4. CHARGED BRANES

· WE WANT TO DETERMINE THE EVOLUTION LAWS OF AN ELECTRICALLY CHARGED p-BRANE INTO A (Dim+1)-DIMENSIONAL MINKOVSKI SPACETIME.

· DISTRIBUTION-VALUED DIFFERENTIAL FORMS [CE 19.1]

· A p-FORM CORRESPONDS TO A COMPLETELY ANTISYMMETRIC TENSOR FIELD OF RANK p.

- 0-FORMS ARE SCALAR FIELDS, 1-FORMS COVECTORS, 2-FORMS ANTISYMMETRIC 2-TENSORS...
- $0 \leq p \leq D$, WHERE D IS THE DIMENSION OF THE SPACE
- DUE TO ANTISYMMETRY, THE VECTOR SPACE OF p-FORMS IN D DIMENSIONS HAS DIMENSION

$$\binom{D}{p} = \frac{D!}{p!\,(D-p)!}$$

- WE CAN DEFINE A CANONICAL BASIS FOR p-FORMS CAN BE WRITTEN AS

$$\underbrace{d^{\mu_p} \wedge \ldots \wedge d^{\mu_1}}_{\text{WEDGE PRODUCT}}$$

WITH $a \wedge b = -b \wedge a$ AND THE CHOICE OF ELEMENTS IS DONE TO BE CONSISTENT WITH d^{μ_1}, BASIS OF 1-FORMS.

- THE INTRINSIC FORM OF A p-FORM READS

$$\Phi = \frac{1}{p!}\, d^{\mu_p} \wedge \ldots \wedge d^{\mu_1}\ \Phi_{\mu_1 \cdots \mu_p}$$

· WE HAVE THAT p-FORMS AND (D-p)-FORM HAVE THE SAME DIMENSION. WE CAN MAP THE TWO FIELDS THROUGH HODGE DUALITY $*$:

$$*\Phi = \frac{1}{(D-p)!}\, dx^{\mu_{D-p}} \wedge \ldots \wedge dx^{\mu_1}\ \underbrace{\frac{1}{p!}\, \varepsilon_{\mu_1 \cdots \mu_{D-p}}{}^{\nu_1 \cdots \nu_p}\, \Phi_{\nu_1 \cdots \nu_p}}_{\tilde{\Phi}_{\mu_1 \cdots \mu_{D-p}}}$$

· AS FOR THE DUAL VECTOR OPERATION:

$$*^2 = (-1)^{(D+1)(p+1)}$$

(IN MINKOVSKI METRIC)

· THE DIFFERENTIAL, OR EXTERNAL DERIVATIVE, IS THE MAP FROM A p-FORM TO A (p+1)-FORM AS

$$d\Phi = \frac{1}{p!}\, d^{\mu_p} \wedge \ldots \wedge d^{\mu_1} \wedge d^{\mu}\ \partial_{[\mu}\, \Phi_{\mu_1 \cdots \mu_p]}$$

· AS FOR THE BOUNDARY OPERATOR, $d^2 = 0$, SO d IS A COMPLEX, SINCE

$$dd\,\Phi = \frac{1}{p!}\, d^{\mu_p} \wedge \ldots \wedge d^{\mu_1} \wedge d^{\mu} \wedge d^{\nu}\ \partial_{[\nu}\, \partial_{\mu}\, \Phi_{\mu_1 \cdots \mu_p]}$$

$$-\partial_\mu \partial_\nu \qquad + \partial_\mu \partial_\nu$$

· A p-FORM IS CLOSED IF $d\Phi_p = 0$

· A p-FORM IS EXACT IF $\Phi_p = d\Lambda_{p-1}$

- THANKS TO d BEING A COMPLEX, ANY EXACT FORM IS CLOSED.

