1' mag 2023

INTRODUCTION TO INTERSECTION

1. GENERAL CONSIDERATIONS

- INTERSECTION THEORY IS A BRANCH OF ALGEBRAIC GEOMETRY WHICH STUDIES THE GENERALISATION AND APPLICATIONS OF THE CONCEPT OF INTERSECTION BETWEEN TWO CURVES OR MANIFOLDS

- THE APPLICATION OF INTERSECTION THEORY TO PARTICLE PHYSICS, AND IN PARTICULAR TO THE FEYNMAN INTEGRALS REDUCTION PROBLEM, HAS PRODUCED A NEW LINE OF RESEARCH, BOTH IN FORMAL AND ALGORITHMICAL FIELDS, AND IS ROOTED ON THE POSSIBILITY TO DEFINE A "SCALAR PRODUCT" AMONG ELEMENTS OF THE SET OF FEYNMAN INTEGRALS, EXPLOITING DIRECTLY ITS STRUCTURE OF VECTOR FIELD.

2. GAUSSIAN INTEGRALS VIA IBPS

- WE CONSIDER THE SET OF INTEGRALS

$$I_n(\alpha) = \int_{-\infty}^{+\infty} e^{-\alpha x^2} x^n \, dx$$

$\alpha > 0$, $n \in \mathbb{N}$.

- IT IS NOT NECESSARY TO EVALUATE EACH ONE OF THEM. WE KNOW THAT

$$\int_{-\infty}^{+\infty} \partial_x \left[e^{-\alpha x^2} x^n \right] dx = \begin{cases} \left[e^{-\alpha x^2} x^n \right]_{-\infty}^{+\infty} = 0 - 0 = 0 \\ n\, I_{n-1}(\alpha) - 2\alpha\, I_{n+1}(\alpha) \end{cases} \longrightarrow \underline{I_{n+1}(\alpha) = \frac{n}{2\alpha} I_{n-1}(\alpha)}$$

THIS CLASS OF RELATIONS IS KNOWN AS INTEGRATION-BY-PART IDENTITIES (IBPS).

- USING IBPS WE CAN ESTABLISH A CHAIN OF LINEAR RELATIONS AMONG n-EVEN AND n-ODD INTEGRALS, MAPPING BACK EVERY INTEGRAL TO

$$I_0(\alpha) = \int_{-\infty}^{+\infty} e^{-\alpha x^2} dx \quad \text{AND} \quad I_1(\alpha) = \int_{-\infty}^{+\infty} e^{-\alpha x^2} x \, dx$$

THESE TWO INTEGRALS ARE CALLED MASTER INTEGRALS FOR THE CLASS OF INTEGRALS UNDER STUDY.

- NOTICE THAT THERE IS FREEDOM IN CHOOSING THE MASTER INTEGRALS (MIS), AS LONG AS THEY ARE INDEPENDENT UNDER IBPS.
- THANKS TO THE LINEARITY OF INTEGRATION, THE SET OF INTEGRALS IS NATURALLY EQUIPPED WITH A VECTOR SPACE STRUCTURE. THE IMPLEMENTATION OF IBPS ENRICHES SUCH STRUCTURE WITH FURTHER RELATIONS.

- MIS CAN BE EVALUATED BY SOLVING DIFFERENTIAL EQUATIONS (DES) OBTAINED BY DIFFERENTIATING THEM W.R.T. UNINTEGRATED VARIABLES AND THEN APPLYING IBPS TO CONSTRUCT A CLOSED SYSTEM OF (PARTIAL) DIFFERENTIAL EQUATIONS

$$\frac{\partial}{\partial\alpha} I_n(\alpha) = -I_{n+2}(\alpha) = -\frac{n+1}{2\alpha} I_n(\alpha) \Rightarrow \frac{\partial}{\partial\alpha}\begin{pmatrix} I_0(\alpha) \\ I_1(\alpha)\end{pmatrix} = \begin{pmatrix} -1/(2\alpha) & 0 \\ 0 & -1/\alpha \end{pmatrix}\begin{pmatrix} I_0(\alpha) \\ I_1(\alpha)\end{pmatrix}$$

$$\begin{pmatrix} I_0(\alpha) \\ I_1(\alpha)\end{pmatrix} = \begin{pmatrix} C_0/\sqrt{\alpha} \\ C_1/\alpha \end{pmatrix}$$

- THE INTEGRATION CONSTANTS CAN BE DETERMINED EITHER BY IMPOSING REGULARITY CONDITIONS (FIXING CONSTANTS SUCH THAT NON-PHYSICAL SINGULARITIES VANISH) OR BY MATCHING THE SOLUTION AT SPECIFIC POINTS TO BOUNDARY FUNCTIONS

$I_1(\alpha)$ IS AN ODD INTEGRAND ON AN EVEN DOMAIN, SO $\forall \alpha \in \mathbb{R}$ $C_1 = 0$ (BY REMOVING SINGULARITIES IN α, OR BY MATCHING $I_1(\alpha=0)=0 \longrightarrow C_1 = 0$)

$I_0(\alpha)$ CAN BE FIXED BY SAYING $I_0(0) = \int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi} \longrightarrow C_0 = \sqrt{\pi}$

$$\begin{pmatrix} I_0(\alpha) \\ I_1(\alpha)\end{pmatrix} = \begin{pmatrix} \sqrt{\pi/\alpha} \\ 0 \end{pmatrix}$$

- IBPS PROVIDE LADDERS OF RELATIONS TO REDUCE INTEGRALS TO MIS. A HUGE BOTTLENECK OF SUCH PROCEDURE IS THE GROOMING OF SUCH RELATIONS (WHOSE NUMBER GROWS DRAMATICALLY FAST WITH THE NUMBER OF PARAMETERS n) IN ORDER TO FIND A PATH FROM ONE INTEGRAL TO ITS EXPRESSION IN TERMS OF A LINEAR COMBINATION OF MIS.
- A FEATURE OF VECTOR SPACES IS THE POSSIBILITY TO DEFINE A SCALAR PRODUCT, WHICH IMMEDIATELY EXTRACTS THE PROJECTION OF A VECTOR ONTO EACH ONE OF THE VECTORS OF THE BASIS.

- WOULD IT BE POSSIBLE TO BYPASS THE REDUNDANCY OF IBPS AND DIRECTLY EXTRACT THE PROJECTIONS ONTO BASIS ELEMENTS?

3. FORMS, CONTOURS, CONTRACTIONS

- WE CONSIDER THE INTEGRAL

$$\int_{\underbrace{\Gamma}_{\text{MANIFOLD OF DIMENSION } k}} \underbrace{\varphi}_{k\text{-FORM}}$$

- A <u>k-FORM</u> ON A VECTOR SPACE V (OF DIMENSION $\dim V = n$) IS A FUNCTION $V^k \longrightarrow \mathbb{R}$ SUCH THAT
  - $\varphi(\ldots, ax+by, \ldots) = a\,\varphi(\ldots, x, \ldots) + b\,\varphi(\ldots, y, \ldots)$ (LINEAR IN EACH ENTRY)
  - $\varphi(\ldots, x, \ldots, y, \ldots) = -\varphi(\ldots, y, \ldots, x, \ldots)$ ("ANTISYMMETRIC")
- A k-FORM ON $\mathbb{R}^n$ IS USUALLY WRITTEN AS

$$dx_{i_1} \underbrace{\wedge}_{\text{WEDGE PRODUCT}} \ldots \wedge dx_{i_k}$$

AND, BY DEFINITION, BELONGS TO THE <u>DUAL SPACE</u> OF $(\mathbb{R}^n)^k$, I.E. IT IS AN OBJECT THAT APPLIED TO AN ELEMENT OF $(\mathbb{R}^n)^k$ (I.E. A k-TUPLE OF VECTORS OF DIMENSION n) RETURNS A SCALAR.

