2. DEFORMATION OF BRANES

· WE MAKE USE OF THE THEORY OF ELASTICITY TO DESCRIBE HOW A p-BRANE REACTS TO THE FORCES APPLIED TO IT.

· DESCRIPTION OF DEFORMATIONS [LL 1]

· WHEN A FORCE IS APPLIED TO A p-BRANE EACH POINT X^μ OF THE BRANE CHANGES ITS POSITION TO X'^μ. THE NEW POSITION DEPENDS ON THE OLD ONE, AND THE DISLOCATION IS DESCRIBED BY THE DEFORMATION VECTOR

$$U^i(X) = X'^i(X) - X^i$$

· THE INFINITESIMAL LENGTH CHANGES UNDER THE ACTION OF THE FORCES AS

$$d\ell^2 = dX^\mu dX_\mu$$

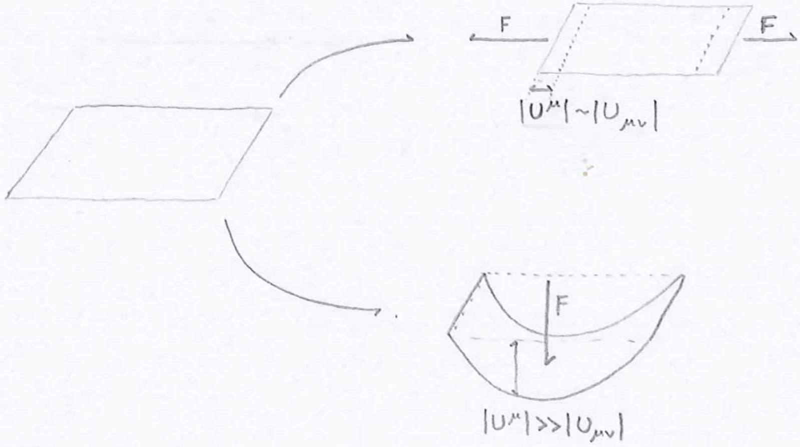
$$d\ell' = dX'^\mu dX'_\mu = (dX^\mu + dU^\mu)(dX_\mu + dU_\mu) = d\ell^2 + 2\underbrace{\left[\frac{\partial U_\mu}{\partial X^\nu} + \frac{1}{2}\frac{\partial U_\alpha}{\partial X^\mu}\frac{\partial U^\alpha}{\partial X^\nu}\right]} dX^\mu dX^\nu$$

DEFORMATION TENSOR $\quad U_{\mu\nu} = \frac{1}{2}\left[\frac{\partial U_\mu}{\partial x^\nu} + \frac{\partial U_\nu}{\partial x^\mu} + \frac{\partial U^\alpha}{\partial x^\mu}\frac{\partial U_\alpha}{\partial x^\nu}\right]$

· THE DEFORMATION TENSOR IS SYMMETRIC BY CONSTRUCTION, AND THEREFORE CAN BE DIAGONALIZED BY AN APPROPRIATE ROTATION OF COORDINATES. IT IS ZERO FOR ISOMETRIES.

· IT IS WORTH TO DISTINGUISH BETWEEN DIRECTIONS INTERNAL TO THE BRANE AND EXTERNAL TO IT (AT LEAST, LOCALLY):

- ALONG INTERNAL DIRECTIONS SMALL DEFORMATIONS (I.E. SMALL ENTRIES IN $U_{\mu\nu}$) IMPLY SMALL CHANGES IN THE BRANE (I.E. SMALL ENTRIES IN U^μ);
- ALONG EXTERNAL DIRECTIONS SMALL DEFORMATIONS MIGHT LEAD TO LARGE CHANGES (I.E. LARGE ENTRIES IN U^μ).



· IT FOLLOWS THAT ALONG INTERNAL DIRECTIONS WE CAN DISCARD HIGHER ORDER DIFFERENTIALS, GETTING

$$U^{(int)}_{\mu\nu} = \frac{1}{2}\left[\frac{\partial U_\mu}{\partial x^\nu} + \frac{\partial U_\nu}{\partial x^\mu}\right]$$

· WE CAN ALWAYS PERFORM A (LOCAL) CHANGE OF VARIABLES SUCH THAT $X^{1,\dots,dim}$ ARE THE "INTERNAL" COORDINATES AND $X^{dim+1\dots Dim}$ ARE THE "EXTERNAL" COORDINATES, ORTHOGONAL TO THE INTERNAL ONES.
THE DEFORMATION VECTOR AND THE DEFORMATION TENSOR TAKE THE FORM

$$U^I = \begin{pmatrix} u^i(x) \\ U^{\hat{\imath}}(x) \end{pmatrix} \begin{matrix} \text{INTERNAL} \\ \text{EXTERNAL} \end{matrix} \qquad U^{IJ} = \begin{pmatrix} u^{ij}(x) & \frac{1}{2}\frac{\partial U^{\hat{\jmath}}}{\partial x_i} \\ \frac{1}{2}\frac{\partial U^{\hat{\imath}}}{\partial x_j} & \underbrace{0}_{\text{NO DEPENDENCE OVER } X^{\hat{\imath}}} \end{pmatrix}$$

· WE EXPECT SOME PROPERTIES BY A DEFORMATION:
- IT SHOULD PRESERVE THE OVERALL STRUCTURE OF MANIFOLDS (SHOULD PRESERVE THE BOUNDARY, THE TOPOLOGY, THE DIFFERENTIABLE STRUCTURE);
- IT SHOULD BE A RESULT OF A CONTINUOUS MODIFICATION OF THE POSITIONS OF THE POINTS OF THE MANIFOLD.

· THE FIRST POINT MEANS THAT THE DEFORMATIONS SHOULD BE DIFFEOMORPHISMS, WHILE THE SECOND ONE ALLOWS US TO CONSIDER DIFFEOMORPHISMS THAT CAN BE WRITTEN AS FLOWS, I.E. SUCH THAT, GIVEN A DIFFEOMORPHISM $\varphi : A \times [0,1] \longrightarrow A$, WHERE A IS THE AMBIENT SPACE, WE HAVE

$$\varphi_0 = X = x$$

$$\varphi_1 = X'(x)$$

· SINCE A DEFORMATION IS A SPECIFIC CASE OF DIFFEOMORPHISM, IT IS ALSO PART OF A LIE GROUP, AND IT ADMITS A LIE ALGEBRA AS A LINEARIZATION AROUND THE IDENTITY. WE HAVE

$$X'^I(X) = X^I + \underbrace{A^I}_{\text{GLOBAL TRANSLATIONS}} + \underbrace{\frac{\partial X'^I}{\partial X^J}}_{\text{INFINITESIMAL GENERATORS}} dX^J + O((dX)^2)$$

