3. RELATIVISTIC BRANES

• RELATIVISTIC FORMULATION [CE 19.2.2, 19.3, ST 6.1 – 6.5, 6.9]

• THE RELATIVISTIC DESCRIPTION NATURALLY EMBEDDS THE CONCEPT OF CAUSALITY AND GIVES A UNIFIED NOTATION TO DISCUSS TIME-DEPENDENT PHENOMENA.

• IN TERMS OF GEOMETRY WE ADD THE TIME COORDINATE AS A 0-TH TERM. TO ENFORCE CAUSALITY WE WILL CONSTRUCT THE SPACETIME METRIC AS

$$\eta_{\mu\nu} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -g_{IJ} \end{array}\right)$$

• WE WANT THE THEORY TO BE COVARIANT AT SIGHT ("LORENTZ INVARIANCE") AND INDEPENDENT OF THE PARAMETRIZATION USED TO DESCRIBE THE BRANES.

• RIPARAMETRIZATION INVARIANCE IS GUARANTEED BY THE MANIFOLD DESCRIPTION OF BRANES.

• "LORENTZ" COVARIANCE IS ENSURED BY THE USAGE OF TENSORS.

• TO IMPLEMENT TIME EVOLUTION ON A p-BRANE $Y^I(\xi)$ WE INCLUDE TIME BOTH IN THE PARAMETERS ξ^a (WHICH WILL NOW DEPEND ON THE TIME) AND AS THE 0-TH COORDINATE OF Y:

$$\underline{Y^\mu(t, \xi(t)) = (t, Y^I(\xi(t)))}$$

• THIS EXPRESSION FOR Y DESCRIBES A NEW MANIFOLD, OF DIMENSION dim+1, IN AN AMBIENT SPACE OF DIMENSION Dim+1. SINCE IT IS A MANIFOLD, WE CAN REPARAMETRIZE IT, SUCH THAT THE 0-TH COORDINATE READS

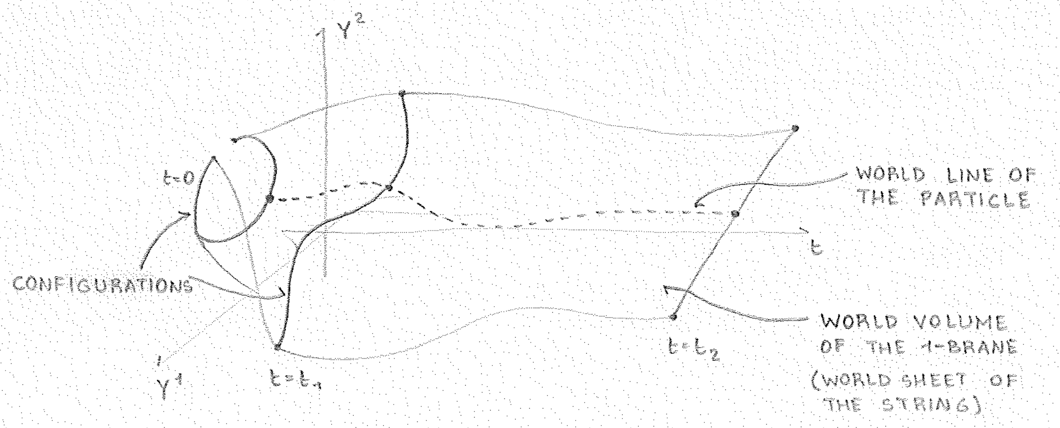
$$\xi^0(t, \xi^a) \Longleftrightarrow t(\xi^\alpha) \equiv Y^0(\xi^\alpha)$$

• WE CALL THE MANIFOLD Y^μ THE WORLD VOLUME, IN ANALOGY WITH THE WORLD LINE OF A PARTICLE.

• WE CALL THE SLICE AT FIXED TIME $t = Y^0$

$$Y^I(\xi(Y^0))$$

THE CONFIGURATION OF THE BRANE AT TIME Y^0.



• TO CONSTRUCT THE ACTION FOR THE p-BRANE WE NEED TO SATISFY:
  - LORENTZ INVARIANCE
  - RIPARAMETRISATION INVARIANCE
  - 0-BRANE HAS ACTION $m \int ds$

• THE WORLD-VOLUME VOLUME SATISFIES ALL THE CHARACTERISTICS ABOVE:

$$\int_{\mathcal{M}_W} dV = \int_{\mathcal{M}_W} \sqrt{|\det \eta|}\; d\xi^0 \dots d\xi^{\dim}$$

· CONSIDERING THE SPACETIME SETTING WE ARE NO LONGER CONSIDERING MAPS FROM $\mathbb{H}^{\dim}$ TO $\mathcal{O}$, BUT FROM $\mathbb{H}^{1,\dim}$ TO $\mathcal{O}_W$ (WHERE THE PEDIX "W" INDICATES THE WORLD VOLUME OF $\mathcal{O}$). THE DIFFERENCE IN SIGN BETWEEN TIME AND SPACE COORDINATES AFFECT THE METRIC, WHICH WILL NOW READ

$$g_{ab} = \frac{\partial X^K}{\partial x^a}\frac{\partial X^K}{\partial x^b} = \frac{\partial X^K}{\partial x^a}\delta_{KL}\frac{\partial X^L}{\partial x^b} \Longrightarrow \underline{\underline{(g_W)_{\alpha\beta} = \frac{\partial X^\mu}{\partial x^\alpha}\eta_{\mu\nu}\frac{\partial X^\nu}{\partial x^\beta}}}$$

· CONSIDER A CHANGE OF PARAMETERS OF THE WORLD VOLUME: $x'(x)$. THE WORLD-VOLUME VOLUME CHANGES AS

$$\int_{\mathcal{M}_W'} dv_W' = \int_{x'^{-1}_W(\mathcal{M}_W)} \sqrt{|\det g'|}\; dx'^0 \cdots dx'^{\dim}$$

$$\hookrightarrow \underbrace{\frac{\partial X^\mu}{\partial x^\gamma}\frac{\partial X_\mu}{\partial x^\delta}}_{g_{\gamma\delta}}\, \frac{\partial x^\gamma}{\partial x'^\alpha}\frac{\partial x^\delta}{\partial x'^\beta}$$

$$= \int_{x^{-1}_W(\mathcal{M}_W)} \sqrt{\underbrace{\left|\det\left(\frac{\partial x}{\partial x'}^T g \frac{\partial x}{\partial x'}\right)\right|}_{\det g \; \det^2 \frac{\partial x}{\partial x'}}}\; dx^0 \cdots dx^{\dim} \left|\det \frac{\partial x'}{\partial x}\right|$$

$$= \int_{x^{-1}_W(\mathcal{M}_W)} \sqrt{|\det g|}\; dx^0 \cdots dx^{\dim}$$

$$= \int_{\mathcal{M}} dv_W$$

SO THE WORLD-VOLUME VOLUME IS REPARAMETRIZATION INVARIANT.