- k-FORMS ON $\mathbb{R}^n$ ARE DEFINED AS DETERMINANTS

$$dx_{i_1} \wedge \ldots dx_{i_k} \underbrace{(\vec{v}_1, \ldots, \vec{v}_k)}_{\substack{k \text{ VECTORS OF } \mathbb{R}^n \\ \text{" } v_1 \wedge \ldots \wedge v_k}} = \begin{vmatrix} v_{1,i_1} & \cdots & v_{k,i_1} \\ \vdots & \ddots & \vdots \\ v_{1,i_k} & \cdots & v_{k,i_k} \end{vmatrix}$$

- WE SEE THAT:
  - IF $k > n$ ALL k-FORMS ARE ZERO
  - 1-FORMS ARE CO-VECTORS

$$(a_1\, dx_1 + \ldots + a_n\, dx_n)\,(\vec{v}_1) = |a_1 v_{1,1}| + \ldots + |a_n v_{1,n}| = (a_1 \ldots a_n) \begin{pmatrix} v_{1,1} \\ \vdots \\ v_{1,n} \end{pmatrix} = \vec{a} \cdot \vec{v}_1$$

  - 0-FORMS ARE PAIRED TO 0-DIMENSIONAL MANIFOLDS (POINTS): THEY ARE SCALAR FUNCTIONS

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}$$

  - n-FORMS ARE PAIRED TO n-DIMENSIONAL MANIFOLDS: THEY ARE (ORIENTED, WEIGHTENED) VOLUME ELEMENTS
  - (n-1)-FORMS ARE ORTHOGONAL TO A 1-DIMENSIONAL SUB-VECTOR SPACE AND ARE PAIRED TO (n-1)-MANIFOLDS (HYPERSURFACES): THEY ARE AREA ELEMENTS AND ARE USED TO COMPUTE FLUXES.
- THE <u>LINE INTEGRAL FORMULA</u> IS A CONCEPTUAL CORNERSTONE TO UNDERSTAND HOW FORMS AND CURVES INTERACT DURING INTEGRATION

$$\int_{\underbrace{\gamma}_{\substack{\text{1-DIMENSIONAL} \\ \text{MANIFOLD: A CURVE}}}} \underbrace{f}_{\text{1-FORM}} = \int_a^b \underbrace{df(\gamma(t))}_{\text{CO-VECTOR}} \cdot \underbrace{\overbrace{\partial\gamma}^{\text{TANGENT BUNDLE OF } \gamma}(t)}_{\text{VECTOR}}\, dt$$

- THIS FORMULA MAKES EVIDENT THAT A k-FORM IS ALWAYS INTERFACED WITH THE "k-UPLE OF INDEPENDENT VECTORS OF $\mathbb{R}^n$" ASSOCIATED TO THE k-(TANGENT BUNDLE) OF THE INTEGRATION MANIFOLD. THE RESULTING SCALAR INTEGRAL IS CARRIED OUT BY 1-DIM INTEGRATION.

E.G. n-VOLUME INTEGRAL

CO-VECTOR FIELD OF CO-COORDINATES

VECTOR FIELD OF COORDINATES

$$\int_V f(\vec{x}) = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \tilde{f}(\vec{x}(\vec{t}))\, \underbrace{dx_1 \wedge \ldots \wedge dx_n \cdot \partial^1 \wedge \ldots \wedge \partial^n V(\vec{t})}_{\underbrace{\partial_1 V dx_1\, \partial_2 V dx_2 \ldots \partial_n V dx_n}_{d^n \vec{x}}}\, dt$$

$$= \int_V f(\vec{x})\, d^n\vec{x}$$

- THE DIMENSION OF THE VECTOR SPACE OF k-FORMS IN $\mathbb{R}^n$ IS GIVEN BY THE NUMBER OF COMBINATIONS WITHOUT REPETITIONS OF n INDICES IN k SLOTS, I.E. $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
- SINCE THE "k-VECTOR FIELD" IS <u>DUAL</u> TO THE k-FORMS FIELDS (I.E. THERE EXISTS A FUNCTION $V^{k*}.V^k \to \mathbb{R}$ THAT APPLIES A k-FORM TO A "k-FIELD") IT HAS THE SAME DIMENSION.
- A k-FORM φ IS <u>CLOSED</u> IFF $d\varphi = 0$
- A k-FORM φ IS <u>EXACT</u> IFF $\varphi = d\xi$, WHERE ξ IS A $(k-1)$-FORM
- SINCE $dd\varphi = 0$ (SCHWARZ SAYS $\frac{\partial^2\varphi}{\partial x_i \partial x_j} = \frac{\partial^2 \varphi}{\partial x_j \partial x_i}$, BUT $dx_i \wedge dx_j = -dx_j \wedge dx_i$), ALL EXACT FORMS ARE ALSO CLOSED. THE VICE VERSA IS IN GENERAL NOT TRUE AND THIS FACT CAN BE EMPLOYED TO STUDY THE SPACE THE FORMS ARE DEFINED ON.

- <u>EXAMPLES</u>
  - 1-FORM OF THE CENTRAL FIELD: $\varphi = \frac{dr}{r^n}$
    - THIS FORM IS CLOSED: $d\varphi = \underbrace{dr \wedge dr}_{0} \frac{-n}{r^{n+1}} + d\Omega \wedge dr \cdot 0 = 0$
    - THIS FORM IS ALSO EXACT:

$$dF = d\left(\frac{1-n}{r^{n-1}}\right) = \frac{dr}{r^n} = \varphi$$

ELECTRIC FIELD FROM A POINT CHARGE IN O

  - 1-FORM ARGUMENT: $\varphi = \frac{1}{x^2+y^2}\left[-y\,dx + x\,dy\right]$
    - THIS FORM IS CLOSED: $d\varphi = \frac{y^2-x^2}{(x^2+y^2)^2}\left[dx \wedge dy + dy \wedge dx\right] = 0$
    - THIS FORM IS <u>NOT</u> EXACT:

$$\int_0^{2\pi} \varphi = \int_0^{2\pi} \begin{pmatrix} -\sin\vartheta & \cos\vartheta \end{pmatrix} \cdot \underbrace{\begin{pmatrix} -\sin\vartheta \\ \cos\vartheta \end{pmatrix}}_{\text{DERIVATIVE OF THE CURVE } \gamma = \begin{pmatrix} \cos\vartheta \\ \sin\vartheta \end{pmatrix}} d\vartheta = \int_0^{2\pi} d\vartheta = 2\pi$$

MAGNETIC FIELD FROM A WIRE ON $+\hat{z}$

THE INTEGRAL OF AN EXACT FORM DEPEND ONLY ON THE EXTREMA, NOT ON THE PATH (AS FOR THE FUNDAMENTAL THEOREM OF CALCULUS), SO THIS FORM IS NOT EXACT.

  - 2-FORM GAUSS: $\varphi = \sin\vartheta\, d\vartheta \wedge d\varphi$
    - THIS FORM IS CLOSED: $d\varphi = 0\, dr \wedge d\vartheta \wedge d\varphi = 0$
    - THIS FORM IS NOT EXACT: ITS INTEGRAL OVER A CLOSED SURFACE AROUND THE ORIGIN IS DIFFERENT FROM ZERO (IT IS THE FLUX OF $\vec{E}$ GENERATED BY A CHARGE IN THE ORIGIN)

ELECTRIC FIELD FROM A POINT CHARGE IN O

- THE ABOVE EXAMPLES SHOW HOW THE SHAPE OF THE DOMAIN OF THE FORM CAN PLAY A ROLE IN THE OUTCOME OF INTEGRATION OR IN THE PROPERTIES OF A FORM, AND ALSO SHOW HOW FORMS CAN BE USED TO INVESTIGATE (WHEN INTEGRATED) THE PROPERTIES OF THE DOMAIN.
- NOT ALL FORMS ARE APPROPRIATE FOR STUDYING THE PROPERTIES OF THE DOMAIN:
  - THE INTEGRATION OF AN EXACT FORM OVER A MANIFOLD DEPENDS ONLY ON THE EXTREMA OF INTEGRATION (SINCE A PRIMITIVE EXISTS, BY DEFINITION), SO WE CANNOT USE IT TO INSPECT THE DOMAIN;
  - IF WE TAKE A NON-CLOSED FORM WE "SPREAD" INFORMATION

CONSIDER THE FOLLOWING 1-FORMS ON $\mathbb{R}^2$

$$\omega = y\, dx + (x+2y)\, dy, \qquad \vartheta = (x+y)\, dx + 3\, dy$$

THEY DO NOT HAVE ANY WEIRD STRUCTURE IN THEIR DOMAIN AROUND $(0,0)$, SO WE WOULD LIKE TO SEE THIS WHEN INTEGRATING THEM.