· $X'(X)$ IS A DIFFEOMORPHISM, SO $\frac{\partial X'}{\partial X}$ MUST BE INVERTIBLE, THUS $\det \frac{\partial X'}{\partial X} \neq 0$. SINCE WE NEED $X'(X)$ TO BE SMOOTHLY CONNECTED TO THE IDENTITY, $\det \frac{\partial X'}{\partial X} > 0$. AS A RESULT

$$\frac{\partial X'}{\partial X} \in GL^+(Dim, \mathbb{R})$$

· A GENERAL MATRIX OF GL^+ CAN ALWAYS BE DECOMPOSED AS

$$\underbrace{M}_{GL^+(Dim)} = \underbrace{K}_{\text{POSITIVE-DEFINITE SYMMETRIC MATRIX}} \underbrace{T}_{SO(Dim)}$$

· K IS A MATRIX OF DILATIONS ($\det K > 0$), WHILE T EMBEDDS ROTATIONS ($\det T = 1$). SINCE K IS SYMMETRIC AND POSITIVE-DEFINITE, IT CAN BE ROTATED INTO A DIAGONAL FORM, REVEALING THE MAIN AXES OF DILATION (NOT NECESSARY ISOTROPICAL).

- INCLUDING TRANSLATIONS A^I, WE CONCLUDE THAT DEFORMATIONS ARE LOCALLY GIVEN BY $ISO^+(\mathcal{M})$ COMBINED WITH DILATIONS.
- EVEN THOUGH LOCALLY DEFORMATIONS ARE TRANSLATIONS, ROTATIONS, AND DILATIONS ONLY, THE DIFFERENT MAGNITUDE OF EACH OF THEM AT DIFFERENT X^I MAKES THE GLOBAL DEFORMATION QUITE INTRICATE.

- A WAY OF DETERMINING HOW $ISO(\mathcal{M})$ LOOKS LIKE IS TO SOLVE THE EQUATIONS RESULTING FROM IMPOSING THAT THE INFINITESIMAL LENGTH IS CONSERVED:

$$dL'^2(X+A) = dL^2(X)$$

$$dX'^I g_{IJ} dX'^J \Big|_{X+A} = dX^I g_{IJ} dX^J$$

$$\frac{\partial X'^I}{\partial X^L} g_{IJ} \frac{\partial X'^J}{\partial X^K}\Big|_{X+A} dX^L dX^K = g_{LK}\, dX^L dX^K$$

GETTING

$$\underline{\underline{\frac{\partial X'^I}{\partial X^L} g_{IJ}(X+A) \frac{\partial X'^J}{\partial X^K} = g_{LK}(X)}}$$

- WHAT WE GET IS SIMILAR TO THE CONDITION E.G. OF $O(m,n)$, WITH TWO IMPORTANT DIFFERENCIES
  - g IS A GENERAL METRIC, WHICH MIGHT VARY FROM POINT TO POINT IN $\mathcal{M}$
  - IN ORDER TO INCLUDE TRANSLATIONS, g IS EVALUATED AT THE TRANSLATED POINT IN THE L.H.S.

EXAMPLE: $ISO(\mathbb{R}^n)$ & $ISO(S_2)$

- IF WE CONSIDER THE EUCLIDEAN SPACE $g_{IJ} = \delta_{IJ}$, THEREFORE $g(X+A) = g(X) = g$, AND THE ISOMETRIES ARE

$$\frac{\partial X'^I}{\partial X^L} \delta_{IJ} \frac{\partial X'^J}{\partial X^K} = \delta_{LK}$$

$$\Updownarrow$$

$$M^T M = \mathbb{1}$$

$$\Updownarrow$$

$$O(n)$$

- NO CONSTRAINTS ARE PUT ON A^I, SO ALL TRANSLATIONS PRESERVES LENGTHS. IMPOSING THAT TRANSFORMATIONS ARE CONNECTED TO THE IDENTITY WE GET

$$ISO^+(\mathbb{R}^n) = \underbrace{SO^+(n)}_{\text{NO REFLEXIONS INCLUDED}} \ltimes \mathbb{R}^n$$

· ON THE SPHERE IN SPHERICAL COORDINATES THE METRIC IS EXPLICITLY DEPENDING ON THE ANGLES:

$$g_{ij} = \begin{pmatrix} 1 & \\ & \sin^2\vartheta \end{pmatrix}$$

· INTUITIVELY, THE ISOMETRIES OF THE SPHERE ARE THE ROTATIONS IN THREE DIMENSIONS CENTERED IN THE CENTRE OF THE SPHERE. WE CAN ALIGN TWO OF THE THREE GENERATORS WITH THE $\varphi = \text{const.}$ OR $\vartheta = \text{const.}$ LINES (GETTING ELEMENTARY TRANSLATIONS) AND USE THE LAST ONE AS GENUINE ROTATIONS.

· TRANSLATIONS ALONG φ

$$X' = \begin{pmatrix} \vartheta \\ \varphi + \alpha \end{pmatrix} \qquad g = \begin{pmatrix} 1 & \\ & \sin^2\vartheta \end{pmatrix}$$

$$\underbrace{\left(\frac{\partial X'}{\partial X}\right)^T}_{1} \underbrace{g(X')}_{g(X)} \underbrace{\frac{\partial X'}{\partial X}}_{1} = g(X)$$

NO CONSTRAINTS: IT IS AN ISOMETRY $\forall \alpha$. AS WE EXPECT, MOVING ALONG PARALLELS DOES NOT CHANGE THE ARC LENGTH SUBTENDED BY ϑ.

· TRANSLATIONS ALONG ϑ

SINCE THE ARC LENGTH IN THE PARALLELS DIRECTION DEPENDS ON ϑ, WE NEED TO INTRODUCE SOME DILATION TO BALANCE THE EFFECT. FOR SMALL TRANSLATIONS WE GET

$$X' = \begin{pmatrix} \vartheta + \alpha \\ \varphi(1 - \alpha \cot\vartheta) \end{pmatrix} + O(\alpha^2)$$

· THE LENGTH DOES NOT CHANGE UP TO $O(\alpha^2)$ TERMS. WE SEE THAT THE HOMOGENEOUS PART IS OF THE FORM

$$\begin{pmatrix} 1 & \\ & 1 - \alpha \cot\vartheta \end{pmatrix}$$

WHICH GOES SMOOTHLY TO 1 FOR $\alpha \to 0$ AND REPRESENTS A DILATION.