· THE METRIC g IS DEFINED STARTING FROM THE TANGENT SPACE TO THE WORLD VOLUME AT EACH POINT, THEREFORE IT CARRIES INFORMATION ABOUT THE VELOCITY OF THE ELEMENTS OF THE BRANE.

· CONSIDER A PARAMETRIZATION FOR THE WORLD VOLUME AS

$$Y^\mu(\xi) = Y^\mu(t, \sigma^i) = \begin{pmatrix} t \\ Y^i(t,\sigma) \end{pmatrix}$$

THE COORDINATE $Y^0 = t$ SLICES THE WORLD VOLUME INTO CONFIGURATIONS.

· THE METRIC READS

$$(g_W)_{\alpha\beta} = \left(\begin{array}{c|c} 1 - \vec{v}^2 & -\vec{v}\cdot\vec{w}_b \\ \hline -\vec{v}\cdot\vec{w}_a & -\vec{w}_a\cdot\vec{w}_b \end{array}\right) \qquad v^i = \frac{\partial Y^i}{\partial t} \qquad w^i_a = \frac{\partial Y^i}{\partial \sigma^a}$$

THEREFORE

$$\det(g_W)_{\alpha\beta} = (-1)^{\dim}\det(\vec{w}_a\cdot\vec{w}_b)\,\det\left[1 - \left(\vec{v}^2 - \vec{v}\cdot\vec{w}_b\,(\vec{w}_a\cdot\vec{w}_b)^{-1}\,\vec{v}\cdot\vec{w}_a\right)\right]$$

· $\vec{v}$ IS THE <u>NOMINAL VELOCITY</u> OF THE POINT $Y^i(t,\xi)$ OF THE BRANE. IN FULL GENERALITY, Y^i CAN MOVE TRANSVERSAL TO THE BRANE OR LONGITUDINAL TO IT. WE CAN ISOLATE THE TRANSVERSAL COMPONENT (CALLED <u>PHYSICAL VELOCITY</u> IN STRING THEORY) AS

$$\vec{v}_\perp = \vec{v} - \sum_a \frac{\vec{v}\cdot\vec{w}_a}{\|\vec{w}_a\|}\frac{\vec{w}_a}{\|\vec{w}_a\|}$$

· TO HAVE CAUSAL MOTION IN THE TRANSVERSAL DIRECTION WE MUST ASSUME

$$\vec{v}_\perp^2 < 1$$

WHICH MEANS

$$\vec{v}^2 - \sum_{a,b} \vec{v}\cdot\vec{w}_a\;\vec{v}\cdot\vec{w}_b\left[\frac{\vec{w}_a\cdot\vec{w}_b}{\|\vec{w}_a\|^2\,\|\vec{w}_b\|^2} - \frac{2\,\delta_{ab}}{\|\vec{w}_a\|\,\|\vec{w}_b\|}\right] < 1$$

· TO RELATE THIS RESULT TO THE FORM OF g_W WE OBSERVE THAT

$$\left[-\frac{\vec{w}_a \cdot \vec{w}_b}{\|\vec{w}_a\|^2 \|\vec{w}_b\|^2} + \frac{2\delta_{ab}}{\|\vec{w}_a\| \|\vec{w}_b\|} \right] \left[\vec{w}_a \cdot \vec{w}_b \right] =$$

$$= -\frac{\mathrm{tr}\left[\vec{w}_a \cdot \vec{w}_b \; \vec{w}_a \cdot \vec{w}_b \right]}{\|\vec{w}_a\|^2 \; \|\vec{w}_b\|^2} + 2 \; \frac{\|\vec{w}_a\|^2}{\|\vec{w}_a\|^2}$$

· MOVING TO A LOCAL ORTHOGONAL BASIS OF PARAMETERS

$$\vec{w}_a \cdot \vec{w}_b = \|\vec{w}_a\|^2$$

$$= 1$$

SO WE SEE THAT THE CONDITION ON THE TRANSVERSE VELOCITY TO BE SMALLER THAN THE SPEED OF LIGHT GIVES

$$\det (g_W)_{\alpha\beta} \; (-1)^{\dim} > 0$$

· WE PUT CONSTRAINTS ON THE WORLD-VOLUME METRIC REQUIRING CAUSALITY. WE GOT:

- $\det (g_W)_{\alpha\beta} \; (-1)^{\dim} > 0$ FROM TRANSVERSAL VELOCITY
- $g_{00} \in \,]0, 1]$ FROM NOMINAL VELOCITY
- $g_{0i} \, g_{0i} < \mathrm{tr} \, g_{ab}$ FROM LONGITUDINAL VELOCITY

· THE LAST TWO CONSTRAINTS ARE VALID ONLY IF WE ADMIT PHYSICAL DEGREES OF FREEDOM ALONG THE BRANE, I.E. IF WE ASSUME THAT WE ARE ABLE TO DISTINGUISH DIFFERENT POINTS ON THE BRANE AND HOW THEY EVOLVE. THIS IS NOT THE CASE E.G. IN STRING THEORY.

· WE FINALLY ARRIVED TO THE RELATIVISTIC ACTION OF A p-BRANE, CALLED THE NAMBU–GOTO ACTION

$$S = -m \int_{\mathcal{M}_W} dv_W = -m \int \sqrt{(-1)^{\dim} \det (g_W)_{\alpha\beta}} \; d^{\dim+1} \xi$$

HERE m IS THE MASS PER dim-VOLUME OF THE BRANE.

· WE CAN DERIVE THE EQUATIONS OF MOTION BY STATIONARY ACTION PRINCIPLE.

$$\delta \sqrt{(-1)^{\dim} \det g} = \frac{(-1)^{\dim} \delta \det g}{\sqrt{(-1)^{\dim} \det g}} \; \frac{1}{2}$$

$$\delta \det g = \det (g + \delta g) = \det g \, \det \left[1 + g^{-1} \delta g \right] = \det g \left(1 + \mathrm{tr} \left(g^{-1} \delta g \right) \right)$$

$$= \sqrt{(-1)^{\dim} \det g} \quad g^{\alpha\beta} \delta g_{\alpha\beta} \frac{1}{2}$$

$$\cdot\ \delta g_{\alpha\beta} = \delta\left(\frac{\partial Y^\mu}{\partial \xi^\alpha} \frac{\partial Y_\mu}{\partial \xi^\beta}\right) = 2 \frac{\partial Y^\mu}{\partial \xi^\alpha} \frac{\partial \delta Y_\mu}{\partial \xi^\beta}$$

$$= \sqrt{(-1)^{\dim} \det g} \quad g^{\alpha\beta} \frac{\partial Y^\mu}{\partial \xi^\alpha} \frac{\partial \delta Y^\mu}{\partial \xi^\beta}$$