- ω IS EXACT (AND THEREFORE CLOSED)

$$\omega = d\left(xy + y^2\right)$$

SO IT CANNOT BE USED TO STUDY THE DOMAIN.

- ϑ IS NOT EXACT, BUT IT IS NOT EVEN CLOSED: $d\vartheta = -dx \wedge dy$. THIS PRODUCES SOME WEIRD BEHAVIOR:

$$\int_{\partial B(0,1)} \omega = \int_0^{2\pi} \left(\cos\vartheta + \sin\vartheta \quad 3 \right) \begin{pmatrix} -\sin\vartheta \\ \cos\vartheta \end{pmatrix} d\vartheta = -\pi$$

$$\int_{B(0,1)} d\omega = -\int_{\substack{B(0,1) \\ r\in[0,1] \\ \vartheta\in[0,2\pi]}} \underbrace{dx \wedge dy \cdot \partial_x \wedge \partial_y}_{r\, dr\, d\vartheta} = -\pi$$

- WE NEED BOTH THE BEHAVIOR OF THE FORM AND OF ITS DERIVATIVE TO SAY SOMETHING ON THE DOMAIN. WITH CLOSED FORMS WE NATURALLY CONDENSATE THE INFORMATION ON A SINGLE INTEGRATION.

- WE RESTRICT OURSELVES TO THE SPACE OF CLOSED FORMS $(d\varphi = 0)$ MODULUS THE SPACE OF EXACT FORMS $(\varphi = d\Phi)$. THIS SPACE OF k-FORMS ON $\mathbb{R}^n$ IS CALLED <u>k-TH DE RHAM CO-HOMOLOGY GROUP $H^k_0(\mathbb{R}^n, +)$</u>.

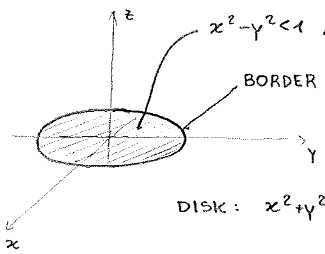
- WE SEE THAT (IN ORDER OF VALIDITY)
  - $\dim H^n_0(\mathbb{R}^n,+) = 1$: ALL VOLUME FORMS ARE CLOSED AND EXACT: $d(\alpha_1 x_1\, dx_2 \wedge \ldots \wedge dx_n + \ldots) = (\alpha_1 + \ldots)\, dx_1 \wedge \ldots dx_n$
  - $\dim H^0_0(\mathbb{R}^n,+) = 0$: ALL SCALAR FUNCTIONS ARE <u>NOT</u> EXACT (THERE IS NOTHING ABOVE TO USE AS $d? = f$) AND ONLY THE CONSTANT FUNCTION IS CLOSED
  - $H^{n-1}_0(\mathbb{R}^n,+) \neq \phi$ AND CONTAINS ALL THE "DIVERGENCIES"
  - $\dim H^1_0(\mathbb{R}^n,+) = 0$ SINCE ALL CLOSED 1-FORMS ARE EXACT IN SIMPLY CONNECTED SPACES (ALL PATHS THAT ARE CLOSED CAN BE SHRUNK TO POINTS)

- WE SAW THAT FORMS AND INTEGRATION DOMAINS ARE DEEPLY INTERTWINED. IS THERE A DUAL WAY TO GET INFORMATION ON THE DOMAIN USING INTEGRAL PROPERTIES?
- THE DUAL THEORY IS CARRIED OUT BY (HYPER-) SURFACES AND (HYPER-) VOLUMES.

- A <u>k-MANIFOLD WITH BORDER</u> IS A k-MANIFOLD EQUIPPED WITH A BORDER, WHICH IS A k-1 DIMENSIONAL SET OF POINTS SUCH THAT THE k-MANIFOLD WITH BORDER IS LOCALLY DIFFEOMORPHIC TO $\underbrace{[0,+\infty[\times \cdots \times [0,+\infty[}_{k \text{ TIMES}}$.
  - EXAMPLE: DISK IN $\mathbb{R}^3$

$x^2 - y^2 < 1 \wedge z = 0$ DOES <u>NOT</u> CONTAIN BORDERS

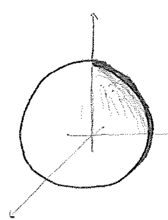
BORDER: $x^2 + y^2 = 1 \wedge z = 0$



DISK: $x^2 + y^2 \leq 1 \wedge z = 0$

LOCALLY DIFFEOMORPHIC TO $[0,+\infty[\times [0,+\infty[$

  - EXAMPLE: SPHERICAL SURFACE IN $\mathbb{R}^3$



- A SPHERICAL SURFACE HAS DIMENSION $k=2$ AND NO BORDER (IF YOU WALK ON IT, IT NEVER ENDS)
- IT IS ALTHOUGH THE BORDER OF THE 3-BALL $x^2+y^2+z^2 = R^2$.

· THE <u>BORDER OPERATOR</u> ∂ ACTS ON MANIFOLDS AND RETURNS THEIR BORDER.

· A k-MANIFOLD M IS A <u>BORDER</u> IF THERE EXISTS A G $(k+1)$-MANIFOLD WITH BORDER SUCH THAT

$$\partial G = M \qquad \longleftrightarrow \qquad F = d\xi$$

· A MANIFOLD M IS <u>BORDERLESS</u> IF

$$\partial M = \phi \qquad \longleftrightarrow \qquad dF = 0$$

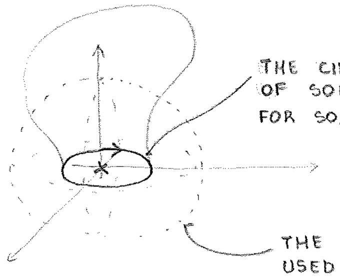
· ALL THE MANIFOLDS WE CAN CONSIDER WHEN STUDYING THE DOMAIN MUST BE ENTIRELY CONTAINED IN THE DOMAIN. GIVEN THIS CONSTRAINT, BORDERS DO NOT REALLY ADD ANY INFORMATION ABOUT THE FORM OF THE DOMAIN, SO WE CAN DISCARD THEM (AS WE DID FOR EXACT FORMS)

· NON-BORDERLESS DOMAINS CAN ALSO BE TRICKY: WE CAN SMOOTHLY DEFORM THEM MOVING THE EXTREMA AROUND UNTIL WE FREE THE SINGULARITIES AND THEN LOOSE INFORMATION.

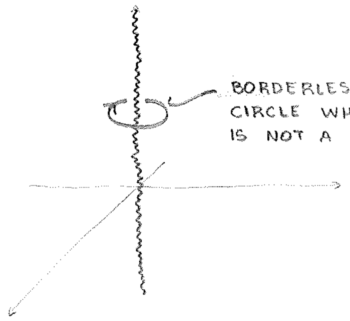
· WE CONSIDER, THEN, ONLY THE SPACE OF BORDERLESS MANIFOLDS $(\partial M = 0)$ MODULUS BORDER MANIFOLDS $(M = \partial \mathcal{M})$, OBTAINING A SPACE OF k-MANIFOLDS THAT CAN BE CALLED THE <u>k-TH DE RHAM HOMOLOGY GROUP $H^0_k(\mathbb{R}^n, \cup)$</u>, WHERE $\cup$ IS THE COMPOSITION OPERATION.