STRESS [LL 2–4]

- FOR EACH INFINITESIMAL ELEMENT OF THE BRANE WE ASSUME THAT

$$U^I(x)=0 \Longleftrightarrow F^I_\Sigma(x)=0$$

- IF $U^I(x)\neq 0$ THE INFINITESIMAL ELEMENT OF THE BRANE IS SUBJECTED TO FORCES, INTERNAL TO THE BRANE, WHICH WILL BE DIRECTED TOWARDS BRINGING THE ELEMENT BACK TO ITS UNDEFORMED POSITION.

- WE ASSUME THAT THE INTERNAL RECALL FORCES ARE CLOSE-RANGE FORCES, AND IN PARTICULAR THAT THEY ACT ONLY ON CONTIGUOUS INFINITESIMAL ELEMENTS AND ONLY THROUGH THE BOUNDARY OF SUCH ELEMENTS.

- GIVEN AN INFINITESIMAL BRANE ELEMENT THE SUM OF ALL INTERNAL FORCES ACTING ON IT IS GIVEN BY

$$F^I_\Sigma=\int \underbrace{F^I(x)}_{\substack{\text{SUM OF ALL}\\ \text{FORCES ACTING}\\ \text{ON THE VOLUME AT } x}}\ \underbrace{dv}_{\substack{\text{VOLUME ELEMENT}\\ \text{OF THE BRANE}}}$$

- SINCE INTERNAL FORCES ACT ON THE BOUNDARY OF THE BRANE ELEMENT AND ARE GIVEN BY DEFORMATIONS WE CAN WRITE

$$\int F^I \underbrace{dv'}_{\substack{\text{DEFORMED}\\ \text{VOLUME ELEMENT}}} = \int \frac{\partial \sigma^{IJ}}{\partial x'^J}\, dv' \underbrace{=}_{\substack{\text{GAUß'}\\ \text{THEOREM}}} \oint \sigma^{IJ}\, ds'_J$$

- WE NEED THE DEFORMED ELEMENT, SINCE TRANSVERSAL DEFORMATIONS CAN BE LARGE.

WHERE σ^{IJ} IS THE STRESS TENSOR.

- σ^{IJ} IS THE I-TH COMPONENT OF THE FORCE ACTING ON THE SURFACE NORMAL TO THE J-TH DIRECTION; IT FOLLOWS THAT σ^{JJ} ENCODES THE NORMAL FORCES AND THE OFF-DIAGONAL TERMS OF σ^{IJ} ENCODE SHEAR FORCES.
- LET US NOW CONSIDER THE BALANCE OF TORQUE. TORQUE IS A (2,0) ANTISYMMETRIC TENSOR DEFINED AS

$$M^{IJ}=\int\left[F^IX^J-X^IF^J\right]dv'$$

$$=\int\left[\frac{\partial\sigma^{IK}}{\partial x'^K}X^J-X^I\frac{\partial\sigma^{JK}}{\partial x'^K}\right]dv'$$

IBPS (ON x)

$$=\oint\left[\sigma^{IK}X^J-X^I\sigma^{JK}\right]ds'_K-\underbrace{\int\left[\sigma^{IJ}-\sigma^{JI}\right]dv'}_{\text{SINCE } \partial X^J/\partial x'^K=g^J{}_K}$$

- IF WE MOVE THE CENTER OF ROTATION FROM THE ORIGIN TO X THE TORQUE MUST BE ZERO. WE GET

$$0=\int\left[\sigma^{IJ}-\sigma^{JI}\right]dv$$

GIVEN THE GENERALITY OF THE INTEGRATION DOMAIN WE GET

$$\sigma^{IJ}=\sigma^{JI}$$

SO THE STRESS TENSOR IS SYMMETRIC.

· WE CONSTRUCT THE WORK DONE BY THE INTERNAL FORCES DUE TO AN INFINITESIMAL VARIATION OF THE DISPLACEMENT OF THE BRANE:

$$\int_{\mathcal{M}} \delta W \, dv' = \int_{\mathcal{M}} F^I \delta U_I \, dv' = \int_{\mathcal{M}} \frac{\partial \sigma^{IK}}{\partial x'^K} \delta U_I \, dv' \underset{\text{IBPS}}{=} -\int_{\mathcal{M}} \sigma^{IK} \frac{\partial \delta U_I}{\partial x'^K} dv' + \underbrace{\int_{\partial \mathcal{M}} \sigma^{IK} \delta U_I \, ds'_K}_{\text{TERM AT THE BOUNDARY}} =$$

VOLUME INTEGRAL OVER THE BRANE

INFINITESIMAL VARIATION OF THE DISPLACEMENT

FORCE CONSIDERED TO BE CONSTANT $(f + \tilde{f}\delta U)\delta U \sim f \delta U$

$$\sigma^{IK} \frac{\partial \delta U_I}{\partial x'^K} = \sigma^{IK} \frac{1}{2}\left(\frac{\partial \delta U_I}{\partial x'^K} + \frac{\partial \delta U_k}{\partial x'^I}\right) \underset{\substack{\partial \text{ AND } \delta \text{ EXCHANGEABLE:} \\ \text{ORTHOGONAL VARIATIONS}}}{=} \sigma^{Ik} \delta U_{Ik} \qquad \text{FOR SMALL VARIATIONS}$$

$$= -\int_{\mathcal{M}} \sigma^{Ik} \delta U_{Ik} \, dv' + \int_{\partial \mathcal{M}} \sigma^{Ik} \delta U_I \, ds'_K$$

· WE CONSIDER THE WORK DONE BY THE INTERNAL FORCES FOR SMALL DISPLACEMENTS FROM THE EQUILIBRIUM CONFIGURATION:

$$\underbrace{-W = V}_{\substack{\text{POTENTIAL ENERGY} \\ \text{OF INTERNAL FORCES}}} = V\Big|_{U=0} + \underbrace{\frac{\partial V}{\partial U^{IJ}}\Big|_{U=0}}_{\frac{\partial(\sigma^{KL}U_{KL})}{\partial U^{IL}}\Big|_{U=0} = 0} U^{IJ} + \frac{1}{2}\frac{\partial^2 V}{\partial U^{IJ}\partial U^{KL}}\Big|_{U=0} U^{IJ} U^{KL} + O(U^3)$$

SINCE $\sigma^{KL} = 0$ IN $U = 0$

· GIVEN A SCALAR WRITTEN AS THE CONTRACTION OF A SYMMETRIC (2,0)-TENSOR WITH ITSELF THE MOST GENERAL DECOMPOSITION FOR IT READS

$$\frac{\lambda}{2} \mathrm{tr}^2 U + \mu U^{IK} U_{IK}$$

HERE, λ AND μ ARE THE LAMÉ COEFFICIENTS

$$= V_0 + \left[\mu\left(U^{IK} - \underbrace{g_{\mathcal{M}}^{IK}}_{\text{MANIFOLD METRIC}} \frac{\mathrm{tr}\, U}{\mathrm{tr}\, g_{\mathcal{M}}}\right)^2 + \frac{K}{2} \mathrm{tr}^2 U\right]\frac{1}{2} + O(U^3)$$

$$K = \lambda + \frac{2}{3}\mu$$

· THE REDEFINITION ABOVE IS BASED ON THE FACT THAT THE INFINITESIMAL GENERATORS OF DISPLACEMENTS ARE DILATIONS AND ISOMETRIES. TO UNDERSTAND WHAT μ AND K MEAN, LET US FIGURE OUT WHAT KIND OF DISPLACEMENTS THE TWO TERMS IN THE EXPANSION DESCRIBE.