THEREFORE

$$\delta S = +m \int_V \frac{\partial}{\partial \xi^\beta}\left[\sqrt{\mathcal{G}} \ g^{\alpha\beta} \frac{\partial Y^\mu}{\partial \xi^\alpha}\right] \delta Y_\mu \, d^{\dim+1}\xi - \underbrace{m \int_{\partial V} \sqrt{\mathcal{G}} \ g^{\alpha\beta} \frac{\partial Y^\mu}{\partial \xi^\alpha} \delta Y_\mu \, d^{\dim} s_\beta}_{\text{BOUNDARY TERM}}$$

WHERE $\mathcal{G} = (-1)^{\dim} \det g_W$ AND TOTAL DERIVATIVES VANISH. WE GET THE EQUATIONS OF MOTION

$$m \frac{\partial}{\partial \xi^\beta}\left[\sqrt{\mathcal{G}} \ g^{\alpha\beta} \frac{\partial Y^\mu}{\partial \xi^\alpha}\right] = 0$$

· 0-BRANES : FOR POINTLIKE OBJECTS WE HAVE NO INTERNAL D.O.F.

$$\cdot\ g_{\alpha\beta} = g_{00} = 1 - \vec{v}^2$$

$$\cdot\ S = -m \int_{[t_0, t_1]} \sqrt{1 - \vec{v}^2} \ dt = -m \int_{\gamma_W} ds$$

γ_W ↳ WORLD LINE

$$\cdot\ m \frac{\partial}{\partial t}\Big[\underbrace{\sqrt{1-\vec{v}^2} \ \frac{1}{1-\vec{v}^2} \frac{\partial Y^\mu}{\partial t}}_{\frac{\partial Y^\mu}{\partial s}}\Big] = \frac{\partial p^\mu}{\partial s} = 0$$

↳ FREE PARTICLE

· 1-BRANES: STRINGS ARE USUALLY PARAMETRIZED IN TERMS OF τ AND σ

$$\cdot\ g_{\alpha\beta} = \begin{pmatrix} \dot{Y}\cdot\dot{Y} & \dot{Y}\cdot Y' \\ \dot{Y}\cdot Y' & Y'\cdot Y' \end{pmatrix} \qquad \dot{Y}^\mu = \frac{\partial Y^\mu}{\partial \tau}, \quad Y'^\mu = \frac{\partial Y^\mu}{\partial \sigma}$$

$$\cdot\ S = -m \int_{[\tau_0, \tau_1]} \int_{[0, \sigma_1]} \sqrt{(\dot{Y}\cdot Y')^2 - \dot{Y}\cdot\dot{Y} \ Y'\cdot Y'} \ d\sigma d\tau$$

$$\cdot\ \frac{\partial \Pi^\tau_\mu}{\partial \tau} + \frac{\partial \Pi^\sigma_\mu}{\partial \sigma} = 0 \qquad \Pi^a_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_a Y^\mu)}$$

· WE STILL HAVE A BOUNDARY TERM IN THE VARIATION OF THE ACTION TO DISCUSS. EVEN THOUGH IT IS REMOVED ONCE BCS ARE FIXED, WE WANT TO SEE HOW TO APPLY THEM.
· LET US FIRST DERIVE THE VARIATION OF THE ACTION IN A SLIGHTLY DIFFERENT WAY:

$$\delta S = \int_{V_w} \Big[\underbrace{\frac{\partial \mathcal{L}}{\partial Y^\mu}}_{\substack{0 \\ \text{IN OUR CASE}}} \delta Y^\mu + \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\alpha Y^\mu)}}_{\Pi^\alpha_\mu \ \text{CONJUGATED MOMENTA}} \delta(\partial_\alpha Y^\mu) \Big] d^{\dim+1}\xi =$$

$$= -\int_{V_w} \partial_\alpha \Pi^\alpha_\mu \, \delta Y^\mu \, d^{\dim+1}\xi + \int_{\partial V_w} \Pi^\alpha_\mu \, \delta Y^\mu \, \underbrace{d^{\dim}\xi_\alpha}_{\text{BOUNDARY ELEMENT}}$$

· VARIATIONS ARE ZERO AT $t = t_0, t_1$

$$\int_{\partial V \times [t_0, t_1]} \Pi^a_\mu \, \delta Y^\mu \, d^{\dim}\xi_a \, dt$$

$$= -\int_{V_w} \partial_\alpha \Pi^\alpha_\mu \, \delta Y^\mu \, d^{\dim+1}\xi + \int_{\partial V \times [t_0, t_1]} \Pi^a_\mu \, \delta Y^\mu \, d^{\dim}\xi_a \, dt$$

· THE FIRST INTEGRAL GIVES

$$\underline{\partial_\alpha \Pi^\alpha_\mu = 0}$$

WHICH IS A GENERALIZATION OF MOMENTUM CONSERVATION IN THE ABSENCE OF A POTENTIAL.

· TO MAKE THE SURFACE INTEGRAL VANISH WE CAN HAVE EITHER $\delta Y^\mu = 0$ OR $\Pi^a_\mu = 0$.

- $\underline{\delta Y^\mu = 0}$: ON THE BORDER THERE IS NO VARIATION, I.E. THE BOUNDARY OF $\mathcal{M}$ DOES NOT CHANGE. WE GET

$$\left. \frac{\partial Y^i}{\partial t} \right|_{\partial \mathcal{M}} = 0 \qquad \text{DIRICHLET BC}$$

NOTICE THAT $Y^0 = t$ IS EXCLUDED FROM THE CONSTRAINT

- $\Pi^a_\mu = 0$: FREE ENDPOINT CONDITION, WHICH IS THE SAME AS

$$\left. \frac{\partial \mathcal{L}}{\partial(\partial_a Y^\mu)} \right|_{\partial \mathcal{M}} = 0 \qquad \text{VON NEUMANN BC}$$

WE SEE THAT THE SET OF BCS THAT WE CAN IMPOSE IS THE SAME AS FOR THE NONRELATIVISTIC CASE. ALSO HERE, WE CAN APPLY DIFFERENT BCS ON DIFFERENT SUBSETS OF $\partial \mathcal{M}$.

