1. BRANES AS MANIFOLDS

- A p-BRANE IS AN OBJECT OF DIMENSION p, "SMOOTH", AND LIVING IN AN AMBIENT SPACE OF DIMENSION D.
- BRANES AND AMBIENT SPACES CAN POSSESS DIFFERENT PROPERTIES W.R.T. THE EUCLIDEAN SPACE. SUCH PROPERTIES CAN ALSO CHANGE DURING THE TIME EVOLUTION OF THE SYSTEM.
- BUILDING ON OUR EXPERIENCE FROM GENERAL RELATIVITY, A POWERFUL AND GENERAL FRAMEWORK TO DISCUSS SUCH OBJECTS IS GIVEN BY DIFFERENTIAL GEOMETRY, WHERE THE NOTION OF MANIFOLD APPEARS.

· MANIFOLDS [CM 4.1; RW 2.1]

- A MANIFOLD IS A SET OF PIECES THAT "LOOK LIKE" OPEN SETS OF $\mathbb{R}^p$, SUCH THAT THESE PIECES ARE "SEWN TOGETHER" SMOOTHLY AND MIGHT HAVE A BOUNDARY.
- WE NEED A DEFINITION THAT DOES NOT REQUIRE A LARGER "SUPERAMBIENT" SPACE, AS IN GENERAL RELATIVITY THE SPACETIME IS NOT EMBEDDED IN A "LARGER SPACETIME".
- CONSIDER $\mathbb{R}^p$ WITH EACH POINT SPECIFIED BY A VECTOR OF p REAL NUMBERS

$$\mathbb{H}^p = \left\{ x \in \mathbb{R}^p : x^p \geq 0 \right\}$$

$$\partial\mathbb{H}^p = \left\{ x \in \mathbb{R}^p : x^p = 0 \right\}$$

$\partial\mathbb{H}^p$ IS CALLED THE BOUNDARY OF $\mathbb{H}^p$

- AN OPEN BALL IN $\mathbb{H}^p$ AROUND THE POINT y OF RADIUS r IS DEFINED AS

$$B_{y,r} = \left\{ x \in \mathbb{H}^p : \sum_{n=1}^{p} (x_n - y_n)^2 < r^2 \right\}$$

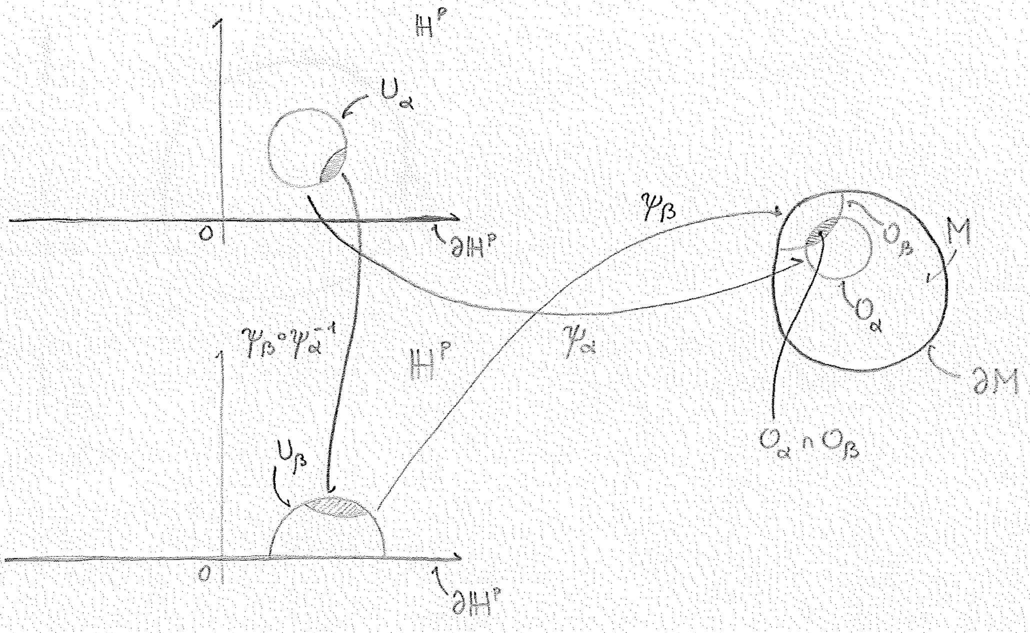
- WE WILL USE OPEN BALLS (WITH EUCLIDEAN NORM) FROM NOW ON TO CONSTRUCT OUR DESCRIPTION OF MANIFOLDS. DIFFERENT DEFINITIONS (I.E. DIFFERENT TOPOLOGIES) WILL LEAD TO DIFFERENT OUTCOMES. WE WILL KEEP THIS DEFINITION TO COMPLY TO OUR INTUITIVE CONCEPT OF DISTANCE.
- A p-DIMENSIONAL $\mathcal{E}^k$ REAL MANIFOLD M IS A SET OF POINTS AND A COLLECTION $\{O_\alpha\}$ OF SUCH POINTS SUCH THAT
  - EACH $P \in M$ BELONGS TO AT LEAST ONE O_α, I.E. $\{O_\alpha\}$ COVERS M
  - FOR EACH α THERE IS A $\mathcal{E}^k$-DIFFEOMORPHISM

$$\psi_\alpha : O_\alpha \longrightarrow U_\alpha$$

  WHERE U_α IS AN OPEN BALL OF $\mathbb{H}^p$.
  - FOR ANY O_α AND O_β WITH $O_\alpha \cap O_\beta \neq \emptyset$

$$\psi_\beta \circ \psi_\alpha^{-1} : \psi_\alpha(O_\alpha \cap O_\beta) \longrightarrow \psi_\beta(O_\alpha \cap O_\beta)$$