· ALSO HERE WE HAVE (IN ORDER OF VALIDITY)
- $\dim H^0_n(\mathbb{R}^n, \cup) = 1$: THE WHOLE n-VOLUME HAS NO BORDER AND IT IS NOT THE BORDER OF ANYTHING IN THE DOMAIN;
- $\dim H^0_0(\mathbb{R}^n, \cup) = 0$: IT IS THE BORDER OF A STRING FROM INFINITY TO THE POINT AND HAS NO BORDER;
- $H^0_{n-1}(\mathbb{R}^n, \cup) \neq \phi$: HYPERSURFACES THAT ENCAPSULATE POINT-LIKE DEFECTS
- $\dim H^0_1(\mathbb{R}^n, \cup) = 0$: ALL BORDERLESS CURVES ARE BOUNDARIES OF SOME OTHER MANIFOLD

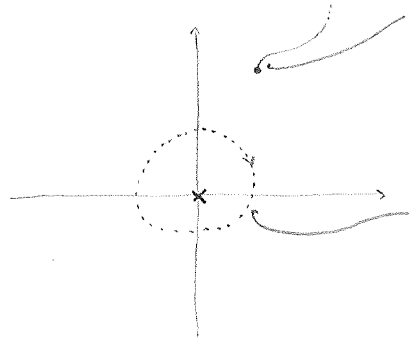
· <u>EXAMPLES</u> · $\mathbb{R}^3 \setminus \{0\}$



· $\mathbb{R}^3 \setminus \{\hat{z}\}$



· $\mathbb{R}^2 \setminus \{0\}$



- INTEGRATION OF k-FORMS OVER k-MANIFOLDS IS THEN OPTIMISED BY INTERFACING THE k-TH HOMOLOGY GROUP $H^{0}_{k}(\Gamma, \cup)$ WITH THE k-TH CO-HOMOLOGY GROUP $H^{k}_{0}(\Gamma, +)$.
  - EXACT FORMS & BORDER MANIFOLDS ARE INSENSIBLE TO DEFECTS (I.E. EITHER GIVES 0 CONTRIBUTION TO THE INTEGRAL;
  - NON-CLOSED FORMS & NON-BORDERLESS MANIFOLDS DILUTE INFORMATION (I.E. FURTHER INTEGRALS ARE NEEDED.
- WE CAN WRAP UP THE CONSIDERATIONS UNTIL THIS POINT IN THE FOLLOWING CHAIN

$$\begin{tikzcd}
 & \text{SCALAR FUNCTIONS 0-FORMS} & \text{1-FORMS} & & \text{(n-1)-FORMS} & \text{n-FORMS} & \\
\text{DE RHAM COHOMOLOGY} \quad 0 \arrow[r, "d"] & H^{0}_{0}(\mathbb{R}^n, +) \arrow[r, "d"] \arrow[d] & H^{0}_{1}(\mathbb{R}^n, +) \arrow[r, "d"] \arrow[d] & \cdots \arrow[r, "d"] & H^{0}_{n-1}(\mathbb{R}^n, +) \arrow[r, "d"] \arrow[d] & H^{0}_{n}(\mathbb{R}^n, +) \arrow[r, "d"] \arrow[d] & 0 \\
 & \text{EVALUATIONS AT A POINT} & \text{LINE INTEGRALS} & \cdots & \text{FLUXES} & \text{VOLUMES} & \\
\text{"DE RHAM HOMOLOGY"} \quad 0 & H^{0}_{0}(\mathbb{R}^n, \cup) \arrow[l, "\partial"'] \arrow[u] & H^{1}_{0}(\mathbb{R}^n, \cup) \arrow[l, "\partial"'] \arrow[u] & \cdots \arrow[l, "\partial"'] & H^{n-1}_{0}(\mathbb{R}^n, \cup) \arrow[l, "\partial"'] \arrow[u] & H^{n}_{0}(\mathbb{R}^n, \cup) \arrow[l, "\partial"'] \arrow[u] & 0 \arrow[l, "\partial"'] \\
 & \text{POINTS} & \text{LINES} & & \text{HYPER-SURFACES} & \text{SPACES} &
\end{tikzcd}$$

- THE "CANONICAL MAP" THAT PAIRS EACH OF THE k-FORMS OF THE BASIS OF k-FORMS TO THE k-TUPLE OF VECTORS OF THE BASIS OF k-TANGENT BUNDLES THROUGH THE "DETERMINANT MAP" TELLS US THAT THE TWO SPACES ARE PAIRED THROUGH AN ISOMORPHISM, SO THEY HAVE THE SAME DIMENSION.

- THE RELATION ABOVE IS ALSO VALID AMONG DE RHAM k-HOMOLOGY AND DE RHAM k-COHOMOLOGY, RELATING SELECTED FORMS TO WAYS OF CUTTING k-MANIFOLDS REMOVING k-DIMENSIONAL PARTS AND OBTAINING DIFFERENT TOPOLOGICAL OBJECTS.

4. TWISTED PERIOD INTEGRALS

- WE CONSIDER NOW THE CLASS OF INTEGRALS ON $\underline{\mathbb{C}^n}$

$$\int_{\underbrace{\Gamma}_{\text{TWISTED CYCLE}}} \underbrace{u}_{\text{TWIST}} \; \underbrace{\varphi}_{\text{TWISTED COCYCLE}}$$

- THE TWIST u IS A MULTIVALUED FUNCTION, WHICH GOES TO 0 FAST ENOUGH ON THE BOUNDARY $\partial\Gamma$ TO KEEP $u\varphi|_{\partial\Gamma}=0$. ITS EFFECT IS TO "SMEAR" THE NOTION OF BOUNDARY AND TO ADAPT FOR MORE GENERAL CONDITIONS ON φ. MOREOVER, THE TWIST MAKES THE PROPERTY OF (CO)HOMOLOGY STRUCTURES LOCALLY DEPENDING ON THE POINTS OF THE DOMAIN.
- THE PRESENCE OF THE TWIST INTRODUCES THE CONDITION

$$0 = (u\underbrace{\xi}_{\substack{(n-1)\text{-FORM}\\ \text{ON } \mathbb{C}^n}})\Big|_{\partial\Gamma} = \int_\Gamma d(u\xi) = \int_\Gamma \left[du\wedge\xi + u\,d\xi\right] = \int_\Gamma u\underbrace{\left[\frac{du}{u}\wedge + d\right]}_{\nabla_\omega}\xi$$

$$\nabla_\omega = d + \omega\wedge = d + d\log u\,\wedge$$

THEREFORE INTEGRALS ARE DEFINED UP TO THE EQUIVALENCE

$$\varphi \longrightarrow \varphi + \nabla_\omega\xi \qquad \text{(MODIFIED EXACTNESS RELATION)}$$

- MOTIVATED BY THE RELATION ABOVE, WE CAN DEFINE THE TWISTED DE RHAM COHOMOLOGY AS THE SET OF ω-CLOSED FORMS MODULUS ω-EXACT FORMS $H^n_\omega(\Gamma)$.
- ANALOGOUSLY, TWISTED CYCLES THAT ARE ω-BORDERLESS AND MODULUS ω-BORDERS GENERATE THE TWISTED DE RHAM HOMOLOGY $H^\omega_n(\Gamma)$. HERE WE USE ∂_ω INSTEAD OF ∂ AS A BORDER OPERATOR, ALLOWING US TO INCLUDE IN THE BORDER REGIONS WHERE $u\varphi$ IS ZERO.