- TO CORRECTLY DESCRIBE THE ELECTRODYNAMICS OF BRANES WE NEED TO USE DISTRIBUTIONS, AS $\delta^{(D)}(x-y)$. INCLUDING SUCH OBJECTS ALLOWS US TO REWRITE PUNCTURES AND SINGULARITIES IN TERMS OF FUNCTIONALS DEFINED ON THE WHOLE SPACE. WE ARE NOW WORKING IN A CONTRACTIBLE SPACE, THEREFORE THANKS TO POINCARÉ LEMMA: A DISTRIBUTION-VALUED p-FORM IS CLOSED IF AND ONLY IF IT IS EXACT.
- WE FORMALLY OVERCAME A VERY PROBLEMATIC MISMATCH BETWEEN CLOSEDNESS AND EXACTNESS AT THE PRICE OF INTRODUCING POSSIBLY PATHOLOGICAL "FUNCTIONS".

- A CONTRACTIBLE SPACE IS A SPACE WHERE ALL SUBSETS CAN BE SMOOTHLY CONTRACTED TO POINTS.

- THE DIFFERENTIAL AND THE BOUNDARY ARE ANALOGOUS:

$d^2=0$		$\partial^2=0$	
$d\Phi=0$	CLOSED	$\partial N=0$	BORDERLESS
$\Phi=d\Lambda$	EXACT	$M=\partial N$	BOUNDARY
EXACT $\longrightarrow$ CLOSED		BOUNDARY $\longrightarrow$ BORDERLESS	

- LET US NOW USE FORMS TO WRITE THE EQUATIONS OF ELECTRODYNAMICS OF POINT PARTICLES.
- $F_{\mu\nu}$ IS A COMPLETELY ANTISYMMETRIC 2-TENSOR, THEREFORE WE CAN ASSIGN IT A 2-FORM AS

$$F=\frac{1}{2}\,d^{\nu}\wedge d^{\mu}\,F_{\mu\nu}$$

- THE CURRENT j_μ ONLY HAS 1 INDEX, SO IT IS ASSOCIATED TO A 1-FORM

$$j=d^{\mu}j_{\mu}$$

- BIANCHI IDENTITY:

$$\partial_{[\mu}F_{\nu\rho]}=0 \iff dF=0$$

AS SEEN PER THE DEFINITION OF THE DIFFERENTIAL OF A FORM. SINCE THIS FORM IS CLOSED, IT IS ALSO EXACT IN THE SPACE OF DISTRIBUTIONS, THEREFORE THERE EXISTS A 1-FORM $A=d^{\mu}A_{\mu}$ SUCH THAT

$$F=dA \iff F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$$

$$F=dA=dA' \iff d(A-A')=0 \iff A-A' \text{ IS A 0-FORM} \iff A'=A+d\Lambda \text{ EQUIVALENT}$$

- $A'=A+d\Lambda \iff A'_\mu=A_\mu+\partial_\mu\Lambda$ IS A GAUGE TRANSFORMATION.

- MAXWELL EQUATION

$$\underbrace{\partial_\mu F^{\mu\nu}}_{\sim 3\text{-FORM}}=\underbrace{j^{\nu}}_{1\text{-FORM}} \iff d*F=*j$$

1-FORM $\xrightarrow{*\text{ IN } D=4}$ 3-FORM

WE NEED $*F$ BECAUSE dF WOULD BE ZERO AND F AND $*F$ ARE BOTH 2-FORMS IN $D=4$

· COMPONENT BY COMPONENT THE EQUATION READS (AFTER APPLYING $*$ ON BOTH SIDES)

$$d^{\mu}\left(\partial^{\nu} F_{\nu\mu}\right) = d^{\mu} j_{\mu}$$

AS EXPECTED.

· APPLYING d ON BOTH SIDES WE GET

$$\underbrace{0 = d * j}_{\substack{\text{4-FORM} \\ \downarrow * \\ \text{0-FORM IN } D=4}} \Longleftrightarrow \partial_{\mu} j^{\mu} = 0$$

· ALL THE "HODGE DUALITY TRICKS" THAT WE USED ABOVE WERE VALID ONLY IN $D=4$. THE SECOND STEP TOWARDS THE ELECRODYNAMICS OF p-BRANES IS THE GENERALIZATION OF THE BIANCHI AND MAXWELL EQUATIONS TO D DIMENSIONS.

