but only the solutions $T_{2\,n+1}(k)$ and $S_{2n}(2\,k)$ are found this way.

Note 2: This gives the general solution (in the natural numbers) of *Pell* equation $x^2\,-\,D\,y^2\,=\,+1$, for $D\,=\,k^2\,-\,1$ (*cf.*[9], §27, p.92 ff. See also [7], p. 354, ch.7.8, Th. 7.26, with $d=k^2\,-\,1$ and minimal solution $(x_+,y_+)\,=\,(k,1)$ rewritten in terms of *Chebyshev* polynomials.)

Note 3: This is the generic form of *Pell* equation $x^2\,-\,D\,y^2\,=\,+1$ with $D\in\mathbb{N}$, not a square. If $D\neq k^2\,-\,1$ then $y_+\,>\,1$, and with the definition $\tilde{D}\,:=\,y_+^2\,D\,=\,x_+^2\,-\,1$ and $\tilde{y}\,:=\,y/y_+$ one has to solve $x^2\,-\,\tilde{D}\,\tilde{y}^2\,=\,+1$ which is of the generic type with minimal solution $(x_+,\tilde{y}_+)\,=\,(x_+,1)$. Hence $x_n\,=\,T_{n+1}(x_+)$ and $\tilde{y}_n\,=\,S_n(2\,x_+)$, *i.e.* $y_n\,=\,y_+\,S_n(2\,x_+)$, are the general solutions. *E.g.* if $D=k^2+1$ then $(x_+,y_+)\,=\,(2\,k^2\,+1,2\,k)$ with general solution $x_n\,=\,T_{n+1}(2\,k^2\,+\,1)$ and $y_n\,=\,2\,k\,S_n(2\,(2\,k^2\,+\,1))$.

b) −1 Pell equation:

$$
x^2 \ - \ (k^2\,+\,1)\,y^2 \ = \ -1, \quad k\in\mathbb{N}_0\ . \tag{25}
$$

$(B1;p=k^2;n),\quad k\in\mathbb{N}_0,\ n\in\mathbb{N}_0$:

$$
\begin{aligned}
x \ \equiv \ x_n(k) \ &= \ k\,\delta_n(2\,(2\,k^2\,+\,1)) \ = \ k\,S_{2n}(2\,\sqrt{k^2\,+\,1}) \\
&= \ k\,\left[S_n(2\,(2\,k^2\,+\,1)) \,+\, S_{n-1}(2\,(2\,k^2\,+\,1))\right] \\
&= \ (-1)^n\,T_{2\,n+1}(i\,k)/i\ ,
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
y \ \equiv \ y_n(k) \ &= \ \gamma_n(2\,(2\,k^2\,+\,1)) \ = \ \frac{1}{\sqrt{k^2\,+\,1}}\,T_{2n+1}(\sqrt{k^2\,+\,1}\,) \\
&= \ S_n(2\,(2\,k^2\,+\,1)) \ - \ S_{n-1}(2\,(2\,k^2\,+\,1)), \\
&= \ (-1)^n\,S_{2n}(i\,2\,k)\ .
\end{aligned} \tag{27}
$$

Note 4: This gives the general solution of *Pell* equation $x^2\,-\,D\,y^2\,=\,-1$, for $D\,=\,k^2\,+\,1$ satisfying the solvability criterion, namely that the regular continued fraction for $\sqrt{D}$ has odd (primitive) period length. (*cf.*[9], §30, p.109 . For the general solution see also [7], ch.7.8, problem *1, p. 356, with $d = k^2+1$ and minimal solution $(x_-, y_-) = (k, 1)$ rewritten in terms of *Chebyshev* polynomials.)

Note 5: This is the generic form of *Pell* equation $x^2 - D\,y^2 = -1$ with $D \in \mathbb{N}$, not a square, satsfying the solvability criterion mentioned in *Note 4*. This is because for $D = k^2 + 1$ the minimal solution is $(x_-, y_-) = (k, 1)$. If $D \neq k^2 + 1$ satisfies the solvability criterion, then $y_- > 1$, and with the definition $\tilde{D} := y_-^2\,D = x_-^2 + 1$ and $\tilde{y} := y/y_-$ one has to solve $x^2 - \tilde{D}\,\tilde{y}^2 = -1$ which is of the generic type with minimal solution $(x_-, \tilde{y}_-) = (x_-, 1)$. Hence the general solution is $x_n = x_-\,S_{2\,n}(\sqrt{x_-^2 + 1})$ and $y_n = y_-\,T_{2\,n+1}(\sqrt{x_-^2 + 1})/\sqrt{x_-^2 + 1}$ with $n \in \mathbb{N}_0$. *E.g.* $D = 13$ with$(x_-, y_-) = (18, 5)$ and the general solution $x_n = 18\,S_{2\,n}(10\sqrt{13})$ and $y_n = 5\,T_{2\,n+1}(5\sqrt{13})/(5\sqrt{13})$. The minimal solution of *Pell* equation $x^2 - (k^2 + 1)\,y^2 = +1$ is $(x_+, y_+) = (2\,k^2 + 1, 2\,k)$ (see [9], pp.94-95, and *Note 3*).