  IS OF CLASS $\mathcal{E}^k$. $\psi_\alpha(O_\alpha \cap O_\beta) \subseteq U_\alpha \subset \mathbb{H}^p$ AND $\psi_\beta(O_\alpha \cap O_\beta) \subseteq U_\beta \subset \mathbb{H}^p$.
- THE UNION OF ALL THE POINTS MAPPED INTO $\partial\mathbb{H}^p$ FROM M IS CALLED THE BOUNDARY OF THE MANIFOLD. IF THE BORDER IS EMPTY, THE MANIFOLD HAS NO BORDER, AND IS CALLED "MANIFOLD WITHOUT BOUNDARY".

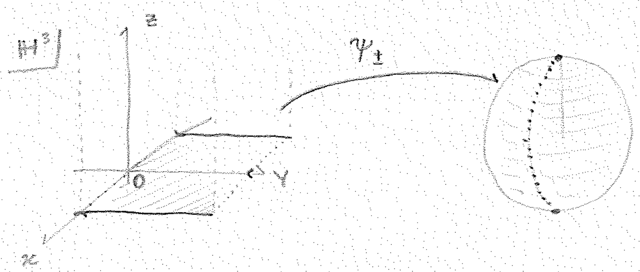


- IF M HAS A BOUNDARY, BOTH $M \setminus \partial M$ AND ∂M ARE MANIFOLDS, SINCE WE RETRIEVE ALL THE REQUIREMENTS GIVEN IN THE MANIFOLD DEFINITION FOR THESE SUBSETS OF POINTS AS WELL. MOREOVER, THEY ARE BOTH MANIFOLDS <u>WITHOUT A BOUNDARY</u>. THIS IS BECAUSE $\mathbb{H}^p \setminus \partial\mathbb{H}^p \sim \mathbb{R}^p$ AND $\partial\mathbb{H}^p \sim \mathbb{R}^{p-1}$.
- THE OPERATION OF "TAKING THE BOUNDARY" IS A <u>COMPLEX</u>, I.E. TAKING IT TWICE RETURNS AN EMPTY SET. THIS IS IN CORRESPONDANCE WITH THE DIFFERENTIATION OPERATION, SINCE AN EXACT FORM IS ALSO CLOSED, OR $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{V}) = 0$ AND $\vec{\nabla}\times(\vec{\nabla} f)$.

- <u>EXAMPLE: SPHERE AND SPHERICAL SURFACE</u>
  - TO PROVE THAT A SPHERE IS A 3-DIMENSIONAL MANIFOLD WITH BOUNDARY WE NEED TO FULFILL THE DEFINITION GIVEN ABOVE, WHICH CAN BE MATCHED CONSIDERING THE COORDINATE MAPS

$$\psi_+ = \begin{cases} \bar{X} = e^{-z} \sin x \cos y \\ \bar{Y} = e^{-z} \sin x \sin y \\ \bar{Z} = e^{-z} \cos x \end{cases} \qquad \psi_- = \begin{cases} \bar{X} = e^{-z} \sin(x+\pi)\cos\left(y+\frac{\pi}{2}\right) & x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\\ \bar{Y} = e^{-z} \sin(x+\pi)\sin\left(y+\frac{\pi}{2}\right) & y \in \left]0, 2\pi\right[\\ \bar{Z} = e^{-z} \cos(x+\pi) & z \in \left[0, +\infty\right[\end{cases}$$

$$\psi_{0+} = \begin{cases} \bar{X} = z \sin x \cos y \\ \bar{Y} = z \sin x \sin y \\ \bar{Z} = z \cos x \end{cases} \qquad \psi_{0-} = \begin{cases} \bar{X} = z \sin(x+\pi)\cos\left(y+\frac{\pi}{2}\right) & x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\\ \bar{Y} = z \sin(x+\pi)\sin\left(y+\frac{\pi}{2}\right) & y \in \left]0, 2\pi\right[\\ \bar{Z} = z \cos(x+\pi) & z \in \left]0, 1\right[\end{cases}$$



$\partial\mathbb{H}^3$ IS MAPPED INTO THE SPHERICAL SURFACE
- WE CONSIDERED O_α AS OPEN SETS, NOT OPEN BALLS, FOR SAKE OF BREVITY.