· WE CONSIDER POINTLIKE PARTICLES. WE ASSUME THAT BIANCHI IDENTITY AND LORENTZ EQUATION IN INDEX NOTATION ARE THE SAME AS IN 4 DIMENSIONS. WE GET THEN

$$\begin{matrix} \cdot\, F = \frac{1}{2} d^{\nu} \wedge d^{\mu} F_{\mu\nu} & & dF = 0 \quad \text{BIANCHI IDENTITY} \\ & \longrightarrow & \\ \cdot\, j = d^{\mu} j_{\mu} & & \underbrace{d * F}_{(D-1)\text{-FORM}} = (-1)^{D} \underbrace{* j}_{(D-1)\text{-FORM}} \quad \text{MAXWELL EQUATION} \end{matrix}$$

· ELECTRODYNAMICS OF BRANES [CE 19.2.3 – 19.2.4]

· WE NEED A CONSERVED ELECTRIC CURRENT ASSOCIATED TO A BRANE IN ORDER TO COUPLE IT TO THE ELECTROMAGNETIC FIELD. SUCH CURRENT MUST BE

- A TENSOR FIELD $j^{\mu\cdots}(x)$ TAKING VALUES ON THE AMBIENT SPACE x;
- MUST BE NON-ZERO ONLY ON THE BRANE WORLD VOLUME;
- MUST BE REPARAMETRIZATION INVARIANT;
- MUST BE CONSERVED: $\partial_\mu j^{\mu\cdots}(x) = 0$

· THE FIRST TWO CONDITIONS INDICATE THAT THE CURRENT MUST CONTAIN A

$$\delta^{(D)}(x - Y(\xi))$$

SUCH THAT UNDER INTEGRATION OVER x ONLY THE p-VOLUME OCCUPIED BY THE BRANE IS SELECTED.

· WE ALREADY KNOW THE FORM OF A 0-BRANE CURRENT:

$$j^\mu(x) = e \int U^\mu_s \; \delta^{(D)}(x - Y(s)) \underbrace{ds}_{\substack{\text{WORLDLINE PARAMETER} \\ \text{(LIKE PROPER TIME)}}}$$

WE EXPECT TO RETRIEVE IT FROM THE GENERAL EXPRESSION. WE ARE THEREFORE LOOKING FOR

$$\int \sigma(U)\, \delta^{(D)}(x - Y(\xi))\, d^{p+1}\xi$$

· WE NEED AT LEAST ONE FREE INDEX μ FROM THIS EXPRESSION, AND IT SHOULD COME FROM U. WE CAN THEN ASSUME THAT $\sigma(U)$ IS A POLYNOMIAL IN U.

· WE NEED REPARAMETRIZATION INVARIANCE. CONSIDER

$$\xi'^\alpha = \frac{1}{k}\xi^\alpha$$

$$\Big\downarrow$$

$$d^{p+1}\xi' = \frac{1}{k^{p+1}} d^{p+1}\xi$$

$$U'^\mu_\alpha = k\, U^\mu_\alpha$$

SO

$$\int \sigma(U)\, \delta^{(D)}(x - Y(\xi))\, d^{p+1}\xi = \frac{1}{k^{p+1}} \int \sigma(kU)\, \delta^{(D)}(x - Y(\xi))\, d^{p+1}\xi$$

$$\Big\downarrow$$

$$\sigma(U) = \frac{1}{k^{p+1}}\sigma(kU) \longrightarrow \sigma \text{ IS HOMOGENEOUS OF DEGREE } p+1$$

· SINCE $d^{p+1}\xi$ AND $(U)^{p+1}$ TRANSFORM ONE THE OPPOSITE OF THE OTHER WE CAN CONSIDER CONSTRUCTING THE $(p+1)$-VOLUME WITH U. THIS TURNS OUT TO BE THE RIGHT CHOICE, SO WE HAVE

$$j^{\mu_1\cdots\mu_{p+1}}(x) = e\int U^{\mu_1}_{\alpha_1} \cdots U^{\mu_{p+1}}_{\alpha_{p+1}}\, \varepsilon^{\alpha_1\cdots\alpha_{p+1}}\, \delta^{(D)}(x - Y(\xi))\, d^{p+1}\xi$$