· THE BOUNDARY OF A p-BRANE WITH VON NEUMANN BCS (OR DIRICHLET WITH A SPACE-FILLING D-BRANE) MUST SATISFY THE CONDITION

$$\underline{\underline{\Pi^a_\mu(t)\Big|_{\partial V} = m\sqrt{\mathcal{G}}\; g^{a\beta}\,\frac{\partial Y_\mu}{\partial \xi^\beta}\Big|_{\partial V} = 0}}$$

· FROM $\mu=0$ WE GET

$$\Pi^a_0 = m\sqrt{\mathcal{G}}\; g^{a\beta}\underbrace{\frac{\partial Y_0}{\partial \xi^\beta}}_{g_{0\beta}} = m\sqrt{\mathcal{G}}\; g^a{}_0 = 0$$

$$g^a{}_0 = -\vec{v}\cdot\underbrace{\vec{w}^a}_{\substack{\text{TANGENT}\\ \text{SPACE TO}\\ \text{THE BRANE}}} = 0 \longrightarrow \vec{v} \text{ IS ORTHOGONAL TO } \vec{w}^a,\ \forall a,\ \text{IF } \vec{v}\neq 0$$

|| THE VELOCITY OF THE BOUNDARY IS TRANSVERSAL TO THE BRANE ||

· FROM $\mu=i$ WE GET

$g^{0a} = g^{a0} = 0$ FROM ABOVE

$$\Pi^a_i = m\underbrace{\sqrt{\mathcal{G}}}_{\det(1-\vec{v}^2)}\; g^{ab}\underbrace{\frac{\partial Y_i}{\partial \xi^b}}_{\text{IN GENERAL NON-ZERO}}$$

$v=1$ ON THE BOUNDARY $\longrightarrow$ || THE BOUNDARY MOVES AT THE SPEED OF LIGHT ||

· THE BOUNDARY HAS dim-MEASURE 0, SO IT IS MASSLESS. IT IS THEREFORE REASONABLE TO THINK THAT $v|_{\partial V} = 1$.

· NOETHER'S THEORY [CE 19.3, ST 8.1–8.5

· NOETHER'S THEOREM STATES THAT FOR EVERY DIFFERENTIABLE SYMMETRY OF A SYSTEM (I.E. A DIFFERENTIABLE TRANSFORMATION THAT LEAVES THE ACTION INVARIANT) THERE EXISTS A CONSERVED CURRENT:

$$S[\Phi'] = S[\Phi] \quad \rightleftharpoons \quad \partial_\mu j^\mu_\Phi = 0$$

· AN INFINITESIMAL TRANSFORMATION WITH BRANES IS SLIGHTLY MORE INVOLVED, SINCE WE HAVE

- THE BRANE PARAMETERS $\xi^\alpha = (t = Y^0, \sigma^i)$
- THE SPACETIME POSITION OF THE BRANE $Y^\mu = Y^\mu(\xi)$
- THE SPACETIME x^μ
- FIELDS PRESENT IN THE SPACETIME $F^{\{r\}}(x)$

AND THEIR INFINITESIMAL TRANSFORMATIONS READ

$$
\begin{aligned}
x'^\mu &= x^\mu + \delta x^\mu &&= x^\mu + \Xi^\mu_{(\nu)}\, \varepsilon^{(x,\nu)} \\
\xi'^\alpha &= \xi^\alpha + \begin{pmatrix} \delta Y^0 \\ \delta \xi^a \end{pmatrix} &&= \xi^\alpha + \Omega^\alpha_{(\beta)}\, \varepsilon^{(\xi,\beta)} \\
Y'^\mu(\xi') &= Y^\mu(\xi) + \begin{pmatrix} \delta Y^0 \\ \bar\delta Y^i(\xi) \end{pmatrix} + \partial_a Y^\mu(\xi)\, \delta\xi^a &&= Y^\mu(\xi) + \Upsilon^\mu_{(\beta)}\, \varepsilon^{(\xi,\beta)} \\
F'^r(x') &= F^r(x) + \underbrace{\bar\delta F^r(x)}_{\text{FORM VARIATION}} + \underbrace{\partial_\mu F^r(x)\, \delta x^\mu}_{\text{POINT VARIATION}} &&= F^r(x) + X^r_{(\nu)}\, \varepsilon^{(x,\nu)}
\end{aligned}
$$

$X^r_{(\nu)}$: GENERATORS OF THE TRANSFORMATION

$\varepsilon^{(x,\nu)}$: INFINITESIMAL VARIATION

Υ AND X CONTAIN MORE Y AND F: $X^r_{(\nu)} = X^r{}_{s(\nu)} F^s$

$$\varepsilon = \begin{pmatrix} \varepsilon^{(\xi,\beta)} \\ \varepsilon^{(x,\nu)} \end{pmatrix}$$

· THE ACTION FUNCTIONAL IS INVARIANT (ON THE EQUATIONS OF MOTION) W.R.T. THE TRANSFORMATION ABOVE. THIS MEANS

$$S[\Phi'] = \int_{V'} \mathcal{L}(\Phi', \partial'\Phi')\, d^n x' = \int_V \mathcal{L}(\Phi, \partial\Phi)\, d^n x = S[\Phi]$$

WHICH TRANSLATES TO

$$\int_V \Bigg[\underbrace{\frac{\partial\mathcal{L}}{\partial\Phi^r}\bar\delta\Phi^r + \frac{\partial\mathcal{L}}{\partial(\partial_\rho\Phi^r)}\bar\delta\partial_\rho\Phi^r}_{\left[\underbrace{\frac{\partial\mathcal{L}}{\partial\Phi^r} - \partial_\rho\frac{\partial\mathcal{L}}{\partial(\partial_\rho\Phi^r)}}_{\text{EOM}\to 0}\right]\bar\delta\Phi^r + \partial\left[\underbrace{\frac{\partial\mathcal{L}}{\partial(\partial_\rho\Phi^r)}}_{\Pi^\rho_r}\bar\delta\Phi^r\right]} + \underbrace{\left(\frac{\partial\mathcal{L}}{\partial\Phi^r}\partial_\sigma\Phi + \frac{\partial\mathcal{L}}{\partial(\partial_\rho\Phi^r)}\partial_\sigma\partial_\rho\Phi^r\right)}_{\partial_\sigma\mathcal{L}}\delta x^\sigma \Bigg] \underbrace{\left[1 + \partial_\psi\delta x^\psi\right]}_{\text{MEASURE VARIATIONS}} d^n x =$$

$$= \int_{\partial V} \left[\Pi^\rho_r \bar\delta\Phi^r + \mathcal{L}\,\delta x^\rho \right] \underbrace{d^{n-1}x_\rho}_{\text{SURFACE ELEMENT}}$$

$$\delta x^\rho = \Xi^\rho{}_{(\sigma)}\, \varepsilon^{(\sigma)}$$

$$\bar\delta\Phi^r = \left(X^r{}_{(\sigma)} - \partial_\rho\Phi^r\, \Xi^\rho{}_{(\sigma)} \right) \varepsilon^{(\sigma)}$$

$$= \int_{\partial V} \left[\Pi^{\rho}_{r} X^{r}_{(\sigma)} + \left(\mathcal{L} g^{\rho}_{\omega} - \Pi^{\rho}_{r} \partial_{\omega} \Phi^{r} \right) \Xi^{\omega}_{\ (\sigma)} \right] \varepsilon^{(\sigma)} d^{n-1} x_{\rho}$$