· NOTICE THAT $\psi_\pm$ INDUCE A MAP ON THE BOUNDARY OF THE SPHERE, OBTAINED BY SETTING $z=0$:

$$\psi_+ = \begin{cases} \bar{X} = \sin x \cos y \\ \bar{Y} = \sin x \sin y \\ \bar{Z} = \cos x \end{cases} \qquad \psi_- = \begin{cases} \bar{X} = \sin(x+\pi)\cos(y+\frac{\pi}{2}) \\ \bar{Y} = \sin(x+\pi)\sin(y+\frac{\pi}{2}) \\ \bar{Z} = \cos(x+\pi) \end{cases} \qquad \begin{matrix} x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\\ y \in \left]0, 2\pi\right[\end{matrix}$$

THIS MAPS COVER ALL THE SPHERICAL SURFACE AND CONTAINS ONLY OPEN SETS OF $\mathbb{R}^2$: THE SPHERICAL SURFACE IS A 2-DIMENSIONAL MANIFOLD WITHOUT BOUNDARY.

· COMING BACK TO THE SPHERE, WE SEE THAT THREE PARAMETERS ARE REQUIRED, SHOWING THAT THIS IS A 3-DIMENSIONAL MANIFOLD. WE CAN ALSO CHECK THAT THERE IS NO WAY, NO MATTER THE SET OF U_α, TO AVOID PARTS OF $\partial \mathbb{H}^3$. IN PARTICULAR, ∂M IS MAPPED ONLY TO OPEN PIECES OF $\partial \mathbb{H}^3$.

· GIVEN A MANIFOLD M, A SUBSET N IS A <u>SUBMANIFOLD</u> OF M IF IT IS A MANIFOLD AND CAN BE DESCRIBED AS THE ZERO OF A FUNCTION

$$F : U \longrightarrow \mathbb{H}^q$$

WITH $U \subset M$, $q = \dim N$.

· TENSORS [RW 2.2, 2.3]

· IN ORDER TO CHARACTERIZE THE PROPERTIES OF A MANIFOLD, SUCH AS ITS "CURVATURE", AROUND EACH ONE OF ITS POINTS, WE NEED TO UNDERSTAND HOW TO CONSTRUCT TANGENT VECTOR SPACES AND HOW TO RELATE TANGENT SPACES AT DIFFERENT POINTS.

· IT IS CLEAR FOR US HOW TO CONSTRUCT A TANGENT VECTOR SPACE TO A POINT IN $\mathbb{R}^p$: GIVEN A COORDINATE SYSTEM WE TAKE THE DIRECTIONAL DERIVATIVE ALONG EACH ONE OF THE COORDINATE DIRECTIONS:

$$\left.\frac{\partial}{\partial x^\mu}\right|_Y : \left.\frac{\partial F}{\partial x^\mu}\right|_Y = \lim_{\varepsilon \to 0^+} \frac{F(Y + \varepsilon\,\hat{x}(\mu)) - F(Y)}{\varepsilon}$$

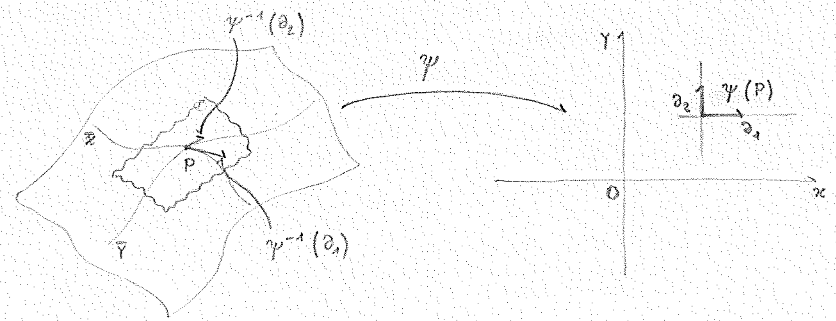
WHERE $\hat{x}^\nu(\mu) = \delta^{\nu\mu}$ AND $F: \mathbb{R}^p \to \mathbb{R}$ IS A SCALAR FIELD. WE CAN THEN DEFINE A VECTOR AS

$$V = V^\mu \frac{\partial}{\partial x^\mu}$$

WHERE $\frac{\partial}{\partial x^\mu}$ IS OBTAINED FROM THE CHOICE OF THE COORDINATE SYSTEM. FROM THIS RESULT WE ALSO SEE THAT $\dim T_Y(\mathbb{R}^p) = p$.

· TO DEFINE TANGENT VECTOR SPACES AT POINTS IN MANIFOLDS (AND THUS, VECTORS ON MANIFOLDS) WE MAKE USE OF ψ TO LINK THE NOTION OF DERIVATIVE ON THE MANIFOLD TO THE BASIS OF VECTORS ON $\mathbb{R}^p$:

$$\left.\frac{\partial}{\partial \bar{x}^\mu}\right|_P : \left.\frac{\partial (F \circ \psi^{-1})}{\partial x}\right|_{\psi(P)}$$



· THE RESULT ABOVE IS ESSENTIALLY A CONSEQUENCE OF THE CHAIN RULE. AS EXPECTED, IF WE CHANGE THE COORDINATE SYSTEM ON THE MANIFOLD FROM ψ_α TO ψ_β WE GET

$$\left.\frac{\partial}{\partial \bar{\bar{x}}^\mu}\right|_P = \frac{\partial \bar{x}}{\partial \bar{\bar{x}}} \left.\frac{\partial}{\partial \bar{x}}\right|_P$$