Note 6: For every D which satisfies the solvability criterion mentioned in *Note 4* the general (positive integer) solution of *Pell* equation $x^2 - D\,y^2 = -1$ and $x^2 - D\,y^2 = +1$ can be combined into the companion sequences [2]

$$x_n(x_-) = (-i)^{n+1}\,T_{n+1}(x_-\,i) \quad , \quad y_n(x_-, y_-) = y_-\,(-i)^n\,S_n(2\,x_-\,i)$$

with the minimal solution (x_-, y_-), where $y_- \geq 1$, of $x^2 - D\,y^2 = -1$. Then $\{x_{2\,n}, y_{2\,n}\}_{n=0}^{\infty}$, *resp.* $\{x_{2\,n+1}, y_{2\,n+1}\}_{n=0}^{\infty}$, provides the general solution for the -1 , *resp.* $+1$, Pell equation. The generating function for the $\{x_n\}$, *resp.* $\{y_n\}$, is $(x_- + x)/(1 - 2\,x_-\,x - x^2) = (x_- + x)\,S(2\,x_-\,i; -i\,x)$, *resp.* $y_-\,S(2\,x_-\,i; -i\,x)$, with the generating function $S(x; y)$ of *Chebyshev*'s $S_n(x)$ polynomials.

c) +4 Pell equation:

$$x^2 - (2\,k+3)\,(2\,k - 1)\,y^2 = +4, \quad k \in \mathbb{N}\ . \tag{28}$$

$(A2; 0, q; k, n) \equiv (A2; k, n),\ n \in \mathbb{N}_0$:

$$\begin{aligned} x \equiv x_n(k) &= \tilde{\alpha}_n(0, q; k)/q = 2\,T_{n+1}\left(\frac{2\,k+1}{2}\right) = S_{n+1}(2\,k + 1) - S_{n-1}(2\,k + 1), \\ &= (2\,k + 1)\,S_n(2\,k + 1) - 2\,S_{n-1}(2\,k + 1), \end{aligned} \tag{29}$$

$$y \equiv y_n(k) = \tilde{\beta}_n(0, q; k)/q = S_n(2\,k + 1) = S_{2\,k+1}(\sqrt{2\,k + 3})/\sqrt{2\,k + 3}\ . \tag{30}$$

Note 7: This gives the general solution of *Pell* equation $x^2 - D\,y^2 = +4$, for $D = (2\,k+3)\,(2\,k - 1) = 8\left(\binom{k+1}{2} - 1\right) + 5 \equiv 5(mod\,8)$ (*cf.*[9], ch. 30, p.107 ff, reformulated for this type of diophantine equation with the minimal solution $(x_+, y_+) = (2\,k + 1, 1)$ and rewritten in terms of *Chebyshev's polynomials* .)

Note 8: This is the generic form of *Pell* equation $x^2 - D\,y^2 = +4$ with $D \in \mathbb{N}$, not a square. Observe that both, x and y have to be odd, hence $D \equiv 1(mod\,4)$. Otherwise there is no solution or the equation reduces to the $+1$ *Pell* case. It is the generic equation because for $D = (2\,k + 3)\,(2\,k - 1)$ the minimal solution is $(x_+, y_+) = (2\,k + 1, 1)$. If $D \neq (2\,k+3)\,(2\,k - 1)$ and $D \equiv 1(mod\,4)$ then $y_+ > 1$, and with the definition $\tilde{D} := y_+^2\,D = x_+^2 - 4 = (2\,k_+ + 1)^2 - 4 = (2\,k_+ + 3)\,(2\,k_+ - 1)$ and $\tilde{y} := y/y_+$ one has to solve $x^2 - \tilde{D}\,\tilde{y}^2 = +4$. This is of the generic type with minimal solution $(x_+, \tilde{y}_+) = (2\,k_+ + 1, 1)$. Hence the general solution is $x_n = 2\,T_{n+1}(x_+/2)$ and $\tilde{y}_n = S_n(x_+)$, *i.e.* $y_n = y_+\,S_n(x_+)$ with $n \in \mathbb{N}_0$. *E.g.* $D \equiv D(k) = 4\,k\,(k + 1) + 5$ with $(x_+, y_+) = (4\,k\,(k + 1) + 3, 2\,k + 1)$ and general solution $x_n = 2\,T_{n+1}((D(k) - 2)/2)$ and $y_n = (2\,k + 1)\,S_n(D(k) - 2)$.