- THE FIRST TERM LEAVES THE VOLUME INVARIANT, SINCE

$$dv' = |\det T|\, dv = |1 + \mathrm{tr}\, T|\, dv = \left|1 + \mathrm{tr}\, U - \frac{\mathrm{tr}\, U}{\mathrm{tr}\, g}\, \mathrm{tr}\, g\right| dv = dv$$

THE DEFORMATION ALTERS THE FORM, NOT THE VOLUME, AND REPRESENTS A SHEAR. μ IS CALLED THE SHEAR MODULE.

- THE SECOND TERM IS ISOTROPIC (IT IS ONLY TRACES), THEREFORE IT CANNOT ALTER THE FORM, BUT ONLY RESCALE THE VOLUME, REPRESENTING A COMPRESSION. K IS CALLED THE COMPRESSION MODULE.
- THE UNDEFORMED FREE BRANE MUST CORRESPOND TO A MINIMUM OF V, THEREFORE

$$\mu > 0 \qquad \wedge \qquad K > 0$$

- LET US NOW COMBINE THE TWO EXPANSIONS ABOVE AND CONSIDER HOW THE POTENTIAL ENERGY NEAR THE EQUILIBRIUM CHANGES FOR SMALL VARIATIONS OF THE DISPLACEMENT. WE HAVE

$$\delta V = \mu\, \delta\left(U^{lk} - g^{lk}_{\nu\mu} \frac{\mathrm{tr}\, U}{\mathrm{tr}\, g_{\nu\mu}}\right)^2 + \frac{K}{2}\, \delta\, \mathrm{tr}^2 U + O(U^3, \delta U^2)$$

$$= \underbrace{\left[2\mu\left(U^{lk} - g^{lk}_{\nu\mu} \frac{\mathrm{tr}\, U}{\mathrm{tr}\, g_{\nu\mu}}\right) + K\, g^{lk}_{\nu\mu}\, \mathrm{tr}\, U\right]}_{\sigma^{lk}} \delta U_{lk} \frac{1}{2}$$

BY COMPARING AGAINST THE FIRST EXPANSION

- WE OBTAINED THE EXPRESSION OF THE STRESS TENSOR NEAR THE UNDEFORMED BRANE AS A FUNCTION OF THE DISPLACEMENT.
- WE CAN INVERT THE RELATION AND WRITE

$$\begin{cases} \mathrm{tr}\, U = \dfrac{1}{K} \dfrac{\mathrm{tr}\, \sigma}{\mathrm{tr}\, g_{\nu\mu}} & \text{(BY TAKING THE TRACE)} \\[2ex] U^{lk} = \dfrac{1}{K} \dfrac{\mathrm{tr}\, \sigma}{\mathrm{tr}^2 g_{\nu\mu}} + \dfrac{1}{2\mu}\left(\sigma^{lk} - g^{lk}_{\nu\mu} \dfrac{\mathrm{tr}\, \sigma}{\mathrm{tr}\, g_{\nu\mu}}\right) \end{cases}$$

- WE FIND HOOKE'S LAW: THE DISPLACEMENT IS A LINEAR FUNCTION OF THE STRESS TENSOR.
- IT IS NOT SURPRISING THAT WE GET HOOKE'S LAW NEAR EQUILIBRIUM: WE TRUNCATED V(U) AT THE SECOND ORDER.

LAGRANGIAN FORMULATION [MA 16.1 – 16.3, ST 4.2, 4.6, LL 3–5]

- TWO TYPES OF FORCES CAN ACT ON/IN A BRANE
  - VOLUME FORCES: PROPORTIONAL TO SOME FUNCTION OF THE VOLUME MEASURE (MASS DENSITY, CHARGE DENSITY...), ACT ON THE WHOLE MANIFOLD.
  - SURFACE FORCES: ACT ON THE BOUNDARY OF THE MANIFOLD, CAN BE REPRESENTED BY STRESS TENSORS.
- THE GENERAL FORM OF THE POTENTIAL ENERGY IN TERMS OF DISPLACEMENTS IS GIVEN BY

$$\mathcal{V} = \int_{\mathcal{M}} \Big[\underbrace{-G^I U_I}_{\text{EXTERNAL VOLUME FORCES}} + \underbrace{\sigma^{IJ}}_{\sigma^{ij}\ \text{INTERNAL REACTION FORCES}} U_{IJ} \Big] dv + \int_{\partial\mathcal{M}} \Big[\underbrace{\sigma^{IJ}}_{\sigma^{ij}\ \text{INTERNAL REACTION FORCES}} - \underbrace{P^{IJ}}_{\text{EXTERNAL SURFACE FORCES}} \Big] U_I \, ds_J + \underbrace{\mathcal{V}_0}_{\text{OVERALL CONSTANT}}$$

- THE KINETIC ENERGY FOR AN INFINITESIMAL VOLUME (SIMIL TO A POINTLIKE PARTICLE) IN Dim DIMENSIONS IS GIVEN BY

$$d\mathcal{T}(x) = \frac{1}{2} \frac{dU^J(x)}{dt} \frac{dU_J(x)}{dt} \, dm(x)$$

WHICH GIVES A LAGRANGIAN OF THE FORM

$$\mathcal{L} = \mathcal{T} - \mathcal{V} = \int_{\mathcal{M}} \Big[\frac{1}{2} \frac{dU^J}{dt} \frac{dU_J}{dt} \rho_m + g^J U_J \rho_g - \sigma^{IJ} U_{IJ} \Big] dv +$$

DENSITIES

$$\underbrace{- \int_{\partial\mathcal{M}} \frac{1}{2} \Big[\sigma^{IJ} - P^{IJ} \Big] U_I \, ds_J}_{\text{TERMS AT THE BOUNDARY}} - \underbrace{\mathcal{V}_0}_{\text{OVERALL CONSTANT FACTOR: IRRELEVANT FOR DYNAMICS}}$$