THE EXPRESSION MUST VANISH $\forall \varepsilon^{(\sigma)}$. GOING BACK TO ITS VOLUME FORMULATION:

$$\partial_{\rho} J^{\rho}_{(\sigma)} = 0$$

$$J^{\rho}_{(\sigma)} = \Pi^{\rho}_{r} X^{r}_{(\sigma)} + \left(\mathcal{L} g^{\rho}_{\omega} - \Pi^{\rho}_{r} \partial_{\omega} \Phi^{r} \right) \Xi^{\omega}_{\ (\sigma)}$$

THIS IS THE NOETHER CURRENT AND IT IS CONSERVED ON THE EQUATIONS OF MOTION. HERE x AND Φ ARE PLACEHOLDERS FOR INDEPENDENT VARIABLES (INTEGRATED OVER TO GET THE ACTION) AND FUNCTIONS, RESPECTIVELY.

· GIVEN A CONSERVED CURRED, ITS ASSOCIATED CHARGE IS

$$Q_{(a)} = \int_V J^{0}_{(a)} d^{n-1} \vec{x}$$

AND IT IS CONSERVED:

$$\partial_0 Q_{(a)} = \int_V \partial_0 J^{0}_{(a)} d^{n-1} \vec{x} = -\int_V \partial_i J^{i}_{(a)} d^{n-1} \vec{x} = -\vec{J} \Big|_{\partial V} = 0$$

ASSUMING SUFFICIENTLY FAST DECREASE AT THE BORDER ($J \underset{>}{\sim} x^{-n}$)

· IN THE DESCRIPTION OF BRANES WE HAVE TWO SETS OF INDEPENDENT PARAMETERS: ξ^{α}, WHICH DESCRIBE THE GEOMETRY OF THE BRANE, AND x^{μ}, WHICH DESCRIBE THE GEOMETRY OF THE AMBIENT SPACETIME. WE CAN THEREFORE LOOK FOR CONSERVED CURRENTS ON EITHER ONE OF THE SETTINGS.

· AS A FIRST STEP, LET US REWRITE THE LAGRANGIAN DENSITY FOR THE p-BRANE SO TO MAKE THE DEPENDENCE OVER ξ^{α} AND x^{μ} EXPLICIT. x SPANS THE WHOLE SPACETIME, BUT WE WANT THE DYNAMICS TO TAKE PLACE ONLY ON $Y(\xi)$, WE CAN ASSURE THIS BY ASKING THAT THE INTEGRAL OVER $d^{\mathrm{Dim}+1} x$ ONLY SELECTS THE SPACETIME OCCUPIED BY $Y(\xi)$: THIS CAN BE ACHIEVED BY A $\delta^{(\mathrm{Dim}+1)}(x - Y(\xi))$ AS

$$S = \int_{V_W} \int_{U_W} \mathcal{L} \, d^{\mathrm{Dim}+1} x \, d^{\mathrm{dim}+1} \xi \, , \qquad \mathcal{L} = -m \sqrt{\mathcal{G}} \, \delta^{(\mathrm{Dim}+1)}(x - Y(\xi))$$

· WE START FROM THE SYMMETRIES OF THE AMBIENT SPACETIME AND SEE HOW THESE CONSTRAIN THE GEOMETRY OF THE BRANE.

· THE CONNECTION POINT BETWEEN THE AMBIENT SPACE AND THE BRANE SPACE IS GIVEN BY THE FACT THAT

$$\bar{\delta} Y^{\mu} = \delta x^{\mu}$$

SINCE x AND Y ARE BOTH THE SAME COORDINATE SYSTEM IN THE AMBIENT SPACE.

· GIVEN A METRIC, WE CAN FIND THE CLASS OF TRANSFORMATIONS THAT LEAVE THE SPACE INVARIANT BY IMPOSING

$$g^{\alpha\beta}(x') = g'^{\alpha\beta}(x')$$

I.E., THAT THE TRANSFORMED METRIC IS EQUAL TO THE ORIGINAL ONE.
EXPANDING THIS REQUIREMENT FOR INFINITESIMAL TRANSFORMATIONS WE GET

$$\left.\begin{aligned} g_{\alpha\beta}(x') &= \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta} g_{\mu\nu}(x) \\ x'^\mu &= x^\mu + \varepsilon \xi^\mu(x) \end{aligned}\right\} \longrightarrow \underline{\underline{g_{\gamma\alpha}\partial_\beta \xi^\gamma + g_{\gamma\beta}\partial_\alpha \xi^\gamma + \xi^\gamma \partial_\gamma g_{\alpha\beta} = 0}}$$

$\xi^\alpha(x)$ ARE THE KILLING VECTOR FIELDS OF THE SPACE $\mathcal{M}_g$.

· IF WE CONSIDER THE FLAT MINKOVSKI SPACETIME AS AMBIENT SPACE, WE GET

$$\partial_\beta \xi^\alpha = -\partial^\alpha \xi_\beta$$

· IF α AND β HAVE THE SAME VALUE WE GET

$$\partial_\alpha \xi^\alpha = 0 \longrightarrow \xi^\alpha = a^\alpha \qquad \text{TRANSLATIONS} \longrightarrow \mathbb{R}^{1,\mathrm{Dim}}$$

· IF α OR β ARE 0 AND THE OTHER NOT

$$\partial_0 \xi^\alpha = \partial_\alpha \xi^0 \longrightarrow \qquad \text{BOOSTS}$$

· $\alpha \neq \beta$, $\alpha \neq 0 \neq \beta$

$$\partial_a \xi^b = -\partial_b \xi^a \longrightarrow \qquad \text{ROTATIONS}$$

$$\left.\begin{matrix}\text{BOOSTS}\\ \text{ROTATIONS}\end{matrix}\right\} SO^+(1,\mathrm{Dim})$$

· TRANSLATIONS ARE MODELLED BY (NOTICE THAT $\mathcal{L}$ IS INVARIANT UNDER TRANSLATIONS)

$$\left.\begin{aligned} x'^\mu &= x^\mu + a^\mu & &\longrightarrow & \Xi^\mu{}_{(\nu)} &= \eta^\mu{}_\nu \\ \xi'^\alpha &= \xi^\alpha & &\longrightarrow & \Omega^\alpha{}_{(\beta)} &= 0 \\ Y'^\mu(\xi') &= Y^\mu(\xi) + a^\mu & &\longrightarrow & \Upsilon^\mu{}_{(\nu)} &= \eta^\mu{}_\nu \\ F'^r(x') &= F^r(x) & &\longrightarrow & X^r{}_{(\nu)} &= 0 \end{aligned}\right\} \underline{\underline{J^{\mu\nu} = m\int \sqrt{\mathcal{G}}\, \frac{\partial Y^\mu}{\partial \xi^\alpha}\frac{\partial Y^\nu}{\partial \xi_\alpha}\, \delta^{(\mathrm{Dim}+1)}(x - Y(\xi))\, d^{\dim+1}\xi = T^{\mu\nu}}}$$

STRESS-ENERGY-MOMENTUM TENSOR OF A p-BRANE

· $T^{\mu\nu} \neq 0$ ONLY ON THE BRANE WORLD VOLUME.