Note 9: The same generic form of this *Pell* equation results from $(B4; p = k\,(k + 1) - 1, n)$ but here not all solutions are covered by $x \equiv \gamma_n(4\,k\,(k + 1) - 1)$ and $y \equiv \delta_n(4\,k\,(k + 1) - 1)$.

d) −4 Pell equation:

$$x^2 - (4\,k\,(k + 1) + 5)\,y^2 = -4, \quad k \in \mathbb{N}_0\ . \tag{31}$$

$(B4; p = k\,(k+1), n)$, $n \in \mathbb{N}_0$:

$$
\begin{aligned}
x \;\equiv\; x_n(k) \;&=\; (2\,k\,+\,1)\,\delta_n(4\,k\,(k+1)\,+\,3) \;=\; (2\,k\,+\,1)\,S_{2\,n}(\sqrt{4\,k\,(k+1)\,+\,5}\,) \\
&=\; (2\,k\,+\,1)\,[S_n(4\,k\,(k+1)\,+\,3)\;+\;S_{n-1}(4\,k\,(k+1)\,+\,3)] \\
&=\; -2\,i\,(-1)^n\,T_{2\,n+1}((2\,k\,+\,1)\,i/2)\ , \qquad (32)
\end{aligned}
$$

$$
\begin{aligned}
y \;\equiv\; y_n(k) \;&=\; \gamma_n(4\,k\,(k+1)\,+\,3) \;=\; T_{2\,n+1}(\sqrt{4\,k\,(k+1)\,+\,5}/2)/(\sqrt{4\,k\,(k+1)\,+\,5}/2) \\
&=\; S_n(4\,k\,(k+1)\,+\,3)\,-\,S_{n-1}(4\,k\,(k+1)\,+\,3)\ , \\
&=\; (-1)^n\,S_{2\,n}(i\,\sqrt{4\,k\,(k+1)\,+\,1}\,)\ . \qquad (33)
\end{aligned}
$$

Note 10: This gives the general solution of *Pell* equation $x^2 - D\,y^2 = -4$, for $D = 4\,k\,(k+1) + 5 = 5(mod\,8)$ (*cf.*[9], Bd. I, ch. 30, pp 107 ff, reformulated for this type of diophantine equation with the minimal solution $(x_-, y_-) = (2\,k\,+\,1, 1)$ and rewritten in terms of *Chebyshev* polynomials.)

Note 11: This is the generic form of *Pell* equation $x^2 - D\,y^2 = -4$ with $D \in \mathbb{N}$, not a square. Observe that both, x and y have to be odd, hence $D \equiv 1(mod\,4)$. Otherwise there is no solution or the equation reduces to the -1 *Pell* case. The solvability criterion for this -4 *Pell* equation is that the (regular) continued fraction of $(\sqrt{D}+1)/2$ has odd period length [9], Satz 3.35, p.109. It is the generic equation because if $D \neq 4\,k\,(k+1) + 5$ and $D \equiv 1(mod\,4)$ satisfies this solvability criterion then $y_- > 1$, and with the definition $\tilde{D} := y_-^2\,D = x_-^2 + 4 = (2\,k_- + 1)^2 + 4 = 4\,k_-\,(k_- + 1) + 5)$ and $\tilde{y} := y/y_-$ one has to solve $x^2 - \tilde{D}\,\tilde{y}^2 = -4$ which is of the generic type with minimal solution $(x_-, \tilde{y}_-) = (2\,k_- + 1, 1)$. Hence the general solution is $x_n = x_-\,S_{2\,n}(\sqrt{x_+^2\,+\,4}\,)$ and $y_n = y_-\,T_{2\,n+1}(\sqrt{x_-^2\,+\,4}\,/2)/(\sqrt{x_-^2\,+\,4}\,/2)$ with $n \in \mathbb{N}_0$. *E.g.* $D = 37$ with minimal solution $(x_-, y_-) = (12, 2)$ and general solution $x_n = 12\,S_{2\,n}(2\,\sqrt{37})$ and $y_n = 2\,T_{2\,n+1}(\sqrt{37})/\sqrt{37}$.