- THE PASSAGE TO TWIST (I.E. LOCAL) DEFINITIONS "SMEARS" THE CONSTRAINTS, POSSIBLY RESULTING EVEN IN THE MODIFICATION OF THE DIMENSION OF THE SPACES. IN PARTICULAR, IT IS POSSIBLE TO DETERMINE THE DIMENSION OF $H^n_\omega(\Gamma\subseteq\mathbb{C}^n)$ USING MORSE THEORY, WHICH SAYS THAT THE DIMENSION OF SUCH SPACE IS EQUAL TO THE NUMBER OF SOLUTIONS OF

$$\partial_i \log u = 0 \qquad i=1,\dots,n$$

- THE ISOMORPHISM BETWEEN HOMOLOGY AND COHOMOLOGY GROUPS IS VALID HERE AS WELL, SO $\dim H^n_\omega(\Gamma) = \dim H^\omega_n(\Gamma)$.
- IN GENERAL WE KEEP Γ FIXED AND VARY φ. WITH THIS APPROACH $\dim H^n_\omega(\Gamma)$ IS ALSO THE NUMBER OF INDEPENDENT n-FORMS, SO THE NUMBER OF MASTER INTEGRALS.
- WE ARE NOW WORKING IN $\mathbb{C}^n$, WHICH IS A $\mathbb{R}^{2n}$ SPACE WITH MORE CONSTRAINTS. WE ARE STILL CONSIDERING THE n-FORMS AND n-CONTOURS THAT WE HAD IN $\mathbb{R}^n$, SO WE STILL "MISS" HALF OF THE COORDINATES. TO SOLVE THIS PROBLEM WE DEFINE DUAL OBJECTS. TO GET INSPIRATION ON HOW TO DO IT, CONSIDER THE TWIST INTEGRAL

$$\int_{\mathbb{R}} e^{-\alpha x^2}\, x^n\, dx$$

- THIS INTEGRAL "LIVES" COMPLETELY IN $\mathbb{R}$. TO INVESTIGATE $i\mathbb{R}$ WE CAN TAKE THE INVERSE OF THE TWIST, WHICH FORCES US TO MOVE THE PATH ONTO THE IMAGINARY AXIS TO KEEP THE VANISHING-AT-THE-BOUNDARY CONDITION VALID

$$\int_{\mathbb{R}i} e^{\alpha x^2}\, x^n\, dx = \int_{i\mathbb{R}} \frac{1}{e^{-\alpha x^2}}\, x^n\, dx = \int_{i\mathbb{R}} \frac{1}{u}\,\varphi$$

SO WE DEFINE THE DUAL INTEGRAL AS AN INTEGRAL WITH A TWIST $\tilde{u}$ WHICH IS THE INVERSE OF THE ORIGINAL ONE.

- A HANDY NOTATION FOR (DUAL) TWISTED (CO)CYCLES IS THE FOLLOWING

SURFACE

$$\langle\varphi|\Gamma] = \int_\Gamma u\,\varphi$$

FORM

IN GENERAL Γ AND φ CHANGE FROM INTEGRALS TO DUAL INTEGRALS

$$[\Gamma|\varphi\rangle = \int_\Gamma \frac{1}{u}\,\varphi$$

DUAL SURFACE — DUAL FORM

- DUAL FORMS AND DUAL CYCLES OBEY CONSTRAINED BASED ON $-\omega$, SINCE THE VANISHING AT THE BOUNDARY IMPLIES

$$0 = \frac{\xi}{u}\bigg|_{\partial\Gamma} = \int_\Gamma \left[d\frac{1}{u} \wedge \xi + \frac{1}{u} d\xi \right] = \int_\Gamma \frac{1}{u} \nabla_{-\omega} \xi$$

$$\nabla_{-\omega} = d - d\log u \wedge$$

- $|\varphi\rangle$ IS AN ELEMENT OF THE DUAL TWISTED COHOMOLOGY GROUP $H^n_{-\omega}(\Gamma)$, WHILE $[\Gamma|$ IS AN ELEMENT OF THE DUAL TWISTED HOMOLOGY GROUP $H^{-\omega}_n(\Gamma)$.

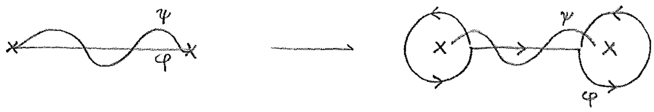
5. INTERSECTION NUMBERS

- INTERSECTION NUMBERS ARE GIVEN BY THE "CONTRACTION" OF TWO ELEMENTS, ONE DUAL OF THE OTHER:
  - $[\Gamma|\Omega]$ INTERSECTION NUMBER OF TWISTED CYCLES Γ AND Ω
  - $\langle\varphi|\psi\rangle$ INTERSECTION NUMBER OF TWISTED COCYCLES φ AND ψ

- THE INTERSECTION NUMBER OF TWISTED CYCLES CAN BE IMMEDIATELY VISUALIZED AS THE SUM OF THE WEIGHTED ORIENTED INTERSECTIONS BETWEEN TWO MANIFOLDS

ORIENTATION

- SINCE CYCLES ONLY INTERSECT "NON TRIVIALLY" IN SINGULAR POINTS, THE REGULARIZATION PRESCRIPTION TO OBTAIN AN UNAMBIGUOUS RESULT IS TO COUNTERCLOCKWISE (POSITIVELY) DEFORM ONE OF THE CYCLES



CLEAR HOW TO INTERSECT

- THE INTERSECTION NUMBER OF TWISTED COCYCLES IS GIVEN BY

$$\langle\varphi|\psi\rangle = \frac{1}{(2\pi i)^n} \int_\Gamma \varphi \wedge \psi = \sum_{x_0} \mathrm{Res}_{x_0}(\Phi\psi)$$

x_0 SINGULAR POINTS OF ω

Φ PRIMITIVE OF φ (AT LEAST LOCALLY) SATISFYING

$$\nabla_\omega \Phi = \varphi$$

THIS RESULT COMES FROM THE REGULARIZATION PROCEDURE NECESSARY TO ADDRESS SINGULARITIES AT THE BORDER OF Γ, SINCE u IS NO MORE PRESENT

- THIS IS ONE OF THE MOST IMPORTANT RESULTS FOR PRACTICAL USE IN INTERSECTION THEORY, AND CAN HOLD FOR THE MULTIVARIATE CASE AS WELL THROUGH THE GENERALIZED RESIDUE THEOREM.
- A PICTORIAL WAY TO UNDERSTAND THE FORMULA IS TO GO BACK TO $\mathbb{R}^n$: HERE DUALS ARE CO-OBJECTS (SINCE THERE ARE NO EXTRA DIMENSIONS) AND WE CAN SAY

$$\langle \varphi | \psi \rangle = \int_X \left(dx_1 \wedge \ldots \wedge dx_n \,.\, \partial^1 \wedge \ldots \wedge \partial^n \right) \hat{\varphi}\, \hat{\psi}$$

FORM ($\langle \varphi |$), VECTOR FIELD ($| \psi \rangle$)

$\hat{\varphi}\,\hat{\psi}$: THESE ARE THE COEFFICIENTS OF THE VOLUME FORM $\longrightarrow$ ψ PLAYS THE ROLE OF "ARC DERIVATIVE" IN THE INTEGRAL

$$= \int_X \hat{\varphi}\; d^n \hat{\psi} = \int_{\partial X} \hat{\Phi}\; \hat{\psi}\; d^{n-1} x = \mathrm{Res}_{x_0 \in X_{sing}} \left(\hat{\Phi}\, \hat{\psi} \right)$$

STOKES' THEOREM $\int_X d\omega = \int_{\partial X} \omega$

- INTERSECTION NUMBERS POSSESS THE FOLLOWING PROPERTIES:
  - $\langle \varphi | \psi \rangle = \langle \varphi + \nabla_{+\omega} \xi | \psi + \nabla_{\omega} \xi \rangle$ (INVARIANCE UNDER EQUIVALENCE CLASSES)
  - $\langle \varphi | \psi \rangle_{\omega} = (-1)^n \langle \psi | \varphi \rangle_{-\omega}$ WHICH FOLLOWS FROM ALTERNANCY OF k-FORMS
  - THE SAME HOLDS FOR CYCLES (THEY ARE ORIENTED)

