# Introduction to Supersymmetry

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# Literature

## Quantum Field Theory and Standard Model


- Michio Kaku
  "Quantum field theory – A modern introduction"

- Michael E. Peskin, Daniel V. Schroeder
  "An Introduction To Quantum Field Theory"

- Okun Lev Borisovich
  "Leptons And Quarks"

- Wolfgang Hollik
  "Quantum field theory and the Standard Model"
  https://arxiv.org/abs/1012.3883


## Supersymmetry


- Stephen P. Martin
  "A Supersymmetry Primer"
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- Manuel Drees, Rohini Godbole, Probir Roy
  "Theory And Phenomenology Of Sparticles"

# Chapter 1

# Motivation to look beyond the SM

The standard model of particle physics (SM) is very successful and experimentally well confirmed. However, some questions can't be addressed within the SM.

## 1.1 Observations

### 1.1.1 Dark Matter

The energy budget of the universe is well known today:

| | | |
|---|---|---|
| Visible Matter | 0.03% | Heavy Elements |
| | 0.3% | Neutrinos |
| | 0.5% | Starts |
| | 4 % | Free hydrogen and helium |
| Dark Matter | 25 % | Weakly interacting new particle (WIMP)? |
| Dark Energy | 70% | ??? |

⇒ The SM can only explain 4.9% of the entire energy in the universe

### 1.1.2 Baryon Asymmetry

We don't see any anti-matter in the observable universe. However, the Big Bang should have produced equal amounts of matter and anti-matter, i.e. the asymmetry must have been introduced later.
In general: one needs interactions which violate CP (charge-parity) to break the symmetry between matter and anti-matter.

⇒ The amount of CP violation in the SM is too small to explain the observed matter–anti-matter asymmetry

## 1.2 Experimental deviations

Not all experiments are in perfect agreement with the SM. In some observables, a sizeable deviation was found

**Anomalous magnetic dipole moment**

The magnetic momentum of an elementary particle is given by

$$m_S = -\frac{g\mu_B S}{\hbar} \tag{1.1}$$

$\mu_B$: Bohr magneton; $S$: Spin
The $g$ factor is predicted to be **2** by Dirac's theory, but higher order effects change this.:

$$\text{Anomalous magnetic moment} \qquad a = \frac{g-2}{2} \tag{1.2}$$

The anomalous magnetic moments are among the best measured and most precisely calculated observables:

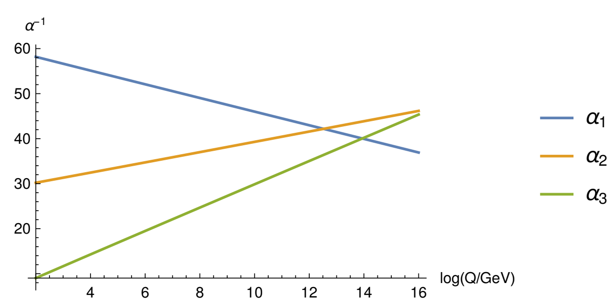
$$a_\mu^{\text{SM}} = 0.001\ 165\ 918\ 04\ (51) \tag{1.3}$$
$$a_\mu^{\text{exp}} = 0.001\ 165\ 920\ 9(6) \tag{1.4}$$

⇒ There is a 3.5 $\sigma$ deviation between the measured anomalous magnetic moment of the myon and the SM prediction

## 1.3 Theoretical Issues

### 1.3.1 Gauge coupling unification

The coupling strength between particles is an energy dependent quantity. The energy dependence is described by the renormalisation group equations (RGEs). For the three gauge couplings of the SM one finds the following behaviour:




⇒ The gauge couplings in the SM don't unify. However, a grand unified theory (GUT) like $SO(10)$ or $SU(5)$ predict such an unification.

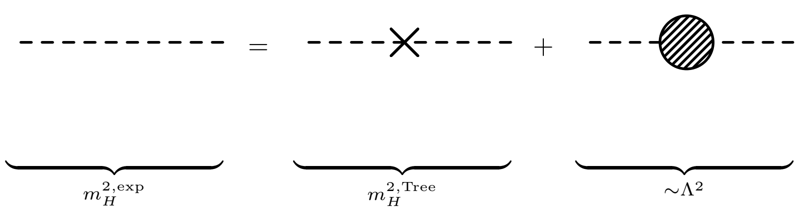
It's not possible to embed the SM in a GUT theory without introducing new matter. It's not clear at which scale the new particles come into play. However, the lighter they are, the bigger their impact is: less particles are needed in low-scale BSM models.