Note 12: For every D which satisfies the solvability criterion mentioned in *Note 10* the general (positive integer) solution of *Pell* equation $x^2 - D\,y^2 = -4$ and $x^2 - D\,y^2 = +4$ can be combined into the companion sequences

$$
x_n(x_-) \;=\; 2\,(-i)^{n+1}\,T_{n+1}(x_-\,i/2)\quad,\quad y_n(x_-, y_-) \;=\; y_-\,(-i)^n\,S_n(x_-\,i)
$$

with the minimal solution (x_-, y_-), where $y_- \geq 1$, of $x^2 - D\,y^2 = -4$. Then $\{x_{2\,n}, y_{2\,n}\}_{n=0}^{\infty}$, *resp.* $\{x_{2\,n+1}, y_{2\,n+1}\}_{n=0}^{\infty}$, provide the general solution for the -4 , *resp.* $+4$, Pell equation. The generating function for $\{x_n\}$, *resp.* $\{y_n\}$, is $(x_- + 2\,x)/(1 - x_-\,x - x^2) = (x_- + x)\,S(x_-\,i; -i\,x)$, *resp.* $y_-\,S(x_-\,i; -i\,x)$, with the generating function $S(x, y)$ of *Chebyshev*'s $S_n(x)$ polynomials.

2 Derivation of the results

Proof of the Cassini-Simson identity eq. 2

Recurrence eq. 1 is related to the transfer matrix $\mathbf{R}(x) := \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}$ with $Det\, R(x) = 1$ for all x. This is because any recurrence of the type $q_n = x\, q_{n-1} - q_{n-2}$ can be rewritten as $\begin{pmatrix} q_n \\ q_{n-1} \end{pmatrix} = \mathbf{R}(x) \begin{pmatrix} q_{n-1} \\ q_{n-2} \end{pmatrix}$ (whence the name transfer matrix). $\mathbf{R}_n(x) := \mathbf{R}^n(x)$ obeys the recurrence $\mathbf{R}_n(x) = \mathbf{R}(x)\, \mathbf{R}_{n-1}(x)$ with offset $\mathbf{R}_0(x) := \mathbf{1}$. Due to eq. 1 $\mathbf{R}_n(x) = \begin{pmatrix} S_n & -S_{n-1} \\ S_{n-1} & -S_{n-2} \end{pmatrix}$ satisfies this recurrence with the correct offset (with $S_{-2} = -1$). In particular, $Det\, \mathbf{R}_n(x) = (Det\, \mathbf{R}(x))^n = 1^n = 1$ which is the desired *Cassini-Simson* identity eq. 2.

If in this identity $S_{n-2}(x)$ is eliminated with the help of recurrence eq. 1 one finds the following corollary.

Corollary 1: Rewritten $(C - s; x, n)$

$$S_n^2(x) \;+\; S_{n-1}^2(x) \;-\; x\, S_{n-1}(x)\, S_n(x) \;=\; 1. \tag{34}$$

Proof of the type B and A identities

Rewrite eq. 34 as combination of two squares using parameters A, B, α, β as follows:

$$A\,(S_n(x) + \alpha\, S_{n-1}(x))^2 + B\,(S_n(x) + \beta\, S_{n-1}(x))^2 \;=\; S_n^2(x) \;+\; S_{n-1}^2(x) \;-\; x\, S_{n-1}(x)\, S_n(x) \;=\; 1\;. \tag{35}$$

Comparing coefficients of S_n^2, S_{n-1}^2 and $S_n\, S_{n-1}$ one has $A + B = 1$, $A\,\alpha^2 + B\,\beta^2 = 1$, and $A\,\alpha + B\,\beta = -x/2$. Two cases are distinguished depending on whether $\alpha^2 - \beta^2 \neq 0$ or $\alpha^2 - \beta^2 = 0$.

Case 1: $\alpha^2 - \beta^2 \neq 0$

$$A \;=\; \frac{1 - \beta^2}{\alpha^2 - \beta^2}\;,\;\; B \;=\; \frac{\alpha^2 - 1}{\alpha^2 - \beta^2}\;, \tag{36}$$

and $(\alpha^2 - \beta^2)\,\frac{x}{2} = (\alpha\,\beta + 1)\,(\beta - \alpha)$, which reduces in this case to the condition

$$\alpha\,\beta \,+\, 1 \,+\, (\alpha \,+\, \beta)\,\frac{x}{2}\;. \tag{37}$$

Case 2: $\alpha^2 - \beta^2 = 0$

Now $\alpha^2 = 1$, $B = 1 - A$, and if $\alpha = \beta = +1$, or -1, the *lhs.* of eq. 35 reduces to $S_n(x) + S_{n-1}(x) = \pm 1$ without any restriction on x. If $\alpha = +1$, $\beta = -1$, then $A = (1 - x/2)/2$, $B = (1 + x/2)/2$. The case $\alpha = -1$, $\beta = +1$ is not considered because it is obtained from the latter one after interchanging A with B. This *case 2* leads thus to the identity

$$\frac{1}{2}\,(1 \,+\, \frac{x}{2})\,(S_n(x) \,-\, S_{n-1}(x))^2 \;+\; \frac{1}{2}\,(1 \,-\, \frac{x}{2})\,(S_n(x) \,+\, S_{n-1}(x))^2 \;=\; 1\;. \tag{38}$$

which is the *type B* identity, eq. 6 with eqs. 7 and 8.