- A GENERALISED VERSION OF THE GRAPHICAL WAY OF COMPUTING INTERSECTION NUMBERS OF TWISTED CYCLES IS TO DEFINE THE SINGLE INTERSECTION AS THE CONTRACTION BETWEEN THE TANGENT SPACE OF $[\Gamma|$ IN THAT POINT WITH THE ORTHOGONAL SPACE TO THE TANGENT SPACE OF $|\omega]$. THIS OPERATION IS ALREADY ORIENTED.
- BEING ABLE TO COMPUTE INTERSECTION NUMBERS, WE CAN NOW DECOMPOSE THE INTEGRALS IN TERMS OF MASTER INTEGRALS. STARTING FROM THE DECOMPOSITIONS

$$\mathcal{I} = \sum_{i=1}^{\nu} c_i J_i \qquad \text{INTEGRALS}$$

$$\tilde{\mathcal{I}} = \sum_{i=1}^{\nu} \tilde{c}_i \tilde{J}_i \qquad \text{DUAL INTEGRALS}$$

AND VARYING EITHER THE (DUAL) CYCLES OR THE (DUAL) COCYCLES WE GET

$$\langle \varphi | = c_i \langle e_i | \qquad \text{COCYCLES}$$

$$| \varphi \rangle = | \tilde{e}_i \rangle \tilde{c}_i \qquad \text{DUAL COCYCLES}$$

$$| \Gamma] = s_i | h_i] \qquad \text{CYCLES}$$

$$[\Gamma | = [\tilde{h}_i | \tilde{s}_i \qquad \text{DUAL CYCLES}$$

- INVERTING THE RELATIONS ABOVE WE CAN EXTRACT $c_i, \tilde{c}_i, s_i, \tilde{s}_i$ USING THE <u>MASTER DECOMPOSITION FORMULAS</u>

$$c_i = \langle \varphi | \tilde{e}_j \rangle \left(C^{-1} \right)_{ji}$$

$$\tilde{c}_i = \left(C^{-1} \right)_{ij} \langle e_j | \varphi \rangle$$

$$C_{ij} = \langle e_i | \tilde{e}_j \rangle \qquad \underline{\text{COCYCLE METRIC}}$$

$$s_i = \left(S^{-1} \right)_{ij} [\tilde{h}_j | \Gamma]$$

$$\tilde{s}_i = [\Gamma | h_j] \left(S^{-1} \right)_{ji}$$

$$S_{ij} = [\tilde{h}_i | h_j] \qquad \underline{\text{CYCLE METRIC}}$$

- THE METRICS ARE $\nu \times \nu$ MATRICES, WHICH IN GENERAL ARE NOT SYMMETRIC.
- NOTICE THAT C_i IS INDEPENDENT OF THE CHOICE OF THE BASIS $\mathcal{E}$. ANALOGOUSLY FOR $\tilde{\mathcal{C}}, S, \mathcal{Z}$ WITH $e, \tilde{h}, h$.
- USING BASES AND METRICS THE INTERSECTION NUMBERS CAN BE WRITTEN AS

$$\langle \varphi | \psi \rangle = \langle \varphi | h_i] \, (S^{-1})_{ij} \, [\tilde{h}_j | \psi \rangle$$

$$[\Gamma | \Omega] = [\Gamma | \tilde{e}_i \rangle \, (C^{-1})_{ij} \, \langle e_j | \Omega]$$

- WE WANT TO UNDERSTAND HOW TO DIFFERENTIATE (DUAL) COCYCLES. IN PARTICULAR, WE WANT TO CONSTRUCT RELATIONS OF THE FORM

$$\partial_x \langle e_i | = \Omega_{ij} \langle e_j |$$

$$\partial_x | \tilde{e}_i \rangle = - | \tilde{e}_j \rangle \, \tilde{\Omega}_{ji}$$

- WE APPLY THE ∂_x OPERATOR TO A (DUAL) INTEGRAL FOR GENERAL (DUAL) CYCLES:

$$\partial_x \langle e_i | \Gamma] = \partial_x \int_\Gamma u \, e_i = \int_\Gamma u \left[\partial_x \log u + \partial_x \right] e_i = \langle \Delta_\omega e_i | \Gamma]$$

$$\partial_x [\tilde{\Gamma} | \tilde{e}_i \rangle = \partial_x \int_\Gamma \frac{1}{u} \tilde{e}_i = \int_\Gamma u \left[-\partial_x \log u + \partial_x \right] \tilde{e}_i = [\tilde{\Gamma} | \Delta_{-\omega} \tilde{e}_i \rangle$$

FOR GENERIC $\Gamma, \tilde{\Gamma}$

$$\partial_x \langle e_i | = \langle \Delta_\omega e_i | = \underbrace{\langle \Delta_\omega e_i | \tilde{e}_j \rangle (C^{-1})_{jk}}_{\Omega_{ik}} \langle e_k |$$

$$\partial_x | \tilde{e}_i \rangle = | \Delta_{-\omega} \tilde{e}_i \rangle = | \tilde{e}_k \rangle \underbrace{(C^{-1})_{kj} \langle e_j | \Delta_{-\omega} \tilde{e}_i \rangle}_{-\tilde{\Omega}_{ki}}$$

SINCE

$$| \tilde{e}_i \rangle \, (C^{-1})_{ij} \, \langle e_j | = \mathbb{1}$$

$$\underbrace{\underbrace{\underbrace{\langle e_k | \tilde{e}_i \rangle}_{C_{ki}} (C^{-1})_{ij}}_{\mathbb{1}_{kj}} \underbrace{\langle e_j | \tilde{e}_\ell \rangle}_{C_{j\ell}}}_{C_{k\ell}} = \underbrace{\langle e_k | \tilde{e}_\ell \rangle}_{C_{k\ell}}$$

- SIMILARLY WE CAN CONSTRUCT DEFORMATION RELATIONS AMONG CONTOURS.
- Ω AND $\tilde{\Omega}$ ARE RELATED THROUGH

$$\partial_x \langle e_i | \tilde{e}_j \rangle = \langle \Delta_\omega e_i | \tilde{e}_j \rangle + \langle e_i | \Delta_{-\omega} \tilde{e}_j \rangle = \Omega_{ik} \underbrace{\langle e_k | \tilde{e}_j \rangle}_{C_{kj}} + \underbrace{\langle e_i | \tilde{e}_k \rangle}_{C_{ik}} (-\tilde{\Omega}_{kj})$$

$$\underline{\partial_x C_{ij} = \Omega_{ik} C_{kj} - C_{ik} \tilde{\Omega}_{kj}}$$

6. GAUSSIAN INTEGRAL VIA INTERSECTION

- LET US GO BACK TO

$$I_n(\alpha) = \int_{-\infty}^{+\infty} \underbrace{e^{-\alpha x^2}}_{\text{"TWIST"}} \underbrace{x^n}_{\text{COCYCLE}} dx$$

- THE "TWIST" IS NOT MULTIVALUED, SO IT DOES NOT HAVE ENOUGH DEFECTS TO PROPERLY INSPECT THE DOMAIN. TO FIX THIS ISSUE, WE ADD A x^ρ TERM, TAKING THE LIMIT $\rho \to 0$ AT THE END TO RETRIEVE OUR INTEGRAL:

$$I_n^\rho(\alpha) = \underbrace{\int_{-\infty}^{+\infty}}_{\text{CYCLE}} \underbrace{e^{-\alpha x^2} x^\rho}_{\text{TWIST } u} \underbrace{x^n dx}_{\text{COCYCLE}}$$

- NUMBER OF MIS

THE NUMBER OF MIS IS GIVEN BY THE NUMBER OF SOLUTIONS OF

$$d \log u = 0$$

$$\frac{\partial}{\partial x} \log\left[e^{-\alpha x^2} x^\rho \right] dx = 0$$

$$x^2 = \frac{\rho}{2\alpha} \longrightarrow x = \pm \sqrt{\frac{\rho}{2\alpha}}$$

THERE ARE TWO SOLUTIONS, THEREFORE $\nu = \dim H^1_\omega(\mathbb{R}) = 2$ AND THERE ARE 2 MIS.