· THE CONSERVED CHARGES

$$P^\mu = \int T^{\mu 0}\, d^{\mathrm{Dim}} x = m\int \sqrt{\mathcal{G}}\, \frac{\partial Y^\mu}{\partial t}\, \delta(Y^0(\xi) - t)\, d^{\dim+1}\xi$$

ARE THE TOTAL MOMENTUM OF THE p-BRANE.

· TRANSLATIONS $Y'^{\mu} = Y^{\mu} + a^{\mu}$ ARE ALSO KILLING VECTOR FIELDS OF THE INDUCED METRIC

$$g_{\alpha\beta} = \frac{\partial Y^{\mu}}{\partial \xi^{\alpha}} \frac{\partial Y_{\mu}}{\partial \xi^{\beta}}$$

ON THE BRANE AND LEAVE $\mathcal{L}$ INVARIANT. IF WE CONSIDER Y^{μ} AS A "FIELD" WITH DEPENDENCE OVER ξ^{α} WE CAN DEFINE THE CURRENT COMING FROM THE "INTERNAL" TRANSLATION SYMMETRY AS

$$j^{\alpha}_{\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_{\alpha} Y^{\mu})} = \Pi^{\alpha}_{\mu}$$

WHOSE CONSERVATION LAW AND CHARGE READ

$$\partial_{\alpha} \Pi^{\alpha}_{\mu} = 0 \qquad \text{EOM OF A FREE BRANE}$$

$$\int_{V(t)} \underbrace{\Pi^{0}_{\mu}}_{\substack{\xi\text{-DENSITY OF SPACETIME} \\ \text{MOMENTUM CARRIED BY} \\ \text{THE BRANE}}} d^{\dim}\xi = p_{\mu} \qquad \text{TOTAL MOMENTUM OF THE BRANE}$$

· CHECKING CONSERVATION EXPLICITELY:

$$\frac{dp_{\mu}}{dt} = \int_{V(t)} \frac{\partial \Pi^{0}_{\mu}}{\partial \xi^{0}} d^{\dim}\xi = - \int_{V(t)} \frac{\partial \Pi^{a}_{\mu}}{\partial \xi^{a}} d^{\dim}\xi = - \Pi^{a}_{\mu} \Big|_{\partial V}$$

AS LONG AS WE CONSIDER A FREE ENDPOINT BC THE MOMENTUM IS CONSERVED FOR A BRANE WITH BORDER. IF WE IMPOSE A DIRICHLET BC THE BOUNDARY IS FIXED TO A D-BRANE AND THE MOMENTUM MAY NOT BE CONSERVED (DEPENDING ON THE FORM OF THE D-BRANE).
CLOSED BRANES CONSERVE MOMENTUM.

· $SO^{+}(1, \mathrm{Dim})$ INVARIANCE OF THE AMBIENT MINKOVSKI SPACETIME IS ASSOCIATED TO THE CONSERVATION OF ANGULAR MOMENTUM.
· $\mathcal{L}$ IS INVARIANT UNDER ROTATIONS:

$$\det g'_{\alpha\beta} = \det\left(\frac{\partial Y'^{\mu}}{\partial \xi^{\alpha}} \frac{\partial Y'_{\mu}}{\partial \xi^{\beta}}\right) = \det\left(\frac{\partial Y^{\rho}}{\partial \xi^{\alpha}} \underbrace{\Lambda^{\mu}{}_{\rho} \Lambda_{\mu}{}^{\sigma}}_{\eta^{\sigma}_{\rho}} \frac{\partial Y_{\sigma}}{\partial \xi^{\beta}}\right) = \det g_{\alpha\beta}$$

· THE GENERATORS READ

$$\left.\begin{aligned} \Xi^{\mu}{}_{(\rho\sigma)} &= \eta^{\mu}{}_{\rho} x_{\sigma} - \eta^{\mu}{}_{\sigma} x_{\rho} \\ X_{(\rho\sigma)} &= \Omega_{\rho\sigma} \end{aligned}\right\} \quad \underbrace{J^{\mu\nu\rho} = T^{\mu\rho} x^{\nu} - T^{\mu\nu} x^{\rho} = M^{\mu\nu\rho}}_{\text{ANGULAR MOMENTUM TENSOR}}$$

· $M^{\mu\nu\rho} = M^{\mu[\nu\rho]}$ AND IT IS ZERO OUTSIDE THE p-BRANE

· DUE TO ITS ANTI-SYMMETRY IN THE LAST TWO INDICES, $M^{\mu\nu\rho}$ PRODUCES ONLY $\frac{D(D-1)}{2}$ INDEPENDENT CONSERVED CHARGES: THE COMPONENTS OF THE D-DIMENSIONAL ANGULAR MOMENTUM.

· AS FOR TRANSLATIONS, WE CAN CONSIDER ROTATIONS AS "INTERNAL TRANSFORMATIONS" OF $Y^\mu(\xi)$, WRITING

$$j^\alpha_{\mu\nu} = Y_\mu \Pi^\alpha_\nu - Y_\nu \Pi^\alpha_\mu$$

· WE GET

$$\partial_\alpha j^\alpha_{\mu\nu} = 0 \qquad \text{CURRENT CURL-FREE}$$

$$\int_{V(t)} j^0_{\mu\nu} = M_{\mu\nu} \qquad \text{TOTAL ANGULAR MOMENTUM OF THE BRANE}$$

· IF THE LINEAR MOMENTUM IS CONSERVED SO IS THE ANGULAR MOMENTUM

· CURRENTS ALL HAVE THE SAME STRUCTURE IN THE VARIATION INDEX:
  - j^0 IS THE DENSITY;
  - j^i IS THE FLOW OF CHARGE PER UNIT TIME THROUGH THE SURFACE NORMAL TO THE i-TH DIRECTION.