The first way to rewrite $\gamma_n(x) := S_n(x) - S_{n-1}(x)$, *resp.* $\delta_n(x) := S_n(x) + S_{n-1}(x)$, follows from bisecting the sequence $\{S_n(x)\}$ into $\{S_{2n}(x)\}$ and $\{S_{2n+1}(x)\}$, *resp.* $\{T_n(x)\}$ into $\{T_{2n}(x)\}$ and $\{T_{2n+1}(x)\}$. For example, the *o.g.f.* for $\{S_{2n}(x)\}_0^\infty$ is $(S(x;\sqrt{y}) + S(x;-\sqrt{y}))/2$, where $S(x;y) = 1/(1 - x\,y + y^2)$ is the *o.g.f.* for $\{S_n(x)\}_0^\infty$. This is $(1 + y)/(1 - (x^2 - 2)\,y + y^2) = (1 + y)\,S(x^2 - 2; y)$. Therefore, $S_{2n}(x) = S_n(x^2 - 2) + S_{n-1}(x^2 - 2)$, or

$$\delta_n(x) \;=\; S_{2n}(\sqrt{2 \,+\, x}\,)\;. \tag{39}$$

Similarly, the *o.g.f.* for $\{T_{2n+1}(x)\}_0^\infty$ is $\left(T(x/2;\sqrt{y}) \,-\, T(x/2;-\sqrt{y})\right)/(2\,\sqrt{y})$ with the *o.g.f.* $T(x/2;y)$ for $\{T_n(x)\}_0^\infty$. Thus, this *o.g.f.* is $(x/2)\,(1 - y)/(1 - (x^2 - 2)\,y + y^2)$, whence

$T_{2n+1}(x) \,=\, (x/2)\,(S_n(x^2 \,-\, 2) \,-\, S_{n-1}(x^2 \,-\, 2))$, or

$$\gamma_n(x) \;=\; \frac{2}{\sqrt{2 \,+\, x}}\, T_{2n+1}(\sqrt{2 \,+\, x}/2) \;. \tag{40}$$

The identities for $\delta_n(x)$ and $\gamma_n(x)$ involving pure imaginary arguments of *Chebyshev*'s polynomials follow from the *Binet-de Moivre* representation of these polynomials, *viz.*

$$T_n(\frac{x}{2})/\frac{x}{2} \;=\; \frac{\lambda_+^n(x) \,+\, \lambda_-^n(x)}{\lambda_+(x) \,+\, \lambda_-(x)} \;\;\text{with}\;\; \lambda_\pm(x) \;=\; \frac{1}{2}\,(x \,\pm\, \sqrt{x^2 \,-\, 4}\,)\;, \tag{41}$$

$$S_n(x) \;=\; \frac{\lambda_+^{n+1}(x) \,-\, \lambda_-^{n+1}(x)}{\lambda_+(x) \,-\, \lambda_-(x)} \;. \tag{42}$$

The third eq. in $\gamma_n(x)$, *resp.* $\delta_n(x)$, from eqs. 7, *resp.* 8, then results from $\lambda_\pm(\sqrt{2 \,+\, x}\,) \,=\, \mp i\,\lambda_\pm(i\,\sqrt{x \,-\, 2}\,)$, with correlated signs.
In *case 1* ($\alpha^2 \,-\, \beta^2 \neq 0$) the subsidiary condition eq. 37 can be solved for $\beta \,=\, \beta(\alpha, x) \,=\, -(1 \,+\, \alpha\, x/2)/(\alpha \,+\, x/2)$. $\alpha \,+\, x/2 \neq 0$ because otherwise $x \,=\, \pm 2$, hence $\alpha \,=\, \mp 1$, which leads to the uninteresting result $(S_n(x) \,\mp\, S_{n-1}(x))^2 \,=\, 1$ due to eq. 34. Conversely, if $\alpha \,=\, +1$ then the subsidiary condition becomes $(1 \,+\, \beta)\,(1 \,+\, x/2)$ because $\beta \neq 1$, $x \,=\, -2$ in *case 1.* Similarly $\alpha \,=\, -1$ implies $x \,=\, +2$. With $\alpha^2 \,-\, \beta^2(\alpha, x) \,=\, (\alpha^2 \,-\, 1)\,(1 \,+\, \alpha^2 \,+\, \alpha\, x)/(\alpha \,+\, x/2)^2$, which implies $\alpha^2 \,-\, 1 \neq 0$ as well as $1 \,+\, \alpha\,(\alpha \,+\, x) \neq 0$, one finds