WHERE INVARIANCE IS ENSURED BY THE DETERMINANT IDENTITY

$$K_{\alpha_1}{}^{\beta_1} \cdots K_{\alpha_n}{}^{\beta_n}\, \varepsilon^{\alpha_1\cdots\alpha_n} = \det K\; \varepsilon^{\beta_1\cdots\beta_n}$$

· NOTICE THAT $j^{\mu_1\cdots\mu_{p+1}}$ IS COMPLETELY ANTISYMMETRIC, SINCE

$$U^{\mu_1}_{\alpha_1}\dots U^{\mu_n}_{\alpha_n}\,\varepsilon^{\alpha_1\cdots\alpha_n} = U^{\mu_1}_{\alpha_2}U^{\mu_2}_{\alpha_1}\dots U^{\mu_n}_{\alpha_n}\,\varepsilon^{\alpha_2\alpha_1\cdots\alpha_n}$$
$$= -U^{\mu_1}_{\alpha_2}U^{\mu_2}_{\alpha_1}\dots U^{\mu_n}_{\alpha_n}\,\varepsilon^{\alpha_1\alpha_2\cdots\alpha_n}$$
$$= -U^{\mu_2}_{\alpha_1}U^{\mu_1}_{\alpha_2}\dots U^{\mu_n}_{\alpha_n}\,\varepsilon^{\alpha_1\cdots\alpha_n}$$

· SINCE j IS COMPLETELY ANTISYMMETRIC IT DOES NOT REALLY MATTER WHAT INDEX WE USE TO CALCULATE ITS DIVERGENCE.

· IN A SLOPPY WAY WE CAN WRITE

$$\partial_{\mu_1} j^{\mu_1\cdots\mu_{p+1}} = e\int \partial_{\mu_1}\left(U^{\mu_1}_{\alpha_1}\cdots U^{\mu_{p+1}}_{\alpha_{p+1}}\right)\varepsilon^{\alpha_1\cdots\alpha_{p+1}}\,\delta^{(D)}\left(x-Y(\xi)\right)d^{p+1}\xi\, +$$
$$+\, e\int U^{\mu_1}_{\alpha_1}\cdots U^{\mu_{p+1}}_{\alpha_{p+1}}\,\varepsilon^{\alpha_1\cdots\alpha_{p+1}}\,\underbrace{\partial_{\mu_1}\delta^{(D)}\left(x-Y(\xi)\right)d^{p+1}\xi}_{\substack{\text{IBPS}\\ -e\int \partial(U\cdots U)\,\varepsilon\,\delta\, d\xi}} = 0$$

SINCE δ ARE ZERO AT THE BOUNDARY

· $j^{\mu_1\cdots\mu_{p+1}}$ IS COMPLETELY ANTISYMMETRIC, THEREFORE IT CAN BE WRITTEN AS A FORM:

$$j = \frac{1}{(p+1)!}\, d^{\mu_{p+1}}\wedge\cdots\wedge d^{\mu_1}\, j_{\mu_1\cdots\mu_{p+1}}$$

THEREFORE

$$\partial_{\mu_1} j^{\mu_1\cdots\mu_{p+1}} \rightleftharpoons d\underbrace{*\underbrace{j}_{(p+1)\text{-FORM}}}_{(D-p-1)\text{-FORM}\,\to\,\text{CLOSED}\,\to\,\text{EXACT}} = 0$$

with $d * j$ a $(D-p)$-FORM.

· THE FORM WE FOUND OF BIANCHI IDENTITY AND MAXWELL EQUATIONS IN TERMS OF DIFFERENTIAL FORMS IS GENERAL AND DOES NOT DEPEND ON THE DEGREE OF THE FORMS, SO WE WANT TO USE IT FOR GENERAL p-BRANES.