- BASIS OF COCYCLES

WE LOOK FOR TWO COCYCLES OF THE FORM $x^n dx$ WHICH ARE INDEPENDENT UNDER DECOMPOSITION VIA INTERSECTION NUMBERS; THEY MUST HAVE A "NON-ZERO" NORM AS WELL. WE CAN TAKE ONE ELEMENT TO BE dx, AND THE OTHER ONE dx/x, INSPIRED BY OUR PREVIOUS CALCULATION VIA IBPS. FOR SIMPLICITY WE WILL ASSUME FOR NOW THAT THESE ARE TRULY A BASIS OF COCYCLES, AND WE WILL CHECK THIS LATER ONCE WE HAVE MORE EXPERTISE WITH COMPUTING INTERSECTION NUMBERS FOR SIMPLE, DEFINITE CASES.

- COMPUTING INTERSECTION NUMBERS AMONG ELEMENTS OF BASES IS COMPUTING THE VARIOUS ENTRIES OF THE METRIC
- WE ALSO NEED A BASIS OF DUAL COCYCLES. WE WILL USE AGAIN dx, dx/x. IN PRINCIPLE WE COULD HAVE TAKEN A DIFFERENT CHOICE, BUT LET US KEEP IT SIMPLE FOR NOW.

$$\langle 1 | 1 \rangle = \sum_{\substack{x_0 \in \text{POLES} \\ \text{OF } \omega}} \text{Res}_{x_0} \left(\underbrace{\Phi_1}_{\substack{\text{A LOCAL SOLUTION} \\ \text{AROUND } x_0 \text{ IS SUFFICIENT}}} \psi_1 \right)$$

- THE POLES OF $\omega = d \log u = \frac{-2\alpha x^2 + \rho}{x}$ ARE $x = 0$ AND $x \to \infty$
- WE ASSUME THAT $\Phi_1(x \sim x_0) = \sum_{j=-N}^{M} a_j (x - x_0)^j$, SO HERE

$$\Phi_1(x \sim 0) = \sum_{j=-N}^{M} a_j x^j$$

· PLUGGING IT INTO THE DIFFERENTIAL EQUATION

$$\nabla_\omega \Phi_1 = \varphi_1$$

$$\left[\frac{\partial}{\partial x} \log\left(e^{-\alpha x^2} x^\rho \right) + \frac{\partial}{\partial x} \right] \sum_{j=-N}^{M} a_j x^j = 1$$

$$\left[\frac{\rho - 2\alpha x^2}{x} \sum_{j=-N}^{M} a_j x^j + \sum_{j=-N}^{M} a_j \, j \, x^{j-1} \right] = 1$$

AND IMPOSING THAT THE EQUATION HOLDS POWER BY POWER IN x WE GET

$$\Phi_1 (x \sim 0) = \frac{x}{1+\rho} + O_0(x^3)$$

WHEN WE HAVE A LAURENT SERIES WE IMMEDIATELY READ THE RESIDUE IN x_0 AS THE COEFFICIENT OF THE $(x - x_0)^{-1}$ TERM, SO HERE WE HAVE

$$\mathrm{Res}_0 \Big[\underbrace{\Phi_1}_{\frac{x}{1+\rho} + O_0(x^3)} \; \underbrace{\psi}_{1} \Big] = 0$$

· NOW WE CONSIDER THE RESIDUE AT INFINITY. SINCE THE RESIDUE AT $x \to \infty$ IS THE SAME AS THE RESIDUE $y \sim 0$ WITH $x = 1/y$ WE TAKE

$$\Phi_1 (y \sim 0) = \sum_{j=-N}^{M} a_j y^j$$

THAT NOW NEEDS TO SATISFY

$$\left[\underbrace{-\frac{1}{y^2}}_{} \frac{\partial}{\partial y} \log\left(e^{-\alpha/y^2} y^{-\rho} \right) + \frac{\partial}{\partial y} \right] \sum_{j=-N}^{M} a_j y^j = 1$$

JACOBIAN OF CHANGE OF COORDINATES

$$dx = -\frac{1}{y^2} dy$$

HERE WE DIFFERENTIATE, SO THE JACOBIAN APPEARS

WE DO NOT SPECIFY THESE FUNCTION IN x COORDINATES, ONLY IN y, SO WE "SKIP" THE JACOBIAN

AGAIN APPLY THE POLYNOMIAL IDENTITY PRINCIPLE WE GET

$$\Phi_1 (y \sim 0) = -\frac{y}{2\alpha} + \frac{1-\rho}{4\alpha^2} y^3 + O_0(y^4)$$

WE COMPUTE THE RESIDUE

$$\mathrm{Res}_0 \left[\underbrace{-\frac{1}{y^2}}_{} \; \underbrace{}_{\psi = 1} \underbrace{\left(-\frac{y}{2} + \frac{1-\rho}{4} y^3 + O_0(y^4) \right)}_{\Phi_1(y)} \right] = \frac{1}{2\alpha}$$

COMING FROM THE CHANGE OF VARIABLES FOR MOVING THE RESIDUE FROM ∞ TO 0. AGAIN, THIS IS DUE TO THE DIFFERENTIAL FORM MAPPING

· THE INTERSECTION NUMBER IS GIVEN BY THE SUM OF THE RESIDUES, SO WE GET

$$\langle 1 | 1 \rangle = \frac{1}{2\alpha}$$

$\langle 1/x | 1/x \rangle$ WE PROCEED IN THE SAME WAY AS FOR $\langle 1|1\rangle$, WITH THE DIFFERENCE THAT NOW THE DES FOR $\Phi_{1/x}(x\sim 0,\infty)$ READ

$$\frac{\rho-2\alpha x^2}{x}\sum_{j=-N}^{M} a_j x^j + \sum_{j=-N}^{M} a_j\, j\, x^{j-1} = \underline{\frac{1}{x}} \longrightarrow \Phi_{1/x}(x\sim 0) = \frac{1}{\rho}+\frac{2x^2}{\rho(\rho+2)}+O_0(x^3)$$

$$\downarrow$$

$$\mathrm{Res}_0\left(\frac{1}{x}\Phi_{1/x}(x\sim 0)\right) = \frac{1}{\rho}$$

$$\left[-\frac{1}{y^2}\frac{\partial}{\partial y}\log\left(e^{-\alpha/y^2}y^{-\rho}\right)+\frac{\partial}{\partial y}\right]\sum_{j=-N}^{M} a_j x^j = y \longrightarrow \Phi_{1/x}(x\sim\infty) = -\frac{y^2}{2\alpha}+O_0(y^4)$$

$$\downarrow$$

$$\mathrm{Res}_0\left(-\frac{1}{y^2}y\left(-\frac{y^2}{2\alpha}+O(y^4)\right)\right)=0$$

$$\underline{\langle 1/x | 1/x\rangle = \frac{1}{\rho}}$$

$\langle 1|1/x\rangle$ GIVES THE DE

$$\frac{\rho-2\alpha x^2}{x}\Phi_1 + d\Phi_1 = \underline{1}$$

RESULTING IN

$$\left.\begin{array}{l}\mathrm{Res}_0(\Phi_1\,\psi_{1/x}) = 0\\ \mathrm{Res}_\infty(\Phi_1\,\psi_{1/x}) = 0\end{array}\right\} \longrightarrow \underline{\langle 1|1/x\rangle = 0}$$