$$A \;=\; A(\alpha, x) \;=\; \frac{1 \,-\, \frac{x^2}{4}}{1 \,+\, \alpha\,(\alpha \,+\, x)} \;,\quad B \;=\; B(\alpha, x) \;=\; \frac{(\alpha \,+\, \frac{x}{2})^2}{1 \,+\, \alpha\,(\alpha \,+\, x)} \;. \tag{43}$$

In a first step the identity in α, x and n ($\alpha \neq \pm 1,\; 1 \,+\, \alpha^2 \,+\, \alpha\, x \,\neq\, 0$)

$$(1 \,-\, \frac{x^2}{4})\,\beta_n^2(\alpha, x) \;+\; (\alpha \,+\, \frac{x^2}{4})^2\, a_n^2(\alpha, x) \;=\; 1 \,+\, \alpha\,(\alpha \,+\, x) \tag{44}$$

ensues with

$$\beta_n(\alpha, x) \;:=\; S_n(x) \,+\, \alpha\, S_{n-1}(x) \;, \tag{45}$$

$$a_n(\alpha, x) \;:=\; S_n(x) \,-\, \frac{1 \,+\, \alpha\, x/2}{\alpha \,+\, x/2}\, S_{n-1}(x) \;. \tag{46}$$

With the redefinition $\alpha_n(\alpha, x) \,:=\, (2\,\alpha \,+\, x)\, a_n(\alpha, x) \;=\; (2\,\alpha \,+\, x)\, S_n(x) \;-\; (2 \,+\, \alpha\, x)\, S_{n-1}(x)$ this coincides with the desired *type A* identity, eqs. 3 - 5. $\alpha \,+\, x/2 \neq 0$ has been assumed.
With the help of the S_n recurrence relation eq. 1, $\alpha_n(\alpha, x)$ can be rewritten as

$$\begin{aligned}
\alpha_n(\alpha, x) \;&=\; \alpha\,(2\, S_n(x) \,-\, x\, S_{n-1}(x)) \;+\; (-2\, S_{n-1}(x) \,+\, x\, S_n(x)) \\
&=\; \alpha\,(S_n(x) \,-\, S_{n-2}(x)) \;+\; (S_{n+1}(x) - S_{n-1}(x)) \\
&=\; 2\,\left(\alpha\, T_n(\frac{x}{2}) \;+\; T_{n+1}(\frac{x}{2})\right) \;.
\end{aligned} \tag{47}$$

This proves eq. 4.

Note 13: If one sends $\alpha \to \pm 1$ in the *type A* identity eqs. 3 - 5 then $(A; \alpha \to 1; x, n) \equiv (B; x, n)$ with $\beta_n(1, x) \,=\, \delta_n(x)$ and $\alpha_n(1, x) \,=\, (2 \,+\, x)\,\gamma_n(x)$ for $x \neq -2$, and $(A; \alpha \to -1; x, n) \equiv (B; x, n)$ with $\beta_n(-1, x) \,=\, \gamma_n(x)$ and $\alpha_n(-1, x) \,=\, (x \,-\, 2)\,\delta_n(x)$ for $x \neq +2$.