· MAXWELL EQUATION

CUSTOMARY OVERALL SIGN

$$d * F = (-1)^{D+p}\, \underbrace{* j}_{(D-p-1)\text{-FORM}}$$

F: $(p+2)$-FORM; $*F$: $(D-p-2)$-FORM

TELLS US THAT F IS A $(p+2)$-FORM

$$F = \frac{1}{(p+2)!}\, d^{\mu_{p+2}}\wedge\cdots\wedge d^{\mu_1}\, F_{\mu_1\cdots\mu_{p+2}}$$

WITH $F_{\mu_1\cdots\mu_{p+2}}$ COMPLETELY ANTISYMMETRIC.

· WE CAN WRITE IN COMPONENTS:

$$\partial_{[\mu_1} F_{\mu_2\cdots\mu_{p+3}]} = 0$$
$$\partial_\mu F^{\mu\mu_1\cdots\mu_{p+1}} = j^{\mu_1\cdots\mu_{p+1}}$$

· BIANCHI IDENTITY

$$dF = 0$$

TELLS US THAT F IS EXACT (AS DISTRIBUTION) AND THEREFORE ADMITS THE PRIMITIVE A, A (p+1) - FORM

$$F = dA$$

$$A = \frac{1}{(p+1)!} d^{\mu_{p+1}} \wedge \cdots \wedge d^{\mu_1} \underbrace{A_{\mu_1 \cdots \mu_{p+1}}}_{\text{ANTISYMMETRIC TENSOR POTENTIAL}}$$

· WE STILL HAVE GAUGE INVARIANCE:

$$A' = A + d\Lambda$$

DELIVERS THE SAME F AS A. THE GENERALIZED GAUGE FUNCTION Λ IS A p-FORM THAT READS

$$\Lambda = \frac{1}{p!} d^{\mu_p} \wedge \cdots \wedge d^{\mu_1} \underbrace{\Lambda_{\mu_1 \cdots \mu_p}}_{\text{ANTISYMMETRIC}}$$

· IN COMPONENTS, THE ABOVE RELATIONS READ

$$F_{\mu_1 \cdots \mu_{p+2}} = (p+2)\, \partial_{[\mu_1} A_{\mu_2 \cdots \mu_{p+2}]}$$

$$A'_{\mu_1 \cdots \mu_{p+1}} = A_{\mu_1 \cdots \mu_{p+1}} + (p+1)\, \partial_{[\mu_1} \Lambda_{\mu_2 \cdots \mu_{p+1}]}$$

· EXAMPLE: POINTS & D=4

· THE ELECRODYNAMICS OF CHARGED POINT PARTICLES IN D DIMENSION IS VERY SIMILAR TO THE ONE IN 1+3 DIMENSIONS. THE ONLY DIFFERENCE IS THAT WE HAVE MORE VARIABILITY OF THE INDICES.

· $E^i = F^{i0}$, SO $\vec{E}$ IS A VECTOR IN D DIMENSIONS
· $B^{ij} = F^{ij}$, AND NOW WE CANNOT IDENTIFY IT WITH A VECTOR THROUGH ε.
· LORENTZ EQUATIONS READ

$$\frac{dp^i}{dt} = e\left(E^i + B^{ij} v^j\right) \qquad \frac{dp^0}{dt} = e\, \vec{v} \cdot \vec{E}$$

· BRANES GENERATE F AS AN ANTISYMMETRIC TENSOR OF RANK (p+2), WHICH HAS $\binom{D}{p+2}$ DOF. WE CAN WRITE THEM AS

$$E^{i_1 \cdots i_{p+1}} = F^{i_1 \cdots i_{p+1} 0} \qquad B^{i_1 \cdots i_{p+2}} = F^{i_1 \cdots i_{p+2}}$$

WHICH ARE THE GENERALIZED ELECTRIC AND MAGNETIC TENSOR FIELDS.

· LET US CONSIDER $D=4$.