- THE TERMS AT THE BOUNDARY CAN BE REMOVED FROM THE LAGRANGIAN, SINCE THEY CAN BE IMPOSED AS BOUNDARY CONDITIONS. FOR SUFFICIENTLY "WELL DEFINED CAUCHY PROBLEMS" WE CAN INCORPORATE THE TERMS AT THE BOUNDARY INTO THE BOUNDARY CONDITIONS, CALCULATE THE UNCONSTRAINED SOLUTION, AND THEN MATCH IT.
- WE SEE NOW THAT WE CAN DEFINE A LAGRANGIAN DENSITY OF THE FORM

$$\mathscr{L} = \frac{1}{2} \frac{dU^J}{dt} \frac{dU_J}{dt} \rho_m \underbrace{- \frac{1}{2} \sigma^{IJ} U_{IJ}}_{\text{EQUIVALENT TO } \frac{\partial \sigma^{IJ}}{\partial x^J} U_I = F^I U_I} + \underbrace{\sum_i g_i^J U_J \rho_i}_{\text{DIFFERENT KIND OF VOLUME FORCES (GRAVITY, EM...)}}$$

· WE IMPOSE THE <u>STATIONARY ACTION PRINCIPLE</u> : THE TIME EVOLUTION OF THE SYSTEM DESCRIBED BY $\mathcal{L}$ CORRESPONDS TO A STATIONARY POINT OF THE FUNCTIONAL

$$S = \int_{t_0}^{t_1} \int_{\mathcal{M}(t)} \mathcal{L} \, dv\, dt$$

WHERE S IS THE ACTION FUNCTIONAL, $\mathcal{M}(t)$ IS THE BRANE CONFIGURATION AT TIME t.

· BY IMPOSING THE ACTION TO BE STATIONARY WE GET

$$\delta S = -\int_{t_0}^{t_1} \int_{\mathcal{M}(t)} \left[\underbrace{\partial_\mu}_{(t,1)} \frac{\delta \mathcal{L}}{\delta(\partial_\mu U^I)} - \frac{\delta \mathcal{L}}{\delta U^I} \right] \delta U^I \, dv\, dt$$
$$+ \underbrace{\left[\int_{\mathcal{M}(t)} \frac{\delta \mathcal{L}}{\delta(\partial_0 U^I)} \delta U^I dv \right]_{t_0}^{t_1}}_{0} + \int_{t_0}^{t_1} \underbrace{\left[\int_{\partial\mathcal{M}(t)} \frac{\delta \mathcal{L}}{\delta(\partial_J U^I)} \delta U^I ds_J \right]}_{BCS} dt$$

· ALL THESE THREE TERMS MUST VANISH INDEPENDENTLY FROM EACH OTHER.

· THE FIRST TERM PRODUCES THE <u>EULER–LAGRANGE EQUATIONS</u>

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu U^I)} - \frac{\delta \mathcal{L}}{\delta U^I} = 0$$

· THE SECOND TERM IS PROPORTIONAL TO δU^I AND EVALUATED AT t_0 OR t_1 : AT THE EXTREMA OF THE TIME INTERVAL δU IS ZERO BY CONSTRUCTION, THEREFORE THE TERM IS ZERO

· THE THIRD TERM IS THE <u>BOUNDARY TERM</u>. FOR IT TO VANISH WE CAN REQUIRE EITHER ONE OF

- $\delta U^I = 0$ <u>DIRICHLET BOUNDARY CONDITION</u>
- $\delta \mathcal{L} / \delta(\partial_J U^I) = 0$ <u>VON NEUMANN BOUNDARY CONDITION</u>

WHERE BOTH CONSTRAINTS ARE UNDERSTOOD ON THE BOUNDARY $\partial\mathcal{M}$.

· <u>DIRICHLET BC</u> INDICATES THAT THERE IS NO MODIFICATION OF DEFORMATION AT THE BOUNDARY THROUGH TIME, I.E. THE BOUNDARY IS FIXED:

$$\partial_0 U^I \big|_{\partial\mathcal{M}} = 0$$

THIS MEANS THAT OUR BRANE HAS (PART OF) ITS BOUNDARY FIXED ONTO A (dim-1) EXTERNAL BRANE, CALLED A D(IRICHLET)-BRANE.

· WE COULD ALSO SAY THAT $\delta U \big|_{\partial t} = 0$ IS A SORT OF "SPACELIKE" D-BRANE.

· <u>VON NEUMANN BC</u> CAN BE UNDERSTOOD AS A REQUIREMENT ON THE BEHAVIOR OF THE POTENTIAL ENERGY FROM VARIATION OF DISPLACEMENTS AT THE BOUNDARY.

· ASSUMING THAT $\mathcal{L}$ DEPENDS ON ∂U_I THROUGH U_{IJ} AND INTERNAL STRESS AS σU WE GET

$$\frac{\partial(\sigma^{AB} U_{AB})}{\partial(U_{IJ})} \, ds_J = 0$$

• IF WE CONSIDER NOW AN ELASTIC BEHAVIOR WE GET

$$\frac{\partial}{\partial U_{LM}}\left[\, 2\mu\left(U^{IK}U_{IK} - \frac{\mathrm{tr}^2 U}{\mathrm{tr}\, g_{\mu\nu}}\right) + K\, \mathrm{tr}^2 U\right] =$$

$$= 4\mu\underbrace{\left(U_{LM} - \frac{\mathrm{tr}\, U}{\mathrm{tr}\, g_{\mu\nu}}\, g_{LM}\right)}_{\widetilde{U}_{Lm}} + 2K\, \mathrm{tr}\, U\, g_{LM}$$

$\widetilde{U}_{Lm}$ "TRACELESS" VERSION OF U_{LM}

THEREFORE

$$4\mu\, \widetilde{U}_{LM}\, ds^M + 2K\, \mathrm{tr}\, U\, ds_L = 0$$

$$\begin{cases} \widetilde{U}_{LM}\, ds^M = 0 \\ \mathrm{tr}\, U = 0 \end{cases}$$