Diophantine analysis

For the diophantine properties of the *type A* or *type B* identities we restrict ourselves to integer values of x. Recurrence eq. 1 then guarantees that $S_n(x)$ is integer for all $n \in \mathbb{Z}$. It is sufficient to consider $n \in \mathbb{N}_0$ because $S_{-n}(x) = S_{n-2}(x)$ which implies $\gamma_{-n}(x) = +\gamma_{n-1}(x)$, $\delta_{-n}(x) = -\delta_{n-1}(x)$ and $\alpha_{-n}(\alpha, x) = \alpha\,\alpha_{n-1}(1/\alpha, x)$, $\beta_{-n}(\alpha, x) = -\alpha\,\beta_{n-1}(1/\alpha, x)$ if $\alpha \neq 0$, and $\alpha_{-n}(0, x) = +\alpha_{n-2}(0, x)$, $\beta_{-n}(0, x) = -\beta_{n-2}(0, x)$. Therefore, negative n corresponds to a relabelling $n \to n-1$ in *type B* and $n \to n-2$ when $\alpha = 0$. If $\alpha \neq 0$ in *type A* then negative n is covered by changing $n \to n-1$ and $\alpha \to 1/\alpha$.
It is also sufficient to consider only non-negative x because due to recurrence eq. 1 $S_n(-x) = (-1)^n\, S_n(x)$ which implies $\gamma_n(-x) = (-1)^n\, \delta_n(x)$, $\delta_n(-x) = (-1)^n\, \gamma_n(x)$, and $\alpha_n(\alpha, -x) = (-1)^{n+1}\, \alpha_n(-\alpha, x)$, $\beta_n(\alpha, -x) = (-1)^n\, \beta_n(-\alpha, x)$. Therefore, negative x is equivalent to a change of $\alpha \to -\alpha$ in *type A* identities.
This explains why we consider in the following only $n \in \mathbb{N}_0$ and $x = l \in \mathbb{N}_0$.

Type B: eqs. 9 to 12

Depending on the congruence class of $x = l \in \mathbb{N}_0$ modulo 4, eq. 6 turns into the eqs. 9 to 10. For example, if $x = l \equiv 2\,(mod 4)$, *i.e.* $l = 2\,(2\,p + 1)$, $p \in \mathbb{N}_0$, eq. 6 turns into eq. 9 . In each case $p \in \mathbb{N}_0$ and in eqs. 7 and 8 the x values have to be chosen according to their congruence class.

Type A: eqs. 13 to 15

In this case we take a rational parameter $\alpha = p/q$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ ($q \neq 0$), and $p \neq \pm q$. For the limit $p \to \pm q$, in which *type B* is reached, see *Note 13.* The change $\alpha \to 1/\alpha$, which is needed because of our restriction to $n \in \mathbb{N}$, is accomplished by the interchange $p \leftrightarrow -q$, and the sign change $\alpha \to -\alpha$, needed because of the $x \geq 0$ restriction, corresponds to the sign change $p \to -p$.
For $\alpha = p/q$ and $x \in \mathbb{N}_0$ eqs. 3 to 5 become eqs. 13 to 15 with the new quantities $\tilde{\alpha}(p, q; x) := q\,\alpha_n(p/q, x)$ *resp.* $\tilde{\beta}(p, q; x) := q\,\beta_n(p/q, x)$. These are just eqs. 14 *resp.* 15

Note 14: From the derivation it is not clear that in every case *all* solutions are covered. See, for example, *Note 1.* In the four *Pell* cases it turned out that in fact all solutions were found. For other cases one should compare the given *Chebyshev* solutions with the ones obtained from all fundamental solutions. The general theory of (indefinite) binary quadratic forms is given in [1]. An English translation of *Gauss'* classical work on this subject is [3]. Chapter 6 of [4] covers this topic as well. Another systematic treatment of these forms can be found in [12]. In *Mathematica 5* [6], section 3.4.9, one finds an explanation of the *Reduce* command which helps to find all fundamental solutions of diophantine binary quadratic eqs. The reader may also try the author's *Maple 9* [5] program [8], based on the reduction of indefinite forms found in [12], to find all fundamental solutions together with the first few derived ones. See also the web page *Diophantine Equation–2nd Powers* [14] where more refs. can be found. See also [11] for *Pell*-equations which represent N.

Note 15: Many instances of sequence pairs $\{x_n, y_n\}$ treated in this paper can be found in [13]. See the four *tables* with some $A - numbers$ for the generic *Pell* equations considered above.

Note 16: It is clear that other orthogonal polynomial systems, which also satify a *Cassini-Simson* type identity, will produce diophantine solutions. However, as was demonstrated above, *Chebyshev*'s polynomials provide already all solutions to the standard *Pell* eqs.

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References

[1] D. A. Buell, *Binary Quadratic Forms*, Springer, New York, Berlin, Heidelberg, Tokyo, 1986

[2] E. I. Emerson, *Recurrent sequences in the equation* $DQ^2 = R^2 + N$, Fib. Quart. $\underline{7}$ (1969) 231-242 (actually $|N| = 1$ in this paper)

[3] C. F. Gauss, *Disquisitiones Arithmeticae*, transl. A. A. Clarke, rev. W. C. Waterhouse, Springer, New York, Berlin, Heidelberg, Tokyo, 1986

[4] A. Hurwitz and N. Kritikos, *Lectures on Number Theory*, Springer, New York, Berlin, Heidelberg, Tokyo, 1986