· A VECTOR V AT A POINT P OF M IS A QUANTITY THAT DOES NOT DEPEND ON THE CHOICE OF COORDINATES. ON THE OTHER HAND, V^μ FROM

$$V = V^\mu \partial_\mu$$

DEPENDS EXPLICITLY ON THE COORDINATE SYSTEM. TO FIND HOW V^μ TRANSFORMS WE USE

$$V = V'$$

$$V^\mu \partial_\mu \qquad V'^\mu \partial'_\mu = V'^\mu \frac{\partial x^\nu}{\partial x'^\mu} \underbrace{\frac{\partial}{\partial x^\nu}}_{\partial_\nu}$$

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu$$

· OBJECTS THAT TRANSFORM IN THIS FASHON ARE CALLED <u>CONTRAVARIANT</u>.

· CONSIDER NOW LINEAR APPLICATIONS ON THE TANGENT VECTOR FIELD, I.E.

$$F : T_p M \longrightarrow \mathbb{R}$$

$$F(\alpha V + \beta W) = \alpha F(V) + \beta F(W)$$

THEREFORE WE CAN ALSO WRITE

$$F(V) \longrightarrow FV$$

AS A CONSEQUENCE, THE F'S POSSESS A STRUCTURE OF VECTOR SPACE.

· WE DERIVED A REPRESENTATION-DEPENDENT FORM FOR $V \cdot V^\mu$. WE WANT TO DO THE SAME FOR F. FIRST OF ALL, WE DEFINE A BASIS FOR F STARTING FROM THE CONDITION

$$dx^\mu \frac{\partial}{\partial x^\nu} = \delta^\mu_\nu$$

(NOTICE THAT HERE d^μ AND ∂_ν ARE JUST PLACEHOLDERS TO INDICATE THE BASIS ELEMENTS). WE CAN NOW WRITE

$$F = F_\mu d^\mu$$

OBTAINING A COORDINATE - DEPENDENT EXPRESSION (d^μ IS FIXED BY ORTHONORMALITY CONDITION FROM ∂_μ, WHICH IS LINKED TO x^μ BY DIRECTIONAL DERIVATIVES). WE NOW HAVE

$$\begin{tikzcd}
& F'V' = FV \arrow[dl] \arrow[dr] & \\
F'_\mu \underbrace{d'^\mu \partial'_\nu}_{\delta^\mu_\nu} V'^\nu \arrow[d] & & F_\mu \underbrace{d^\mu \partial_\rho}_{\delta^\mu_\rho} V^\rho \arrow[d] \\
F'_\mu \frac{\partial x'^\nu}{\partial x^\rho} V^\rho \arrow[dr] & & F_\rho V^\rho \arrow[dl] \\
& F'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} F_\nu &
\end{tikzcd}$$

· OBJECTS THAT TRANSFORM IN THIS FASHON ARE CALLED <u>COVARIANT</u>.

· CONTRAVARIANT OBJECTS ARE ALSO CALLED VECTORS, COVARIANT ONES 1-FORMS. 1-FORMS ARE THE DUALS OF VECTORS, I.E. ARE OBJECTS IN A VECTOR FIELD THAT APPLIED TO VECTORS GIVE SCALARS.

· THE DUAL OF 1-FORMS IS VECTORS, WHICH CAN BE SEEN BY APPLYING THE ABOVE ALGORITHM TO 1-FORMS. THIS MEANS THAT THERE ARE ONLY TWO KIND OF LINEAR OBJECTS WITH ONE INDEX: COVARIANT AND CONTRAVARIANT QUANTITIES.

· NOTICE THAT IN THIS FRAMEWORK QUANTITIES ARE <u>COVARIANT AT SIGHT</u>: THE POSITION AND NUMBER OF INDICES DETERMINE HOW SUCH QUANTITIES TRANSFORM. THE ONLY EXCEPTION IS GIVEN BY COORDINATES, WHICH ARE THE GROUND BASIS AND TRANSFORM LIKE

$$x'^\mu = x'^\mu(x)$$

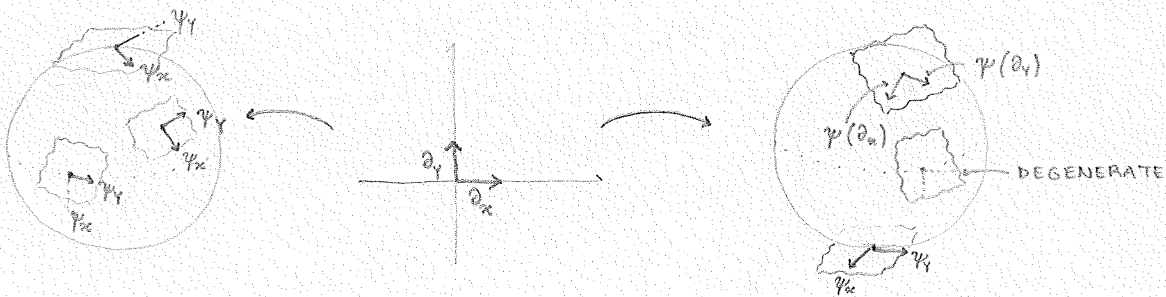
<u>CONTRACTIONS ALWAYS OCCOUR BETWEEN A COVARIANT AND A CONTRAVARIANT INDEX.</u>