$\longrightarrow$ ON THE BOUNDARY THERE CANNOT BE COMPRESSIONS / EXPANSIONS

• THE FIRST CONDITION CAN BE BETTER UNDERSTOOD IF WE CONSIDER A BOUNDARY $x_1 = 0$ OF A dim MANIFOLD IN AMBIENT DIMENSION

$$ds^M = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\widetilde{U}_{Lm} = \left(\begin{array}{cccc|ccc} 0 & \partial_1 U_2 + \partial_2 U_1 & \partial_1 U_3 + \partial_3 U_1 & \cdots & \partial_1 U_{dim+1} & \cdots & \partial_1 U_{Dim} \\ \partial_2 U_1 + \partial_1 U_2 & 0 & \partial_2 U_3 + \partial_3 U_2 & \cdots & & & \\ \vdots & \vdots & 0 & & & & \\ \hline \partial_1 U_{dim+1} & & & & & & \\ \vdots & \cdots & & & & 0 & \\ \partial_1 U_{Dim} & & & & & & \end{array}\right)$$

$$\widetilde{U}_{Lm}\, ds^m = \left(\begin{array}{c} \partial_2 U_1 + \partial_1 U_2 \\ \vdots \\ \partial_{dim} U_1 + \partial_1 U_{dim} \\ \hline \partial_1 U_{dim+1} \\ \vdots \\ \partial_1 U_{Dim} \end{array}\right) = 0$$

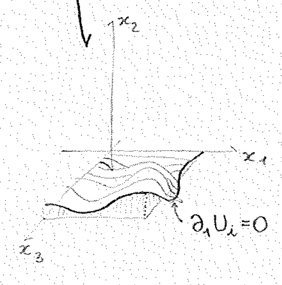
• $\underline{\partial_1 U_i = 0}$ DISPLACEMENT IN DIRECTIONS EXTERNAL TO THE MANIFOLD GOES TO ZERO APPROACHING THE BOUNDARY

• $\underline{\partial_i U_1 = -\partial_1 U_i}$ THE BOUNDARY TRANSFORMS THROUGH ISOMETRIES

$$d\ell'^2 = \partial_1 U_1\, dx^1 dx^1 + \dots + \partial_{dim} U_{dim}\, dx^{dim} dx^{dim}$$

$$\wedge$$

$$\mathrm{tr}\, U = 0$$



- CONSIDER A STRING OF LENGTH L IN Dim DIMENSIONS, ALONG x_1, EUCLIDEAN METRIC.

$$\mathcal{L} = \underbrace{\frac{1}{2}\frac{dU^J}{dt}\frac{dU_J}{dt}\rho}_{\text{TIME EVOLUTION}} + \underbrace{g^J U_J \rho}_{\text{EXTERNAL FORCES}} - \underbrace{\sigma^{IJ} U_{IJ} \frac{1}{2}}_{\text{INTERNAL REACTIONS}}$$

- NOTICE THAT THE INTERNAL FORCES ARE GENERATED BY DEFORMATIONS THAT ARE BEYOND ISOMETRIES, AS ENCODED BY U_{IJ}.
- ELASTIC INTERNAL REACTIONS CAN BE DECOMPOSED AS

$$\sigma^{IJ} U_{IJ} = 2\mu \underbrace{\tilde{U}_{IJ}}_{\text{TRACELESS } U_{IJ}} \tilde{U}^{IJ} + K\,\mathrm{tr}^2 U = \underbrace{\frac{1}{2}\mu\, \partial_1 U_i\, \partial_1 U^i}_{\text{HARMONIC OSCILLATOR IN EACH TRANSVERSAL DIRECTION}} + \underbrace{K\left(\partial_1 U_1 + \frac{1}{2}\partial_1 U_A \partial_1 U^A\right)^2}_{\text{HARMONIC COMPRESSIONS IN THE LONGITUDINAL DIRECTION}}$$

$$U_{IJ} = \left(\begin{array}{c|c} \partial_1 U_1 + 1/2\, \partial_1 U_A \partial_1 U^A & 1/2\, \partial_1 U_j \\ \hline 1/2\, \partial_1 U_i & 0 \end{array}\right)$$

• STATIONARY SOLUTIONS ARE SOLUTIONS TO THE EQUATIONS OF MOTION THAT EXHIBIT EITHER TIME INVARIANCE OR SOME PERIODICITY IN TIME.

• EQUILIBRIUM CONFIGURATIONS [LL 7, 11-12, MA 16.4-16.5]

• EQUILIBRIUM SOLUTIONS ARE TIME INDEPENDENT, THEREFORE THE STATIONARY ACTION PRINCIPLE TAKES THE FORM

$$\delta S = 0 \longrightarrow \int_{\mathcal{M}'} \delta \mathcal{V} \, dv' = 0$$

WHERE $\mathcal{V}$ IS THE POTENTIAL ENERGY.

• WE GET

$$\int_{\mathcal{M}'} \delta \mathcal{V} \, dv' = \int_{\mathcal{M}'} \delta \left[\frac{1}{2} \sigma^{IJ} U_{IJ} + \underbrace{P^K U_K}_{\text{EXTERNAL VOLUME FORCES}} \right] dv'$$

$$= \int_{\mathcal{M}'} \left[\left(\underbrace{\frac{\partial \sigma^{IJ}}{\partial U_{LM}}}_{\substack{\sigma^{IJ} \text{ CAN ONLY} \\ \text{DEPEND ON RELATIVE} \\ \text{DEFORMATIONS } U_{IJ}}} U_{IJ} + \sigma^{LM} \right) \delta U_{LM} + P^K \delta U_K \right] dv'$$

• ASSUMING AN ELASTIC RESPONSE AND SMALL DEFORMATIONS, g INDEPENDENT OF x

$$\left. \begin{aligned} \sigma^{IJ} &= 2\mu \underbrace{\left(U^{IJ} - g^{IJ} \frac{\operatorname{tr} U}{\operatorname{tr} g} \right)}_{\tilde{U}^{IJ}} + K g^{IJ} \operatorname{tr} U \\ U_{IJ} &= \frac{1}{2} \left(\frac{\partial U_I}{\partial x^J} + \frac{\partial U_J}{\partial x^I} \right) \end{aligned} \right\} \quad \frac{\partial \sigma^{IJ}}{\partial U_{LM}} U_{IJ} = \sigma^{LM}$$

WE EXTRACT THE EQUATION

$$\mu \, \partial_a \partial^a U^B + \left(\mu - \frac{2\mu - K}{\operatorname{tr} g} \right) \partial^b \partial_a U^a - P^B = 0$$

THIS EQUATION IS FAR FROM EASY TO SOLVE IN FULL GENERALITY. LET US CONSIDER $P^B /\!/ -\hat{x}^{Dim}$ AND $\dim < \mathrm{Dim}$. LET US DISCUSS SOME LIMIT CASES

• NO SHEAR: $\mu = 0$. WE GET

$$\partial^b \partial_a U^a = \tilde{P} \, \delta^{b \, Dim}$$

IF NO DIRECTIONS INTERNAL TO THE MANIFOLD HAVE A NON-ZERO SCALAR PRODUCT WITH P^B NOTHING HAPPENS. IN CASE $\dim = \mathrm{Dim}$ WE CAN ASSUME THAT THE DEFORMATION IS A FUNCTION OF x^{Dim} ONLY (LOCALLY THE PROBLEM IS INVARIANT FOR TRANSFORMATIONS ORTHOGONAL TO P^B), GIVING

$$\partial^b \partial_{Dim} U^{Dim} = \tilde{P} \, \delta^{b \, Dim}$$

SO

$$U^{Dim} = \tilde{P} (x^{Dim})^2 + x^{Dim}$$

WHERE WE HAVE USED THE FACT THAT FOR $\tilde{P} = 0$ $U = x$.