[5] http://www.maplesoft.com/

[6] http://www.wolfram.com/products/mathematica/index.html

[7] I. Niven, H. S. Zuckerman and H. L. Montgomery, *An Introduction to the Theory of Numbers*. 2nd ed., Wiley, New York, 1966; 5th ed., Wiley, New York, 1991

[8] W. Lang, http//:www.-itp.physik.uni-karlsruhe.de/˜wl/BinQuadForm.html

[9] O. Perron, *Die Lehre von den Kettenbrüchen*, Bd.I, Teubner, 1954, 1957

[10] Th. J. Rivlin, *Chebyshev polynomials: from approximation theory to algebra and number theory*, 2. ed., Wiley, New York, 1990

[11] John Robertson, *Solving the generalized Pell equation*, http://hometown.aol.com/jpr2718/pell.pdf

[12] A. Scholz und B. Schoeneberg, *Einführung in die Zahlentheorie*, Sammlung Göschen, Band 5131, de Gruyter, Berlin, New York, 1973

[13] N.J.A. Sloane's On-Line Encyclopedia of Integer Sequences, http//:www.research.att.com/˜njas/sequences/index.html

[14] http://mathworld.wolfram.com/DiophantineEquation2ndPowers.html

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TAB. 1: +1 Pell Equations for generic D: $\mathbf{x^2 - D\,y^2 = +1}$

$\mathbf{D = k^2 - 1}$, from OEIS A005563, $[\mathbf{3, 8, 15, 24, 35, 48, 63, 80, 99, 120, 143, 168, 195, 224}]$

D	A-number of $x-$sequence	A-number of $y-$sequence
3	A001075	A001353
8	A001541	A001109
15	A001091	A001090
24	A001079	A004189
35	A023038	A097308
48	A011943	A057655
63	A001081	A077412
80	A023039	A049660
99	A001085	A075843
120	A077422	A077421
143	A077424	A077423
168	A097308	A0973089
195	A097310	A097311
224	A097312	A097313

TAB. 2: -1 Pell Equations for generic D: $\mathbf{x^2 - D\,y^2 = -1}$

$\mathbf{D = k^2 + 1}$, from OEIS A002522, $[\mathbf{2, 5, 10, 17, 26, 37, 50^*, 65, 82, 101, 122, 145, 170, 197}]$

D	A-number of $x-$sequence	A-number of $y-$sequence
2	A002315	A001653
5	A075796	A007805
10	A097314	A097315
17	A097723	A097724
26	A097726	A0977277
37	A097729	A097730
50*	A097732	A097733
65	A097735	A097736
82	A097738	A097738
101	A097741	A097742
122	A097766	A097767
145	A097769	A097770
170	A097772	A097773
197	A097775	A097776

* not square-free.

TAB. 3: +4 Pell Equations for generic D: $\mathbf{x^2 - D\,y^2 = +4}$

$\mathbf{D = (2\,k + 3)\,(2\,k - 1)}$, from OEIS A078371,
$[\mathbf{5, 21, 45^*, 77, 117^*, 165, 221, 285, 357, 437, 525^*, 621^*, 725^*, 837^*}]$

D	A-number of $x-$sequence	A-number of $y-$sequence
5	A005248	A001906
21	A003501	A004254
45*	A056854	A004187
77	A056918	A018913
117*	A057076	A004190
165	A078363	A078362
221	A078365	A078364
285	A078367	A078366
357	A078369	A078368
437	A097777	A092499
525*	A09779	A09778
621*	A090733	A097780
725*	A090248	A09778
837*	A090251	A097782

* not square-free.

TAB. 4: -4 Pell Equations for generic D: $\mathbf{x^2 - D\,y^2 = -4}$

$\mathbf{D = (4\,k\,(k+1) + 5)}$, from OEIS A078370,
$[\mathbf{5, 13, 29, 53, 85, 125^*, 173, 229, 293, 365, 445, 533, 629, 733}]$

D	A-number of $x-$sequence	A-number of $y-$sequence
5	*A002878*	A001519
13	$3 \cdot$ *A097783*	A078922
29	$5 \cdot$ *A097834*	A097835
53	$7 \cdot$ *A097837*	A097838
85	$9 \cdot$ *A097840*	A097841
125*	$11 \cdot$ *A097842*	A097843
173	$13 \cdot$ *A097845*	A098244
229	$15 \cdot$ *A098246*	A098247
293	$17 \cdot$ *A098249*	A098250
365	$19 \cdot$ *A098252*	A098253
445	$21 \cdot$ *A098255*	*A098256
533	$23 \cdot$ *A098258*	A098259
629	$25 \cdot$ *A098261*	A098262
733	$27 \cdot$ *A098291*	A098292

* not square-free.