· 0-BRANES GENERATE $\vec{E}$ AND $\vec{B}$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

· 1-BRANES GENERATE $E^{ij} \xrightarrow{*} \vec{E}$ $\quad$ $F_{\mu\nu\rho} = \partial_\mu A_{\nu\rho} + \partial_\nu A_{\rho\mu} + \partial_\rho A_{\mu\nu} \xrightarrow{*} F_\mu$

$B^{ijk} \xrightarrow{*} B$

· 2-BRANES GENERATE $E^{ijk} \xrightarrow{*} E$ $\quad$ $F_{\mu\nu\rho\sigma} = \partial_\mu A_{\nu\rho\sigma} + \partial_\nu A_{\rho\sigma\mu} + \partial_\rho A_{\sigma\mu\nu} + \partial_\sigma A_{\mu\nu\rho}$

$B^{ijkl} = 0$ $\quad$ $\Big\downarrow *$

F

· 3-BRANES GENERATE $E^{ijkl} = 0$ $\quad$ $F_{\mu\nu\rho\sigma\tau} = 0$

$B^{ijklm} = 0$

· 4-BRANES GENERATE $A^{ijklm} = 0$ $\quad$ $F_{\mu\nu\rho\sigma\tau\eta} = 0$

$B^{ijklmn} = 0$

· WE SEE THAT CHARGED BRANES CONVEY A COMPLETELY DIFFERENT TYPE OF INTERACTION, THAT CANNOT BE REDUCED TO THE ONE COMING FROM POINTLIKE PARTICLES. THE REASON FOR THAT CAN BE SEEN TO BE REPARAMETRIZATION INVARIANCE, WHICH IS A STRONGER REQUIREMENT THAN HAVING A DISTRIBUTION OF POINTLIKE PARTICLES ARRANGED IN THE FORM OF A BRANE: AS WE HAVE SEEN, REPARAMETRIZATION INVARIANCE REMOVES BRANE-PARALLEL COMPONENTS FROM THE PHYSICAL VELOCITY, DELIVERING A COMPLETELY NEW OBJECT.

· VARIATIONAL METHOD

· THE DYNAMICS OF THE CHARGED p-BRANE IS GIVEN BY THREE PARTS:
- EVOLUTION OF THE EM FIELD
- EVOLUTION OF THE p-BRANE
- INTERACTION

· TAKING INSPIRATION FROM THE 0-BRANE ACTION

$$\mathcal{S}_0^{(D)}[A,F,Y,U] = \int_{V_W} \Big[\underbrace{-\frac{1}{4} F^{\mu\nu} F_{\mu\nu}}_{\text{EVOLUTION OF THE FIELD}} \underbrace{- A_\mu j^\mu}_{\text{INTERACTION}} \underbrace{- m \int_{\xi_1}^{\xi_2} \delta^{(D)}(x - Y(\xi)) \sqrt{\frac{\partial Y^\mu}{\partial \xi} \frac{\partial Y_\mu}{\partial \xi}}\, d\xi}_{\text{EVOLUTION OF THE PARTICLE}} \Big] d^D x$$

WE WRITE THE ACTION FOR THE p-BRANE IN D DIMENSIONS AS

$$\mathcal{S}_p^{(D)}[A,F,Y,U] = \int_{V_W} \Big(\underbrace{\frac{(-1)^{p+1}}{(p+1)!}}_{\text{MINUS SIGN TO ENSURE POSITIVITY OF ENERGY}} \Big[\frac{1}{2(p+2)} \underbrace{F^{\mu_1 \cdots \mu_{p+2}} F_{\mu_1 \cdots \mu_{p+2}}}_{①} + \underbrace{A_{\mu_1 \cdots \mu_{p+1}} j^{\mu_1 \cdots \mu_{p+1}}}_{②} \Big] \underbrace{- m \int_{\Xi} \delta^{(D)}(x - Y(\xi)) \sqrt{G}\, d^{p+1}\xi}_{③} \Big) d^D x$$