### 1.3.2 Hierarchy problem

The Higgs particle is the only fundamental scalar in the SM. While fermion and vector boson masses are protected by symmetries (chiral and gauge symmetries) against large radiative corrections, the masses of scalars don't have such a protection mechanism. Therefore, the observable mass is given by

$$m^{2,\text{obs}} = m^{2,\text{Tree}} + \delta m^2 \tag{1.5}$$
$$\simeq m^{2,\text{Tree}} + \Lambda^2 \tag{1.6}$$

where $m^{2,\text{Tree}}$ is the mass parameter in the Lagrangian and $\Lambda$ is the scale of new physics. We know that (at least) one scale exists at which new interactions come into play: the Planck scale ($M_P \sim 10^{18}$ GeV) at which gravity becomes important.




(1.7)

⇒ The SM has no natural explanation why the observed Higgs mass is $\sim 125$ GeV, but it demands a cancellation of 32 digits between unrelated parameters.

## 1.4 Why supersymmetry?

Supersymmetry (SUSY) provides possible explanations for all these questions:

- New Particles can form the DM
- New sources of CP violation to generate the Baryon asymmetry
- New loop contributions to $a_\mu$
- Changes the running of gauge couplings → Unification!
- The Higgs mass is protected by the new symmetry and naturally light

Because of these reasons, minimal supersymmetry was for a long time the top candidate for an extensions of the SM. However, with the negative searches at LHC the picture is changing: heavier SUSY masses introduce a new (small) hierarchy problem in the theory. Nevertheless:

- Other benefits of SUSY (dark matter, gauge coupling unification, CP violation) are hardly affected
- The corrections to the Higgs mass are only logarithmic dependent on the SUSY scale, not quadratic as in the SM alone

- There are still unexplored corners in which light SUSY particles are possible within minimal supersymmetry
- There is an increasing interest in non-minimal SUSY models which avoid the small hierarchy problem

# Chapter 2

# Basics

## 2.1 Notations and conventions

- Natural units (formally $\hbar = c = 1$) are used everywhere.
- Lorentz indices are always denoted by Greek characters, $\mu, \nu, .. = 0, 1, 2, 3$.
- Four-vectors for space–time coordinates and particle momenta are written as

$$x = (x^\mu) = (x^0, \vec{x}), \quad x^0 = t\,,$$
$$p = (p^\mu) = (p^0, \vec{p}), \quad p^0 = E = \sqrt{\vec{p}^{\,2} + m^2}\,.$$

- Co- and Contravariant vectors are related by

$$a_\mu = g_{\mu\nu}\, a^\nu,$$

with the metric tensor

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- The 4-dimensional scalar product is

$$a^2 = g_{\mu\nu}\, a^\mu a^\nu = a_\mu a^\mu, \quad a \cdot b = a_\mu b^\mu = a^0 b^0 - \vec{a} \cdot \vec{b}\,.$$

- Covariant and contravariant components of the derivatives are written as

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = g_{\mu\nu}\, \partial^\nu, \quad \partial^\nu = \frac{\partial}{\partial x_\nu} \qquad [\, \partial^0 = \partial_0, \;\; \partial^k = -\partial_k \,]\,,$$
$$\Box = \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \Delta\,.$$

## 2.2 Group Theory

### 2.2.1 Axioms

A collection of elements $g_i$ form a group if the following conditions are fulfilled:

**a)** **Closure** under a multiplication operator; i.e., if $g_i$ and $g_j$ are members of the group, then $g_i \cdot g_j$ is also a member of the group

**b)** **Associativity** under multiplication; i.e.

$$g_i \cdot (g_j \cdot g_k) = (g_i \cdot g_j) \cdot g_k \tag{2.1}$$

**c)** **An identity element**; i.e., there exist an element **1** such that

$$\mathbf{1} \cdot g_i = g_i \cdot \mathbf{1} = g_i \tag{2.2}$$

**d)** **An inverse**; i.e. every element $g_i$ has an element $g_i^{-1}$ such that

$$g_i \cdot g_i^{-1} = \mathbf{1} \tag{2.3}$$

### 2.2.2 Lie Groups

#### 2.2.2.1 Definition

Lie Groups are both groups and differentiable manifolds.