• AS AN EXAMPLE, CONSIDER THE SPHERICAL SURFACE WITH THE TWO DIFFERENT PARAMETRISATIONS

$$\begin{cases} \bar{X} = \sin x \cos y \\ \bar{Y} = \sin x \sin y \\ Z = \cos x \end{cases} \qquad \begin{cases} \bar{X} = x \\ \bar{Y} = y \\ \bar{Z} = \pm\sqrt{1-x^2-y^2} \end{cases}$$

A BASIS FOR THE TANGENT VECTOR SPACE OBTAINED FROM THE TANGENT VECTOR SPACE OF $\mathbb{R}^2$ IN CARTESIAN COORDINATES $\langle \partial_x, \partial_y \rangle$ IS GIVEN BY

$$\begin{cases} \partial_{\bar{X}} = \dfrac{\partial x}{\partial \bar{X}} \partial_x + \dfrac{\partial y}{\partial \bar{X}} \partial_y = \dfrac{1}{\cos x \cos y} \partial_x - \dfrac{1}{\sin x \sin y} \partial_y \\ \partial_{\bar{Y}} = \dfrac{\partial x}{\partial \bar{Y}} \partial_x + \dfrac{\partial y}{\partial \bar{Y}} \partial_y = \dfrac{1}{\cos x \sin y} \partial_x + \dfrac{1}{\sin x \cos y} \partial_y \\ \partial_{\bar{Z}} = \dfrac{\partial x}{\partial \bar{Z}} \partial_x \qquad\qquad = -\dfrac{1}{\sin x} \partial_x \end{cases} \qquad \begin{cases} \partial_{\bar{X}} = \partial_x \\ \partial_{\bar{Y}} = \partial_y \\ \partial_{\bar{Z}} = \mp \dfrac{x\,\partial_x + y\,\partial_y}{\sqrt{1-x^2-y^2}} \end{cases}$$



• EVEN THOUGH WE ARE CONSIDERING THE SAME MANIFOLD, THE IMAGES OF THE SAME TANGENT SPACE FROM $\mathbb{R}^2$ ARE VERY DIFFERENT.

• A <u>RANK (m,n) TENSOR</u> IS A MULTILINEAR (I.E. LINEAR IN EACH ENTRY) MAP FROM m 1-FORMS AND n VECTORS INTO $\mathbb{R}$, I.E. IT CARRIES m CONTRAVARIANT INDICES AND n COVARIANT ONES.

• THANKS TO THE PARADIGM OF COVARIANCE AT SIGHT WE CAN INTUITIVELY UNDERSTAND HOW TO OPERATE WITH TENSORS:

• CONTRACTION

$$T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n} V^{\nu_1} = S^{\mu_1 \dots \mu_m}{}_{\nu_2 \dots \nu_n}$$

$$T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n} F_{\mu_1} = S^{\mu_2 \dots \mu_m}{}_{\nu_1 \dots \nu_n}$$

(CONTRACTED INDICES "ANNIHILATE" WHEN IT COMES TO TRANSFORMATIONS)

• OUTER PRODUCT

$$T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n} R^{\rho_1 \dots \rho_r}{}_{\sigma_1 \dots \sigma_s} = S^{\mu_1 \dots \mu_m \rho_1 \dots \rho_r}{}_{\nu_1 \dots \nu_n \sigma_1 \dots \sigma_s}$$

(ALL FREE INDICES COUNT)

· TENSOR TRANSFORMATION LAW

$$T'^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n} = T^{\rho_1 \dots \rho_m}{}_{\sigma_1 \dots \sigma_n} \frac{\partial x'^{\mu_1}}{\partial x^{\rho_1}} \dots \frac{\partial x'^{\mu_m}}{\partial x^{\rho_m}} \frac{\partial x^{\sigma_1}}{\partial x'^{\nu_1}} \dots \frac{\partial x^{\sigma_n}}{\partial x'^{\nu_n}}$$

(EACH INDEX TRANSFORMS INDEPENDENTLY)

· THE REASON WHY WE PICKED THE SYMBOL dx^μ TO INDICATE THE BASIS OF CO-VECTORS TOGETHER WITH THE CONDITION $\partial_\mu d^\nu = \delta^\nu_\mu$ CAN BE FOUND IN THE CHAIN RULE FOR CALCULATING THE TOTAL VARIATION OF A FUNCTION

$$\begin{tikzcd} f : \mathbb{R} \arrow[rr] \arrow[dr] & & \mathbb{R} \\ & M \arrow[ur] & \end{tikzcd}$$

$$\frac{df}{dt}$$

$$\frac{\partial f}{\partial x^\mu} \frac{dx^\mu}{dt}$$

GIVEN THE GENERALITY OF $f(t)$ AND THE FACT THAT df/dt IS A SCALAR WE GET

$$\partial_\mu d^\nu = \delta^\nu_\mu$$

· METRIC [CM 4; RW 2.3; ST 5,6]