- $\partial_a U^a = 0$. THE EQUATION READS

$$\partial_a \partial^a U^{Dim} = 0 \longrightarrow U^{Dim} = \frac{P}{\mu} r^2 + br + c \quad \text{IN TWO DIMENSIONS}$$

- DIRICHLET BCS: $\partial \mathcal{M} = \bar{x}^{Dim}$ IMPLIES

$$U^{Dim}(r=R) = 0 \Longrightarrow \underline{U^{Dim} = \frac{P}{\mu}(r^2 - R^2)} \quad \text{PARABOLA}$$

- VON NEUMANN BCS: $\partial_r U^{Dim}(r=R) = 0 \Longrightarrow P = 0$ FREE-FALLING STRING

- WE WOULD EXPECT THE PROFILE OF A MASSIVE STRING OF CONSTANT LENGTH TO BE A CATENARY (I.E. $\cosh r$), WHILE HERE WE GET A PARABOLA. WE CAN LIST SOME MOTIVATIONS:
  - AT SECOND ORDER

$$\cosh x = 1 + \frac{x^2}{2} + O(x^4)$$

  - THE CATENARY SATISFIES THE SET OF EQUATIONS

$$\begin{cases} \displaystyle\int_{x_0}^{x_1} \sqrt{1 + y^2(x)}\, dx = \ell & \text{CONSTANT LENGTH} \\ \displaystyle\partial \int_{x_0}^{x_1} y(x) \sqrt{1 + y^2(x)}\, dx \,/\, \partial y = 0 & \text{MINIMAL OF THE POTENTIAL ENERGY} \end{cases}$$

$$\Big\downarrow$$

$$\underline{y(x)\, y''(x) = 1 + (y'(x))^2}$$

NOTICE THAT HERE THERE IS NO MENTION OF THE MODEL OF THE INTERNAL FORCES (WHICH ALSO CONTRIBUTE TO THE ENERGY BALANCE). OUR MODEL IS THE FIRST NON-CONSTANT TERM APPROXIMATION OF ELASTIC FORCES, EXACT ONLY UNTIL x^2 TERMS.

· ELASTIC WAVES [LL 22,24]

· CONSIDER A FREE BRANE. THE LAGRANGIAN DENSITY READS

$$\mathcal{L} = \frac{1}{2}\frac{dU^J}{dt}\frac{dU_J}{dt} - \frac{1}{2}\sigma^{IJ}U_{IJ}$$

WE CONSIDER SMALL RELATIVE DISPLACEMENTS ($\sigma \propto U$) AND AN EUCLIDEAN AMBIENT SPACE.
THE EOM READ

$$\sigma^{LM} = 2\mu\,\tilde{U}^{LM} + K\,g^{LM}\,\mathrm{tr}\,U$$

$$\underbrace{\frac{d^2U^L}{dt^2}}_{\ddot{U}^L}\,\underbrace{\rho}_{\text{MASS DENSITY}} = \mu\,\partial\cdot\partial U^L + \underbrace{\left(\mu - \frac{2\mu - K}{\mathrm{tr}\,g}\right)}_{\Omega}\partial^L\partial\cdot U$$

· WE CAN DECOMPOSE THE FIELD U^L USING <u>HELMHOLZ DECOMPOSITION</u> AS

$$U^L = T^L + L^L$$

· $\partial_L T^L = 0$ SOLENOIDAL FIELD ⟶ CLOSED (dim − 1) - FORM

· $\partial^L L^M = \partial^M L^L$ GRADIENT FIELD ⟶ EXACT 1-FORM

THIS DECOMPOSITION IS VALID IN Dim = dim, AND WE WILL CONSIDER AN INFINITELY LARGE BRANE

$$\left(\ddot{T}^M + \ddot{L}^M\right)\rho = \mu\left(\partial\cdot\partial T^M + \partial\cdot\partial L^M\right) + \Omega\,\partial^M\partial\cdot L \qquad (*)$$

· WE APPLY ∂_M:

$$\partial_M\left[\rho\,\ddot{L}^M - (\mu+\Omega)\,\partial\cdot\partial L^M\right] = 0$$

· WE CONSTRUCT THE COEFFICIENTS OF THE 2-FORM $dx^I \wedge dx^J$ FOR THE EXPRESSION IN SQUARE BRACKETS

$$\partial^L[\cdots]^M - \partial^M[\cdots]^L = 0$$

· WE HAVE A FIELD WHICH IS EXACT, WHOSE INTEGRAL AROUND ANY CLOSED LOOP IS ZERO (CLOSED), GOING TO ZERO AT INFINITY AND DIFFERENTIABLE: FOR COMPLEX ANALYSIS IT MUST BE ZERO.
WE GET

$$\underbrace{\ddot{L}^M = c_L^2\,\partial\cdot\partial L^M}_{\text{WAVE EQUATION FOR LONGITUDINAL OSCILLATIONS}} \qquad \underbrace{c_L = \sqrt{\frac{\mu+\Omega}{\rho}}}_{\text{LONGITUDINAL SPEED OF SOUND}}$$

· ON THE OTHER HAND, WE FIRST CONSTRUCT THE 2-FORM OF (*)

$$\left(\partial^L\ddot{T}^M - \partial^M\ddot{T}^L\right)\rho = \mu\,\partial\cdot\partial\left(\partial^L T^M - \partial^M T^L\right)$$

· APPLYING NOW ∂_M TO $\ddot{T}^M - \mu\,\partial\cdot\partial T^M$ WE GET ZERO, SO AGAIN WE HAVE A SOLENOIDAL FIELD WITH NO SOURCE, WHICH MUST BE ZERO.

· WE GET

$$\ddot{T}^M = c_T^2\, \partial\cdot\partial T^M \qquad c_T = \sqrt{\frac{\mu}{\rho}}$$

WAVE EQUATION FOR TRANSVERSAL OSCILLATIONS

TRANSVERSAL SPEED OF SOUND

· WE HAVE $\partial\cdot T = 0$, SO THIS KIND OF DEFORMATIONS DOES NOT CHANGE THE VOLUME. SINCE LONGITUDINAL WAVES CAN PROPAGATE ONLY THROUGH A VARIATION OF VOLUME, THIS PROPERTY EXPLAINS THE NAMES ABOVE.