· $T^{\mu\nu}$ IS SYMMETRIC:
  - $T^{0i} = T^{i0}$: THE i-TH COMPONENT OF THE MOMENTUM IS THE FLOW OF ENERGY IN THE i-TH DIRECTION
  - $T^{ij} = T^{ji}$: SHEAR OF i ALONG j IS AS SHEAR OF j ALONG i

· DYNAMICS [ST 7.3 – 7.5]

· WE WANT TO STUDY THE DYNAMICS OF A p-BRANE IN MORE DETAILS, UNDERSTANDING WHAT CLASS OF SOLUTIONS TO EXPECT.

· FIRST, SINCE WE CONSIDER p-BRANES WITH NO INTERNAL D.O.F., WE NEED TO CONSIDER THE NORMAL VELOCITY $\vec{V}_\perp(Y)$ OF EACH POINT FOR PHYSICAL PROBLEMS. TO OBTAIN $V_\perp$ WE STARTED FROM $\vec{V}$ AND REMOVED ALL COMPONENTS PARALLEL TO ANY $\vec{w}_a$, WHICH BY DEFINITION SPAN THE TANGENT BUNDLE OF THE p-BRANE. WE CAN LOSE SOME OF THE RIPARAMETRIZATION INVARIANCE OF THE p-BRANE TO MAKE $\vec{V} = \vec{V}_\perp$. IN THIS WAY

$$\vec{V}\cdot\vec{w}_a = 0 \longrightarrow \frac{\partial Y^i}{\partial t}\frac{\partial Y_i}{\partial \sigma^a} = 0$$

$$g_{\alpha\beta} = \begin{pmatrix} 1-\vec{v}^{\,2} & 0 \\ 0 & \vec{w}_a\cdot\vec{w}_b \end{pmatrix}$$

· IN THIS WAY THE TIME $t = Y^0 = \xi^0$ IS COMPLETELY SEPARATED FROM THE GEOMETRY OF THE BRANE PARAMETRIZATION.

· WE PERFORM A CHANGE OF VARIABLE OF ξ^a SUCH THAT $\sqrt{G} = 1$. THE CONDITION FOR THIS IS

$$\det \vec{w}^a\cdot\vec{w}^b = 1-\vec{v}^{\,2}$$

$$\downarrow$$

$$\det \partial^a\vec{Y}\cdot\partial^b\vec{Y} + \partial^0\vec{Y}\,\partial^0\vec{Y} = 1$$

· THE EQUATION OF MOTION GETS SIMPLIFIED TO

$$\partial_\alpha\left(\sqrt{1}\;\partial^\alpha Y^\mu\right) = 0$$

$$\mu=0:\quad \partial_0\partial^0 t = 0 \qquad\qquad \mu=m:\quad \partial\cdot\partial\vec{Y} = 0$$

· AT LAST, WE KNOW THAT ON THE BOUNDARY THE MOTION IS TRANSVERSAL TO THE BRANE

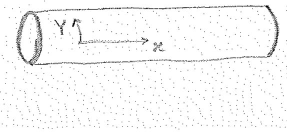
$$\partial_a\vec{Y}\Big|_{\partial V} = 0$$

· THESE FOUR EQUATIONS ALLOW US TO DESCRIBE THE MOTION OF THE BRANE.

· WE CAN HAVE COORDINATES ON THE BRANE WHICH ARE PERIODIC (NO BOUNDARY, "CLOSED" DIRECTIONS) AND SOME WHICH ARE NOT (BOUNDARY, "OPEN" DIRECTIONS). SINCE THE DYNAMICAL EQUATION IS THE WAVE EQUATION WE CAN CONSIDER THE EVOLUTION OF THE BRANE SEPARATELY IN EACH DIRECTION, GETTING THE FULL SOLUTION AS A SUPERPOSITION OF SINGLE SOLUTIONS.

· THE DECOMPOSITION IN DIFFERENT DIRECTIONS DEPENDS ON THE SYMMETRIES OF THE PROBLEM: IF THE BRANE IS A PRODUCT OF INTERVAL WE WILL USE SUPERPOSITIONS OF PLANE WAVES, IF IT HAS ROTATIONAL SYMMETRY WE WILL USE SPHERICAL HARMONICS.

· FOR SIMPLICITY WE CONSIDER A BRANE WHICH IS A PRODUCT OF INTERVALS. SOME DIRECTIONS WILL BE CLOSED, SOME OPEN.



A CYLINDER IS A PRODUCT OF INTERVALS $[0,\ell]\times[0,2\pi[$. THE $[0,\ell]$ DIRECTION IS OPEN, THE $[0,2\pi[$ ONE IS CLOSED.

- IN THE OPEN DIRECTIONS THE WAVE EQUATION GIVES THE SOLUTION

$$\vec{Y}(t,\sigma) = \frac{1}{2}\left[\vec{F}_+(t+\sigma) + \vec{F}_-(t-\sigma)\right]$$

- THE BC AT $\sigma = 0$ GIVES

$$\left.\frac{\partial \vec{Y}}{\partial \sigma}\right|_{\sigma=0} = 0 \longrightarrow \vec{F}_+'(t) - \vec{F}_-'(t) = 0 \longrightarrow \underbrace{\vec{F}_-(t)}_{\forall t \to \forall (t\pm\sigma) \to \text{VALID EVERYWHERE}} = \vec{F}_+(t) + \underbrace{\vec{a}_0}_{\text{CONSTANT TERM}}$$

WE HAVE NOW (WE REABSORBED $\vec{a}_0$ IN $\vec{F}$)

$$\vec{Y}(t,\sigma) = \frac{1}{2}\left[\vec{F}(t+\sigma) + \vec{F}(t-\sigma)\right]$$

- THE BC IN $\sigma = \ell$ GIVES

$$\left.\frac{\partial \vec{Y}}{\partial \sigma}\right|_{\sigma=\ell} = 0 \longrightarrow \vec{F}'(t+\ell) - \vec{F}'(t-\ell) = 0 \longrightarrow \vec{F}(t\pm\sigma+2\ell) = \vec{F}(t\pm\sigma) + 2\ell\,\underbrace{\vec{v}_0}_{\text{CONSTANT VELOCITY}}$$