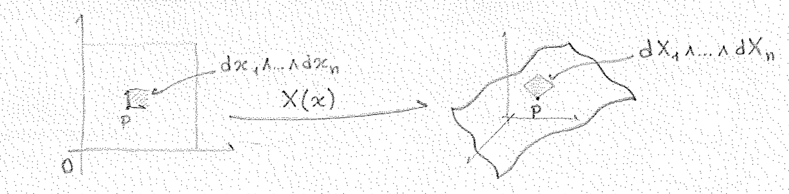
· WE WANT TO COMPUTE THE "MEASURE" OF MANIFOLDS (LENGTH FOR CURVES, AREA FOR SURFACES...). WE WANT THESE MEASURES TO BE INDEPENDENT OF THE PARAMETRIZATION.

· THE INFINITESIMAL MEASURE ELEMENT OF DIMENSION n CORRESPONDS TO THE n-VOLUME GIVEN BY n LINEARLY INDEPENDENT DIRECTIONAL DERIVATIVES.

· GIVEN n VECTORS WITH p COMPONENTS EACH ($n \leq p$), THE n-VOLUME IS GIVEN BY

$$\sqrt{\left|\det\left[A^t A\right]\right|}$$

WHERE $A = (v_1 \dots v_n)$. YOU CAN THINK OF IT AS THE GENERALISATION OF THE MIXED PRODUCT $\varepsilon_{\mu_1 \dots \mu_n} v_1^{\mu_1} \dots v_n^{\mu_n}$, IN A BASIS WHERE $v_i = (v_{i,1} \dots v_{i,n}\ 0 \dots 0)^t$, WHOSE NORM IS TAKEN AS THE SQUARE ROOT OF (THE ABSOLUTE VALUE OF) THE VECTOR SQUARED.



MANIFOLD PARAMETER SPACE x AMBIENT SPACE X

· THE n VECTORS WITH p COMPONENTS dX_i ARE WRITTEN IN TERMS OF THE DIRECTIONAL DERIVATIVES dx_j OF THE PARAMETER SPACE AS

$$dX^\mu = \underbrace{\frac{\partial X^\mu}{\partial x^\nu}}_{\text{CHAIN RULE}} \underbrace{dx^\nu}_{\text{UNIT VECTORS}}$$

THEREFORE THE n-VOLUME IS GIVEN BY THE $\frac{\partial X^\mu}{\partial x^\nu}$ VECTORS. WE GET THEN

$$d^nX = \sqrt{\left|\det\left[\underbrace{\begin{pmatrix} \frac{\partial X^1}{\partial x^1} & \cdots & \frac{\partial X^p}{\partial x^1} \\ \vdots & & \vdots \\ \frac{\partial X^1}{\partial x^n} & \cdots & \frac{\partial X^p}{\partial x^n} \end{pmatrix} \begin{pmatrix} \frac{\partial X^1}{\partial x^1} & \cdots & \frac{\partial X^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial X^p}{\partial x^1} & \cdots & \frac{\partial X^p}{\partial x^n} \end{pmatrix}}_{\begin{pmatrix} \frac{\partial X^\mu}{\partial x^1}\frac{\partial X^\mu}{\partial x^1} & \cdots & \frac{\partial X^\mu}{\partial x^1}\frac{\partial X^\mu}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial X^\mu}{\partial x^n}\frac{\partial X^\mu}{\partial x^1} & \cdots & \frac{\partial X^\mu}{\partial x^n}\frac{\partial X^\mu}{\partial x^n} \end{pmatrix}}\right]\right|}\; dx^1 \cdots dx^n$$

n-DIMENSIONAL MEASURE IN AMBIENT SPACE

TENSOR OF DIMENSION $n \times n$ WITH TWO COVARIANT INDICES

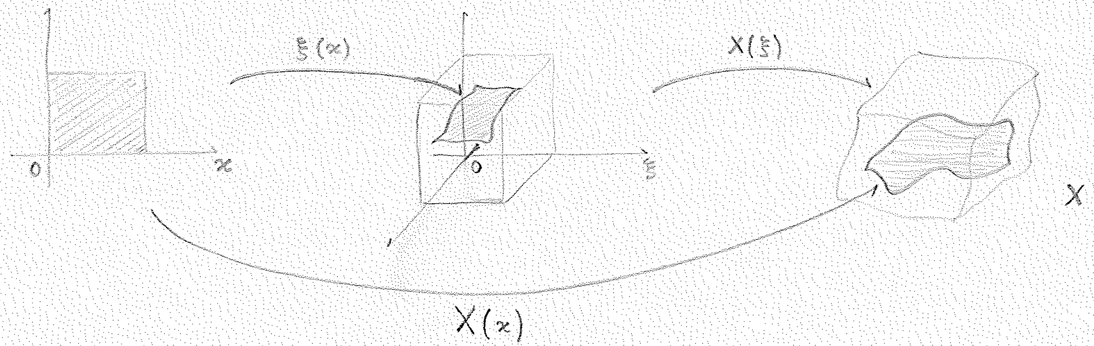
$$\boxed{\frac{\partial X^\mu}{\partial x^a}\frac{\partial X^\mu}{\partial x^b} = g_{ab}} \qquad a,b = 1,\dots,n \quad \mu = 1,\dots,p \qquad (1)$$