· THE ACTION IS LORENTZ AND REPARAMETRIZATION MANIFESTLY INVARIANT, WE NEED TO CHECK THAT IT IS GAUGE INVARIANT AS WELL. WE PROVED THAT F IS GAUGE INVARIANT, SO IS ①. ③ DOES NOT HAVE F'S OR A'S, SO IN GAUGE INVARIANT. FOR ②:

$$\int_{V_W} A'_{\mu_1 \cdots \mu_{p+1}} j^{\mu_1 \cdots \mu_{p+1}} d^D x = \int_{V_W} A_{\mu_1 \cdots \mu_{p+1}} j^{\mu_1 \cdots \mu_{p+1}} d^D x + \underbrace{\int_{V_W} \partial_{\mu_1} \Lambda_{\mu_2 \cdots \mu_{p+1}} j^{\mu_1 \cdots \mu_{p+1}} d^D x}_{\text{IBPS}}$$

$$- \int_{V_W} \Lambda_{\mu_2 \cdots \mu_{p+1}} \underbrace{\partial_{\mu_1} j^{\mu_1 \cdots \mu_{p+1}}}_{0} d^D x + \int_{V_W} \partial_{\mu_1} \underbrace{\left[\Lambda_{\mu_2 \cdots \mu_{p+1}} j^{\mu_1 \cdots \mu_{p+1}} \right]}_{0 \text{ ON THE BOUNDARY}} d^D x$$

$$= \int_{V_W} A_{\mu_1 \cdots \mu_{p+1}} j^{\mu_1 \cdots \mu_{p+1}} d^D x$$

SO ALL THE PIECES OF THE ACTION ARE GAUGE INVARIANT.

· NOTICE THAT THE BRANE IS NOT A FIELD: IT DOES NOT TRANSFORM UNDER GAUGE.

· THIS ACTION DEFINES THE LAGRANGIAN DENSITY

$$\mathcal{L} = \frac{(-1)^{p+1}}{(p+1)!} \left[\frac{1}{2(p+2)} F^{\mu_1 \cdots \mu_{p+2}} F_{\mu_1 \cdots \mu_{p+2}} + A_{\mu_1 \cdots \mu_{p+1}} j^{\mu_1 \cdots \mu_{p+1}} \right] - m \int_{\Xi} \sqrt{G}\, \delta^{(D)}(x - Y(\xi))\, d^{p+1}\xi$$

· TAKING THE EULER–LAGRANGE EQUATIONS:

· VARIATIONS W.R.T. A, F : $\partial_{\mu_1} F^{\mu_1 \cdots \mu_{p+2}} = 0$ → MAXWELL EQUATION

· VARIATIONS W.R.T. Y, U : $m\, \partial_\beta \left[\sqrt{G}\, \partial^\beta Y^\mu \right] = (-1)^p e \underbrace{F^{\mu \mu_1 \cdots \mu_{p+1}} U^{\alpha_1}_{\mu_1} \cdots U^{\alpha_{p+1}}_{\mu_{p+1}} \varepsilon_{\alpha_1 \cdots \alpha_{p+1}}}_{\text{LORENTZ EQUATION}}$

· WE CAN DETERMINE THE STRESS-ENERGY-MOMENTUM TENSOR OF THE INTERACTING THEORY AS

$$T^{\mu\nu} = \underbrace{T_{em}^{\mu\nu}}_{\substack{\text{SYMMETRIC} \\ \text{EM PART}}} + \underbrace{T_b^{\mu\nu}}_{\text{BRANE PART}}$$

· NOTICE THAT SYMMETRISING THE EM PART REMOVES CONTRIBUTIONS FROM INTERACTIONS, AS LONG AS LORENTZ EQUATION IS SATISFIED.

· WE HAVE

$$T_{em}^{\mu\nu} = \frac{(-1)^{p+1}}{(p+1)!}\left[F^{\mu\alpha_1\cdots\alpha_{p+1}} F^{\nu}{}_{\alpha_1\cdots\alpha_{p+1}} - \frac{1}{2(p+2)}\eta^{\mu\nu} F^{\alpha_1\cdots\alpha_{p+2}} F_{\alpha_1\cdots\alpha_{p+2}} \right]$$

· T_{em} AND T_b ARE GAUGE AND REPARAMETRISATION INVARIANT

· THE $(-1)^{p+1}$ IS INCLUDED TO ENSURE POSITIVITY OF THE ACTION.