· THANKS TO THE SUPERPOSITION PROPERTY OF ELASTIC WAVES, WE CAN WRITE ANY WAVE AS A SUM OF MONOCHROMATIC WAVES

$$U^L = \mathrm{Re}\left[\underbrace{\bar{U}^L(x)}_{\text{WAVE PROFILE}}\, e^{-i\overbrace{\omega}^{\text{FREQUENCY}} t}\right]$$

WHICH GIVES

$$\partial\cdot\partial L^M + k_L^2\, L^M = 0 \qquad k_L^M = \frac{\omega}{c_L}\hat{u}^M$$

$$\partial\cdot\partial T^M + k_T^2\, T^M = 0 \qquad k_T^M = \frac{\omega}{c_T}\hat{u}^M$$

WAVE VECTORS

VERSOR NORMAL TO WAVE SURFACES

· SINCE $\mu > 0 \wedge k > 0$, $\Omega > 0$ FROM $\mathrm{trg} \geq 2$ ON. THEREFORE WE ALWAYS HAVE (EUCLIDEAN, AT LEAST TWO DIMENSIONS FOR L AND T)

$$\underline{\underline{c_L > c_T}}$$

· IF $\dim < \mathrm{Dim}$ IT IS POSSIBLE TO STUDY THE DYNAMICS OF ELASTIC WAVE IN THE VICINITY OF THE BOUNDARY. LET US ASSUME THE BRANE TO HAVE AN EQUATION OF THE FORM

$$x^{\mathrm{Dim}} \leq 0$$

· A RAYLEIGH WAVE (MONOCHROMATIC, PLAIN, ALONG x^1) IS WRITTEN AS

$$\underline{\underline{U^M = F^M(x^{\mathrm{Dim}})\, e^{i(kx^1 - \omega t)}}}$$

WHERE U CONTAINS BOTH "LONGITUDINAL" AND "TRANSVERSAL" WAVES. PLUGGING THE EXPRESSION INTO THE WAVE EQUATION WE GET

$$\partial_\perp \partial^\perp F^M(x_\perp) = \left[k^2 - \frac{\omega^2}{c^2}\right] F^M(x_\perp)$$

WHERE $x_\perp = x_{\mathrm{Dim}}$ IS THE DIRECTION NORMAL TO THE BOUNDARY AND $x_{/\!/} = x_1$ IS THE DIRECTION OF PROPAGATION OF THE WAVE.

· IF $k^2 - \omega^2/c^2 < 0$ WE HAVE $\partial^2 F = -F$, SO THE SOLUTION IS PERIODIC ALONG $x_\perp$: THIS IS A STANDARD WAVE THAT PROPAGATES AWAY FROM THE BOUNDARY.

· IF $k^2-\omega^2/c^2>0$ WE GET $\partial^2 F = F$, SO THE SOLUTION IS A COMBINATION OF EXPONENTIAL FUNCTIONS

$$F^M(x_\perp) = K_1^M e^{x_\perp \chi} + K_2^M e^{-x_\perp \chi}$$

$$\chi = \sqrt{k^2 - \frac{\omega^2}{c^2}}$$

SINCE WE WANT WAVES THAT STAY CLOSE TO THE BOUNDARY WE SELECT ONLY THE DECAYING ONE:

$$\underline{\underline{U^M = K^M e^{x_\perp \overset{\text{DECAY RATE}}{\chi}} e^{i(kx_{/\!/}-\omega t)}}}$$

· U^M ADMITS A DECOMPOSITION INTO L^M AND T^M, BUT ON THE BOUNDARY THESE TWO TERMS ARE NOT INDEPENDENT, SINCE

$$\underline{\sigma^{1\perp} = 0}$$

(NO FORCES ACTING ON THE BOUNDARY). WE GET THEN

$$\sigma^{1\perp} = 2\mu U^{1\perp} = 0 \longrightarrow \frac{\partial U^1}{\partial x_\perp} = 0 \Rightarrow K^1 = 0 \qquad 1 \neq /\!/$$

$$\searrow \frac{\partial U^{/\!/}}{\partial x_\perp} = -\frac{\partial U^\perp}{\partial x_{/\!/}} \qquad 1 = /\!/$$

THE SAME RESULTS AS PER VON NEUMANN CONDITIONS

$$\sigma^{\perp\perp} = 2\mu U^{\perp\perp} - \left(\frac{2\mu}{\mathrm{tr}\, g} - K\right) \mathrm{tr}\, U = 0$$

WE GET THAT THE DEFORMATIONS ARE CONTAINED IN THE $\langle \perp, /\!/ \rangle$ PLANE ONLY.

· THE "TRANSVERSAL" PART MUST SATISFY $\partial \cdot L = 0$, GIVING

$$\left\| \begin{aligned} T_{/\!/} &= K_T \chi_T e^{ikx_{/\!/} + x_\perp \chi_T - i\omega t} \\ T_\perp &= -i K_T k\, e^{ikx_{/\!/} + x_\perp \chi_T - i\omega t} \\ \chi_T &= \sqrt{k^2 - \omega^2/c_T^2} \end{aligned} \right\|$$

· THE "LONGITUDINAL" PART IS A 1-FORM, GIVING

$$\left\| \begin{aligned} L_{/\!/} &= K_L k\, e^{ikx_{/\!/} + x_\perp \chi_L - i\omega t} \\ L_\perp &= -i K_L \chi_L e^{ikx_{/\!/} + x_\perp \chi_L - i\omega t} \\ \chi_L &= \sqrt{k^2 - \omega^2/c_L^2} \end{aligned} \right\|$$

· INSERTING $U^M = T^M + L^M$ INTO THE CONDITIONS FOR $\sigma^{M\perp}$ WE GET

$$\left. \begin{aligned} K_T (k^2 + \chi_T^2) + 2K_L k \chi_L &= 0 \\ K_L (k^2 + \chi_T^2) + 2K_T k \chi_T &= 0 \end{aligned} \right\} \Longleftrightarrow (k^2 + \chi_T^2)^2 = 4k^2 \chi_T \chi_L$$

· FROM THIS RELATION WE SEE THAT $\omega = c_T \xi\left(\frac{c_T}{c_L}\right) k$, SO THE VELOCITY OF SURFACE WAVES IS LINKED TO THE VELOCITY OF VOLUME WAVES, BUT IT IS NOT THE SAME.