PULL-BACK METRIC

$$= \sqrt{\left|\det g_{ab}\right|}\; dx^1 \cdots dx^n$$

· THE PULL-BACK METRIC DESCRIBES "HOW MUCH THE MANIFOLD DIFFERS FROM $\mathbb{R}^n$", SHOWING HOW THE NOTION OF MEASURE CHANGES.

- THE PULL-BACK METRIC RELATES ELEMENTS OF THE PARAMETER SPACE THROUGH THE DEFORMATIONS PROPER OF THE AMBIENT SPACE.
- WE CAN CAST THE PULL-BACK METRIC IN THE USUAL FORM OF THE METRIC (A TENSOR ACTING ON ELEMENTS OF THE AMBIENT SPACE) BY FIRST SENDING THE n-DIMENSIONAL MANIFOLD TO MEASURE TO THE PARAMETER SPACE OF THE AMBIENT SPACE. THIS ALSO GIVES THE PUSH-FORWARD METRIC, AS SEEN LATER.



$$d^n X = \sqrt{\left|\det\left[\frac{\partial X^\mu}{\partial \xi^\nu}\frac{\partial \xi^\nu}{\partial x^a}\frac{\partial X^\mu}{\partial \xi^\rho}\frac{\partial \xi^\rho}{\partial x^b}\right]\right|}\quad dx^1\cdots dx^n$$

$$= \sqrt{\left|\det\left[\underbrace{\frac{\partial \xi^\nu}{\partial x^a} g_{\nu\rho}\frac{\partial \xi^\rho}{\partial x^b}}_{g_{ab}}\right]\right|}\quad dx^1\cdots dx^n$$

- NOTICE THAT WE ARE NOT CHANGING THE NUMBER OF PARAMETERS OF OUR DESCRIPTION: $g_{\nu\rho}$ WILL STILL DEPEND ON n PARAMETERS ONLY, EVEN THOUGH IT IS A $p\times p$ MATRIX.
- NOTICE THAT BY CONSTRUCTION g IS SYMMETRIC.

- SOME MEASURE ELEMENTS:
  - VOLUME $\sqrt{|\det g|}$
  - HYPERSURFACE $\left\|\varepsilon_{\mu\alpha_1\cdots\alpha_{p-1}}\frac{\partial X^{\alpha_1}}{\partial x^1}\cdots\frac{\partial X^{\alpha_{p-1}}}{\partial x_{p-1}}\right\|$
  - LENGTH $\sqrt{\frac{\partial X^\mu}{\partial x}\frac{\partial X^\mu}{\partial x}}$ NOTICE THAT THIS IS $d\ell$. TO GET THE USUAL $\frac{\partial \xi^\mu}{\partial x}\frac{\partial \xi_\mu}{\partial x}$ WE NEED THE METRIC IN X.

- FROM THE DEFINITION (1) WE CAN DETERMINE THE TRANSFORMATION LAW FOR g_{ab}. CONSIDER A CHANGE OF VARIABLES $x'(x)$ IN THE PARAMETER SPACE

$$d^n X = \sqrt{|\det g'|}\quad d^n x'$$

$$= \sqrt{\left|\det\left[\frac{\partial X^\mu}{\partial x'^a}\frac{\partial X^\mu}{\partial x'^b}\right]\right|}\quad\left|\det\frac{\partial x'}{\partial x}\right| d^n x$$

$$= \sqrt{\left|\det\left[\frac{\partial X^\mu}{\partial x^c}\frac{\partial x^c}{\partial x'^a}\frac{\partial X^\mu}{\partial x^d}\frac{\partial x^d}{\partial x'^b}\right]\right|}\quad\left|\det\frac{\partial x'}{\partial x}\right| d^n x$$

$$\frac{\partial x^c}{\partial x'^a} g_{cd}\frac{\partial x^d}{\partial x'^b}$$

$$= \sqrt{\left|\det g\,\det\frac{\partial x}{\partial x'}\,\det\frac{\partial x}{\partial x'}\right|}\quad\left|\det\frac{\partial x'}{\partial x}\right| dx = \sqrt{|\det g|}\quad dx$$

THE MEASURE IS INVARIANT IN FORM AND

$$g'_{ab} = g_{cd} \frac{\partial x^c}{\partial x'^a} \frac{\partial x^d}{\partial x'^b}$$

THE SAME HOLDS FOR THE ξ COORDINATES.