Any group element continuously connected to the identity can be written

$$U = e^{i\Theta_a T^a} \tag{2.4}$$

where the $\Theta_a$ is a real parameter and the $T^a$ are the group generators, which live in the Lie Algebra.

The generators $T^a$ , which generate infinitesimal group transformations, form the Lie Algebra.

The Lie algebra is defined by its commutation relations

$$[T^a, T^b] = if^{abc}T_c \tag{2.5}$$

where $f^{abc}$ are known as the structure constants.

By definition they are antitsymmetric

$$f^{abc} = -f^{acb} \tag{2.6}$$

We are interested in so called semi-simple Lie groups as $SU(N)$ and $SO(N)$. We focus in the following on $SU(N)$. These groups preserve a complex inner product. Finite dimensional representations of semi-simple Lie algebras are always Hermitian, so one can build quantum theories which are unitarity based on such algebras. The complex inner product is

$$U^\dagger U = 1 \tag{2.7}$$

defined on $N$ dimensional complex vector spaces, for $U(N)$. Note that in all cases we can write $U(N) = SU(N) \times U(1)$ where the $U(1)$ represents an overall phase. There are $N^2 - 1$ generators for $SU(N)$. To see this, let us write the identity infinitesimally as

$$0 = 1 - e^0 \tag{2.8}$$
$$= 1 - e^{[i\Theta_a T_a + (i\Theta_a T_a)^\dagger]} \tag{2.9}$$
$$= 1 - (1 + i\Theta_a T_a)(1 - i\Theta_a T_a^\dagger) \tag{2.10}$$
$$= - i\Theta_a (T^\dagger)_a + i\Theta_a T_a \tag{2.11}$$
$$\Rightarrow T = T^\dagger \tag{2.12}$$

so we can count the generators by counting $N \times N$ Hermitian matrices. Such matrices have $\frac{1}{2}N(N-1)$ imaginary components and $\frac{1}{2}N(N+1)$ real components, but then we subtract the identity matrix, which just generates $U(1)$. Thus, we find for the number of generators

$$\#(T_a) = \frac{1}{2}N(N-1) + \frac{1}{2}N(N+1) - 1 = N^2 - 1 \tag{2.13}$$

#### 2.2.2.2 Representations

The groups and algebras discussed above are abstract mathematical objects. We want to have these groups act on quantum states and fields, which are vectors, so we need to represent the groups as matrices. There are an infinite number of different representations for a given simple group. However, there are two obvious and most important representations, which occur most often in physics settings. They are

**a)** the fundamental representations

**b)** the adjoint representations

> The fundamental representation is the representation defining $SU(N)$ and $SO(N)$ as $N \times N$ matrices acting on $N$ dimensional vectors. To write the fundamental formally, we say that $N$ fields transform under it as
>
> $$\phi_i \to \phi_i + i\alpha_a (T_f^a)_i^j \phi_j \tag{2.14}$$
>
> where $i = 1, \ldots, N$, $a = 1, \ldots N^2 - 1$ and the $\alpha_a$ are real numbers. The complex conjugate fields transform in the anti-fundamental $\overline{f}$, which is just the conjugate of this
>
> $$\phi_i^* \to \phi_i^* - i\alpha_a (T_f^{a*})_i^j \phi_j^* \tag{2.15}$$

Since $T_f^a$ are Hermitian, we have $T_{\bar{\text{f}}} = (T_f)^*$.
The normalisation of generators is arbitrary and is usually chosen so that

$$\text{Tr} T_f^a T_f^b = \frac{1}{2}\delta_{ab} \tag{2.16}$$

> The other important representation is the adjoint. The point is to think of the generators themselves as the vectors. Thus, the generators are
>
> $$(T_{\text{adj}}^a)_c^b = -if^{abc} \tag{2.17}$$

How can we see that the $T_{\text{adj}}$ actually satisfy the Lie algebra, and thus are really a representation? This is given immediately by the Jacobi identity

$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0 \tag{2.18}$$

written as

$$\begin{aligned}
0 &= [T^a, f^{bcd}T_d] + [T^b, f^{cad}T_d] + [T^a, f^{abd}T_d] && (2.19)\\
&= f^{bcd}[T^a, T_d] + f^{cad}[T^b, T_d] + f^{abd}[T^a, T_d] && (2.20)\\
&= f^{bcd}f^{ade}T_e + f^{cad}f^{bde}T_e + f^{abd}f^{ade}T_d && (2.21)\\
\Rightarrow & f^{cbd}f^{ade} - f^{abd}f^{cde} = f^{cad}f^{dbe} && (2.22)\\
\Rightarrow & [T^c_{\text{adj}}, T^a_{\text{adj}}] = if^{cad}T^d_{\text{adj}} && (2.23)
\end{aligned}$$

The dimension of the adjoint representation is $N^2-1$ for $SU(N)$.