$\langle 1/x|1\rangle$ GIVES

$$\frac{\rho-2\alpha x}{x}\Phi_{1/x}+d\Phi_{1/x} = \frac{1}{x}$$

$$\left.\begin{array}{l}\mathrm{Res}_0(\Phi_{1/x}\,\psi_1) = 0\\ \mathrm{Res}_\infty(\Phi_{1/x}\,\psi_1) = 0\end{array}\right\} \longrightarrow \underline{\langle 1/x|1\rangle = 0}$$

- WE CAN NOW CONSTRUCT THE METRIC FOR THE BASES $\langle \langle 1|, \langle 1/x| \rangle, \langle |1\rangle, |1/x\rangle\rangle$

$$\underline{C = \begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{\rho}\end{pmatrix} \qquad C^{-1} = \begin{pmatrix} 2 & 0\\ 0 & \rho\end{pmatrix}}$$

- NOTICE THAT THE COMPUTATION OF INTERSECTION NUMBERS IS <u>ASYMMETRIC</u>: WE NEED TO GET A (LOCAL) <u>PRIMITIVE OF THE COCYCLE</u> $\langle e|$, SO IN GENERAL EXPRESSIONS THAT LOOK SYMMETRIC MIGHT PRODUCE DIFFERENT RESULTS (DEPENDING ON u): THE METRIC IS IN GENERAL NOT SYMMETRIC.

- NOTICE THAT THE METRIC IS SINGULAR FOR $\rho \to 0$ (I.E. WHEN GETTING BACK TO OUR ORIGINAL PROBLEM) BUT THE METRIC IS AN "INTERMEDIATE" RESULT, NOT AN "OBSERVABLE" SO IT IS FINE.
- SINCE $\operatorname{rank} C = 2$ THE MIS THAT WE CHOSE ARE GOOD CANDIDATES AND FORM A BASIS.

- GIVEN AN INTEGRAL $I_n^{\rho}(\alpha)$ WE CAN WRITE DOWN ITS DECOMPOSITION IN TERMS OF A LINEAR COMBINATION OF MIS USING THE <u>MASTER DECOMPOSITION FORMULAS</u>

$$\langle x^n| = c_0^n \langle 1| + c_{-1}^n \langle 1/x| \qquad\qquad |x^n\rangle = \tilde{c}_0^n |1\rangle + \tilde{c}_{-1}^n |1/x\rangle$$

$$c_0^n = \langle x^n|1\rangle\, 2 \qquad\qquad \tilde{c}_0^n = \langle 1|x^n\rangle\, 2$$

$$c_{-1}^n = \langle x^n|1/x\rangle\, \rho \qquad\qquad \tilde{c}_{-1}^n = \langle 1/x|x^n\rangle\, \rho$$

- WE CAN CHOOSE THE MORE APPROPRIATE BASIS $\langle \langle 1|, \langle x| \rangle, \langle |1\rangle, |x\rangle \rangle$, GETTING

$$C = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{\rho}{4\alpha^2} \end{pmatrix} \qquad C^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & \frac{4\alpha^2}{\rho} \end{pmatrix}$$

GENERATES A SINGULAR COEFFICIENT IN $\rho \to 0$

- WE CAN NOW COMPUTE DIFFERENTIAL RELATIONS FOR THE MIS

$$\partial_\alpha \langle 1| = \langle \partial_\alpha \log(e^{-\alpha x^2} x^{\rho})| = -\langle x^2| = -\underbrace{\langle x^2|1\rangle}_{\frac{1+\rho}{4\alpha}} \underbrace{C_{0,0}^{-1}}_{2} \langle 1| - \underbrace{\langle x^2|x\rangle}_{0} C_{1,1}^{-1} \langle x| = -\frac{1+\rho}{2\alpha} \langle 1|$$

$$\partial_\alpha \langle x| = \langle \partial_\alpha \log(e^{-\alpha x^2} x^{\rho})\, x + \partial_\alpha x| = -\underbrace{\langle x^3|1\rangle}_{0} C_{0,0}^{-1} \langle 1| - \langle x^3|x\rangle \underbrace{C_{1,1}^{-1}}_{\frac{4\alpha^2}{\rho}} \langle x| = -\frac{2+\rho}{2\alpha} \langle x|$$

- WE SEE THAT $\langle x^n|x^{n+1}\rangle = 0$, SO WE COULD HAVE NOT CHOSEN $\langle \langle 1|, \langle x^2| \rangle$ AS A BASIS. THIS IS A SIMILAR RESULT TO THE "JUMPING" RELATION WE FOUND VIA IBPS.
- IN THE LIMIT $\rho \to 0$ WE RETRIEVE OUR CASE OF STUDY, AND WE GET

$$\partial_\alpha \begin{pmatrix} \langle 1| \\ \langle x| \end{pmatrix} = \underbrace{\begin{pmatrix} -1/(2\alpha) & 0 \\ 0 & -1/\alpha \end{pmatrix}}_{\Omega} \begin{pmatrix} \langle 1| \\ \langle x| \end{pmatrix}$$

AS PREVIOUSLY FOUND.

- WE CAN USE THE RELATION $\partial_\alpha C = \Omega C - C\tilde{\Omega}$ TO DETERMINE $\tilde{\Omega}$, GETTING

$$\tilde{\Omega} = \begin{pmatrix} -\frac{1+\rho}{2\alpha} & 0 \\ 0 & \frac{\rho-6}{2\alpha} \end{pmatrix} \xrightarrow{\rho \to 0} \begin{pmatrix} -1/(2\alpha) & 0 \\ 0 & -3/\alpha \end{pmatrix}$$

MAYBE DUE TO DIFFERENT INTEGRATION PATH

· NOTICE THAT C^n_1 IS SINGULAR FOR $\rho \to 0$. THIS PROBLEM IS REMOVED ONCE WE REALISE THAT $\int_{-\infty}^{+\infty} e^{-\alpha x^2} x \, dx = 0$.

· 7. APPLICATIONS

· FEYNMAN INTEGRALS: IN BAIKOV REPRESENTATION (CHANGE OF VARIABLES FROM LOOP MOMENTA TO PROPAGATORS) FEYNMAN INTEGRALS ASSUME THE FORM

$$\int \underbrace{B^{\gamma}}_{\text{TWIST}} \underbrace{\frac{d^n z}{z_1^{\alpha_1} \cdots z_n^{\alpha_n}}}_{n\text{-FORM}}$$

$\left.\begin{array}{l} B : \text{BAIKOV POLINOMIAL} \\ \gamma : \text{NON-INTEGER POWER IN DIM. REG.} \end{array}\right\}$ MULTIVARIATE FUNCTION

THEREFORE INTERSECTION THEORY CAN BE APPLIED

· GREEN'S FUNCTIONS, EFT OPERATORS: THEY ASSUME THE FORM

$$\int \underbrace{e^{-S[\varphi]}}_{\text{"TWIST"}} \underbrace{\varphi_1^{N_1} \cdots \varphi_k^{N_k} \, [d\varphi_1] \cdots [d\varphi_k]}_{n\text{-FORM}}$$

PLUS SOME PUNCTURING FUNCTION TO MAKE THE TWIST MULTIVALUED AS FOR THE GAUSSIAN INTEGRAL

· STATISTICS: PROBABILITY DENSITY FUNCTIONS AND THEIR MOMENT ARE OBTAINED THROUGH INTEGRALS OF THE FORM

$$\int_D \underbrace{F(\vec{x})}_{\text{"TWIST"}} \underbrace{\vec{x}^{\,n} \, d\vec{x}}_{\text{DIFFERENTIAL FORM}}$$

(THEY ARE LIMITED, SINCE $\int_D F(\vec{x}) \, d\vec{x} = 1$)

· THANKS TO FEDERICO GASPAROTTO, MARTIN LANG, FABIAN LANGE, CHIARA SIGNORILE-SIGNORILE, AUGUSTIN VESTNER.

· MORE INFO: · FEDERICO GASPAROTTO'S PHD THESIS
· PIERPAOLO'S, FEDERICO'S, SEBASTIAN'S PAPERS