- ANOTHER WAY OF DEFINING A METRIC FOR A (SUB)MANIFOLD IS TO RESTRICT THE AMBIENT METRIC TO THE (SUB)MANIFOLD BY IMPOSING THE EQUATIONS (LOCALLY) DEFINING IT. WE CALL THIS THE PUSH-FORWARD METRIC, OR INDUCED METRIC. SINCE IT IS OBTAINED BY RESTRICTING THE AMBIENT METRIC IT TRANSFORM AS THE METRIC ABOVE.
- THE METRIC SATISFIES THE PROPERTIES:
  - SYMMETRY $g_{ab} = g_{ba}$
  - (0,2)-TENSOR IT TRANSFORMS IN A COVARIANT WAY FOR BOTH INDICES
  - SIGNATURE IT IS ALWAYS POSSIBLE TO FIND A BASIS WHERE THE METRIC IS LOCALLY EQUAL TO A DIAGONAL MATRIX WITH m "+1" AND $n-m$ "−1". $(m, n-m)$ IS THE SIGNATURE OF THE METRIC.
  - 1-FORM VECTOR CORRESPONDANCE THE METRIC TENSOR CAN BE USED TO "LOWER INDICES" OF VECTORS, MAKING A CORRESPONDANCE BETWEEN VECTORS AND 1-FORMS:

$$V_\mu = g_{\mu\nu} V^\nu$$

THIS ALSO WORKS FOR SINGLE INDICES IN TENSORS.

  - INVERSE THE INVERSE OF THE METRIC IS A (2,0)-TENSOR SUCH THAT

$$g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho$$

AND CAN BE USED TO RAISE INDICES.

- EXAMPLE:

- CONSIDER THE 2-DIMENSIONAL SPHERICAL SURFACE IN POLAR COORDINATES

$$X = \begin{pmatrix} \sin\vartheta \cos\varphi \\ \sin\vartheta \sin\varphi \\ \cos\vartheta \end{pmatrix}$$

- THE PULL-BACK METRIC INDUCED BY THE COORDINATES IS

$$x = \begin{pmatrix} \vartheta \\ \varphi \end{pmatrix} \qquad g_{ab} = \frac{\partial X^\mu}{\partial x^a} \frac{\partial X^\mu}{\partial x^b} = \begin{pmatrix} 1 & \\ & \sin^2\vartheta \end{pmatrix}$$

- WE CAN VIEW THE SPHERICAL SURFACE AS A SUBMANIFOLD OF THE UNIT SPHERE (ITS BOUNDARY). THE SPHERE IS PARAMETRIZED AS

$$X = \begin{pmatrix} r \sin\vartheta \cos\varphi \\ r \sin\vartheta \sin\varphi \\ r\cos\vartheta \end{pmatrix}$$

WITH PULL-BACK METRIC

$$g_{ab} = \begin{pmatrix} 1 & & \\ & r^2 & \\ & & r^2 \sin^2\vartheta \end{pmatrix}$$

BY RESTRICTING OURSELVES TO $\rho = 1$ WE GET

$$X = \begin{pmatrix} \sin\vartheta \cos\varphi \\ \sin\vartheta \sin\varphi \\ \cos\vartheta \end{pmatrix}$$

$$g_{\mu\nu} = \begin{pmatrix} 0 & & \\ & 1 & \\ & & \sin^2\vartheta \end{pmatrix}$$

WHICH IS THE PUSH-FORWARD METRIC ON THE SPHERICAL SURFACE.

· FEW REMARKS

· THE DEFINITION OF THE PULL-BACK METRIC

$$\frac{\partial X^\mu}{\partial x^a} \frac{\partial X^\mu}{\partial x^b} = g_{ab}$$

LOOKS UNPLEASANT, SINCE BOTH AMBIENT COORDINATES X^μ HAVE AN "UPPER" INDEX. THIS IS NOT A PROBLEM AT ALL AND IS THE RIGHT EXPRESSION, SINCE

- X^μ ARE COORDINATES, AND THEY ARE DEFINED WITH UPPER INDICES;
- WE CANNOT USE X_μ, SINCE WE NEED THE METRIC TO MOVE INDICES;
- THE STARTING SPACE IS EUCLIDEAN $(\mathbb{H}^n)$, SO THERE X^μ AND X_μ ARE THE SAME;
- IF WE WANT TO INTRODUCE A METRIC ON X^μ WHICH IS NOT EUCLIDEAN WE FIRST USE AN INTERMEDIATE AMBIENT SPACE AND FROM THERE WE MOVE TO THE FINAL SPACE

$$\mathbb{H}^n \longrightarrow \mathbb{H}^{m,n-m} \longrightarrow X^\mu$$

· GIVEN MANIFOLDS M, N AND SMOOTH FUNCTIONS f, φ

PUSH-FORWARD : CONNECTS TM TO TN

$$\begin{tikzcd} TM \arrow[r, "d\varphi"] & TM & \\ M \arrow[u, "\partial"] \arrow[r, "\varphi"'] \arrow[rr, bend right, "\varphi * f"'] & M \arrow[u, "\partial"'] \arrow[r, "f"'] & \mathbb{R} \end{tikzcd}$$

$\varphi * f$

$\downarrow$

$f(\varphi(x))$ PULL-BACK : CONNECTS 1-FORMS ON N TO 1-FORMS ON M