QUASI-PERIODIC FUNCTION: PERIODIC UP TO A CONSTANT TERM

- THE REMAINING TWO CONDITIONS CAN BE COMBINED. ASSUMING ALL BUT THE CURRENT σ PARAMETER TO BE CONSTANT (I.E. WE ARE EVOLVING ONLY ALONG THE OPEN COORDINATE UNDER STUDY)

$$\frac{\partial \vec{Y}}{\partial \sigma}\cdot\frac{\partial \vec{Y}}{\partial \sigma} \pm 2\,\frac{\partial \vec{Y}}{\partial \sigma}\cdot\frac{\partial \vec{Y}}{\partial t} + \frac{\partial \vec{Y}}{\partial t}\cdot\frac{\partial \vec{Y}}{\partial t} = 1$$

$$\downarrow$$

$$\left(\frac{\partial \vec{Y}}{\partial \sigma} \pm \frac{\partial \vec{Y}}{\partial t}\right)^2 = 1$$

$$\downarrow$$

$$\left\|\frac{d\vec{F}(u)}{du}\right\|^2 = 1 \longrightarrow$$

u IS THE LENGTH PARAMETER OF THE CURVE $\vec{F}(u) = \text{const.}$

u IS A PLACEHOLDER VARIABLE

$\vec{F}(u+du) - \vec{F}(u) = d\vec{F}$, $\|d\vec{F}\| = du$ SO du IS THE DIFFERENCE $\|d\vec{F}\|$

- PUTTING ALL TOGETHER WE GET

$$\vec{Y}(t,\sigma) = \frac{1}{2}\left[\vec{F}(t+\sigma) + \vec{F}(t-\sigma)\right]$$

$$\vec{F}(u+2\ell) = \vec{F}(u) + 2\ell\,\vec{v}_0$$

$$\left\|\frac{d\vec{F}}{du}\right\|^2 = 1$$

IT IS ENOUGH TO KNOW $\vec{F}$ FOR $u \in [0, 2\ell]$ TO DETERMINE $\vec{F}$ $\forall u$

- GIVEN THE FORM ABOVE, WE CAN TAKE $\sigma = 0 \longrightarrow u = t$ AND STUDY THE MOTION OF JUST ONE BOUNDARY FOR $t \in [0, 2\ell]$.
- USING PERIODICITY

$$\vec{F}(0) = \vec{F}(2\ell) + 2\ell\,\vec{v}_0 \longrightarrow \vec{v}_0 = \frac{\vec{F}(0) - \vec{F}(2\ell)}{2\ell}$$

AVERAGE VELOCITY OF $\sigma = 0$

EXAMPLE: ROTATING STRING

$$\vec{F} = \frac{\ell}{2}\begin{pmatrix}\cos(\omega u)\\ \sin(\omega u)\end{pmatrix} \longrightarrow \vec{Y} = \frac{\ell}{2}\begin{pmatrix}\cos(\omega t)\\ \sin(\omega t)\end{pmatrix}$$

- NOTICE THAT THE OSCILLATION $\vec{F} = \frac{\ell}{2}\begin{pmatrix}0\\ \cos(\omega u)\end{pmatrix}$ IS NOT VALID, SINCE $\left\|\frac{d\vec{F}}{du}\right\|^2 \neq 1$.

- IN THE CLOSED DIRECTIONS WE HAVE

$$\vec{Y}(t,\sigma) = \frac{1}{2}\left[\vec{F}_+(\underbrace{t+\sigma}_{u}) + \vec{F}_-(\underbrace{t-\sigma}_{v})\right]$$

- TAKING DERIVATIVES

$$\left.\begin{aligned}\vec{F}'_+(u) &= \frac{\partial\vec{Y}}{\partial\sigma} + \frac{\partial\vec{Y}}{\partial t}\\ \vec{F}'_-(v) &= \frac{\partial\vec{Y}}{\partial t} - \frac{\partial\vec{Y}}{\partial\sigma}\end{aligned}\right\} \xrightarrow{\left(\frac{\partial\vec{Y}}{\partial\sigma}\pm\frac{\partial\vec{Y}}{\partial t}\right)^2 = 1} \|\vec{F}'_+(u)\|^2 = 1 = \|\vec{F}'_-(v)\|^2$$

- IN THE CLOSED DIRECTION WE DO NOT HAVE BCS BUT PERIODICITY CONDITIONS

$$\vec{Y}(t,\sigma+\ell) = \vec{Y}(t,\sigma)$$

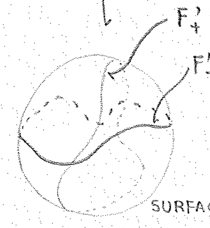
$$\Big\downarrow$$

$$\vec{F}_+(u+\ell) - \vec{F}_+(u) = \vec{F}_-(v-\ell) - \vec{F}_-(v)$$

- $\vec{F}_\pm$ MUST NOT BE PERIODIC OF PERIOD ℓ, BUT THEIR DIFFERENCE MUST BE

- TAKING PARTIAL DERIVATIVES:

$$\left.\begin{aligned}\vec{F}'_+(u+\ell) &= \vec{F}'_+(u)\\ \vec{F}'_-(u+\ell) &= \vec{F}'_-(u)\end{aligned}\right. \longrightarrow \vec{F}'_\pm \text{ ARE PERIODIC UNIT VECTORS}$$



F'_+, F'_- CAN BE PARAMETRIZED AS MOVING POINTS ON A $\mathcal{S}_3$, WITH CLOSED TRAJECTORIES

SURFACE OF t, σ.

- THERE CAN BE POINTS ON THE SPHERE WHERE THE CURVES INTERSECT: WE HAVE

$$\vec{F}'_+(u_0) = \vec{F}'_-(v_0)$$

THESE CURVES DETERMINE THE MOTION UP TO CONSTANTS OF INTEGRATION

• IN THE INTERSECTION POINTS WE GET

$$\left.\frac{\partial \vec{\gamma}}{\partial t}\right|_{t_0,\sigma_0} = \vec{F}'_+(u_0)$$

SINCE $\vec{F}'_+$ IS A UNIT VECTOR THE POINT $\vec{\gamma}$ CORRESPONDING TO σ_0 AT t_0 MOVES AT $v=c$.

• MOREOVER, WE GET

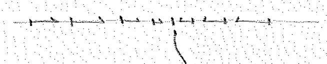
$$\left.\frac{\partial \vec{\gamma}}{\partial \sigma}\right|_{t_0,\sigma_0} = 0 \longrightarrow$$ SINGULARITY ALONG THE CLOSED DIRECTION

THE EXPANSION IN σ AROUND (t_0, σ_0) GIVES

$$\vec{\gamma}(t_0,\sigma) = \vec{\gamma}(t_0,\sigma_0) + \frac{1}{2}\left.\frac{\partial \vec{\gamma}}{\partial \sigma}\right|_{t_0,\sigma_0}(\sigma-\sigma_0)^2 + \dots$$

INVERSION OF THE DIRECTION OF PARAMETRIZATION: <u>CUSP</u>

• GIVEN A CUSP AT (t_0, σ_0), AN INFINITE SET OF CUSPS $(t_0 + m\ell, \sigma_0 + n\ell)$ IS GENERATED



ZERO OF THE TANGENT BUNDLE: INVERSION OF PARAMETRIZATION DIRECTION.