#### 2.2.2.3 Group constants

The quadratic Casimir is defined as

$$T^a_R T^a_R = C_2(R)\mathbf{1} \tag{2.24}$$

This must be proportional to the identity (when acting on a single given irreducible representation) because it commutes with all generators of the group, which follows from

$$\begin{aligned}
[T^a_R T^a_R, T^b_R] =& T^a_R T^a_R T^b_R - T^b_R T^a_R T^a_R && (2.25)\\
=& T^a_R([T^a_R, T^b_R] + T^b_R T^a_R]) - ([T^b_R, T^a_R] + T^a_R T^b_R)T^a_R && (2.26)\\
=& T^a_R(if^{abc}T^c_R) - (if^{bac}T^c_R T^a_R) && (2.27)\\
=& if^{abc}T^a_R T^c_R + if^{abc}T^c_R T^a_R && (2.28)\\
=& 0 && (2.29)
\end{aligned}$$

because of anti-symmetry of $f^{abc}$.
Another important quantity is the Dynkin index $I(R)$

$$\mathrm{Tr}[T^a_R T^b_R] = I(R)\delta_{ab} \tag{2.30}$$

The quantity $I(R)$ is the index of the representation. We have that

$$I(f) = \frac{1}{2} \tag{2.31}$$

and

$$I(G) = N \tag{2.32}$$

for $SU(N)$ and our normalisation. The Dynkin index and the quadratic Casimir are related

$$d(R)C_2(R) = I(R)d(G) \tag{2.33}$$

where $d(R)$ is the dimension of the representation, and $d(G)$ of the algebra, namely $N^2-1$ for $SU(N)$. Thus

$$\begin{aligned}
C_2(f) =& \frac{N^2-1}{N} && (2.34)\\
C_2(G) =& N && (2.35)
\end{aligned}$$

#### 2.2.2.4 Examples

##### 2.2.2.4.1 $SU(2)$

For $SU(2)$ the common generators for the fundamental representation $T_f^a$ are related to the Pauli matrices $\sigma^a$ $(i = 1, 2, 3)$ by

$$T_f^a = \frac{1}{2}\sigma^a \tag{2.36}$$

with

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.37}$$

For later, it is also helpful to introduce

$$\sigma^0 = \overline{\sigma}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.38}$$

and $\overline{\sigma}^i = -\sigma^i$. The Lie algebra

$$[\sigma^a, \sigma^b] = i f^{abc} \sigma_c \tag{2.39}$$

is fulfilled for

$$f^{abc} = \epsilon^{abc} \tag{2.40}$$

where $\epsilon^{abc}$ is the Levi-Civita tensor. And we have

$$d(f) = 2 \qquad d(a) = 3 \tag{2.41}$$

$$C_2(f) = \frac{3}{2} \qquad C_2(a) = 3 \tag{2.42}$$

$$I(f) = \frac{1}{2} \qquad I(a) = 2 \tag{2.43}$$

##### 2.2.2.4.2 SU(3)

The common representation for $SU(3)$ are given by the Gell-Mann Matrices $\lambda^a$

$$T_f^a = \frac{1}{2}\lambda^a \tag{2.44}$$

With

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.45}$$

$$\lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{2.46}$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.47}$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{2.48}$$

And we have

$$d(f) = 3 \qquad d(a) = 8 \tag{2.49}$$

$$C_2(f) = \frac{4}{3} \qquad C_2(a) = 3 \tag{2.50}$$

$$I(f) = \frac{1}{2} \qquad I(a) = 3 \tag{2.51}$$

### 2.2.3 Other groups relevant in particle physics

**a) Lorentz Group**: the Lorentz group is the set of all $4 \times 4$ real matrices that leave the line element in Minkowski space invariant:

$$s^2 = (x^0)^2 - (x^i)^2 = x^\mu g_{\mu\nu} x^\nu \tag{2.52}$$

It is parametrised by

$$x'^\mu = \Lambda^\mu_\nu x^\nu \tag{2.53}$$

The Lorentz group has six generators:

- three generators $J^i$ creating rotations
- three generators $K^i$ creating boosts

**b) Poincare Group**: the Poincare group is the generalisation of the Lorentz group including translation:

$$x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu \tag{2.54}$$

The generator of the translation is the four momentum operator $p_\mu$

## 2.3 Quantum Field Theory

### 2.3.1 Lagrangian formalism

We are working with the Lagrangian formalism of quantum field theory. The basic features are

- space–time symmetry in terms of Lorentz invariance, as well as internal symmetries like gauge symmetries
- causality
- local interactions

Particles are described by fields that are operators on the quantum mechanical Hilbert space of the particle states, acting as creation and annihilation operators for particles and antiparticles. We need in the following particles characterised by their spin:

- spin-0: complex or real scalar fields $\phi(x)$, $\varphi(x)$
- spin-$\frac{1}{2}$: fermions, described by two- or four component spinor fields $\psi_{L,R}$, $\psi(x)$.
- spin-1: vector bosons $A_\mu(x)$

The dynamics of the physical system involving a set of fields $\Phi$ is determined by the Lorentz-invariant Lagrangian $\mathcal{L}$. The action is given by

$$S[\Phi] = \int \mathrm{d}^4x \, \mathcal{L}\big(\Phi(x)\big) \,, \tag{2.55}$$

The equations of motions follow as Euler–Lagrange equations from Hamilton's principle,

$$\delta S = S[\Phi + \delta\Phi] - S[\Phi] = 0 \,. \tag{2.56}$$

Let's go back to mechanics: for $n$ generalised coordinates $q_i$ and velocities $\dot{q}_i$ the Lagrangian reads: $L(q_1, \dots \dot{q}_1, \dots)$ The equations of motion are calculated from $(i = 1, \dots n)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \,. \tag{2.57}$$

Going to field theory, one has to perform the replacement

$$q_i \to \Phi(x) \,, \quad \dot{q}_i \to \partial_\mu \Phi(x) \,, \quad L(q_1, \dots q_n, \dot{q}_1, \dots \dot{q}_n) \to \mathcal{L}(\Phi(x), \partial_\mu \Phi(x)) \tag{2.58}$$

The equations of motion become field equations which are calculated from

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} - \frac{\partial \mathcal{L}}{\partial \Phi} = 0 \,, \tag{2.59}$$

### 2.3.2 Free quantum fields

#### 2.3.2.1 Scalar fields

The equation of motion for a scalar field is known as K̂lein–Gordon equation:

$$(\partial_\mu \partial^\mu + m^2)\, \phi = 0 \,. \tag{2.60}$$

The solution can be expanded in terms of the complete set of plane waves $e^{\pm ikx}$,

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3k}{2k^0} \left[a(k)\, e^{-ikx} \,+\, a^\dagger(k)\, e^{ikx}\right] \tag{2.61}$$

with $k^0 = \sqrt{\vec{k}^2 + m^2}$. Here, we used annihilate and creation operators $a^\dagger$, $a$:

$$\begin{aligned} &a^\dagger(k)\,|0\rangle = |k\rangle \\ &a(k)\,|k'\rangle = 2k^0\,\delta^3(\vec{k} - \vec{k}\,')\,|0\rangle\,. \end{aligned} \tag{2.62}$$

The Lagrangian for a free real or complex scalar field with mass $m$ is

$$\mathcal{L}_{\text{real}} = \frac{1}{2}\,(\partial_\mu\phi)^2 - \frac{m^2}{2}\,\phi^2 \tag{2.63}$$

$$\mathcal{L}_{\text{complex}} = (\partial_\mu\phi)^\dagger(\partial^\mu\phi) - m^2\,\phi^\dagger\phi \tag{2.64}$$

One can easily check that they give us the Klein–Gordon Equation as equation of motion. A complex scalar field $\phi^\dagger \neq \phi$ has two degrees of freedom. It describes spin-less particles which carry a charge and can be interpreted as particles and antiparticles.
So far, we have considered particles without any space–time restrictions. Now, we want to consider the case that a particle propagates from a point-like source at a given space-time point. This is described by the inhomogeneous field equation

$$(\partial_\mu\partial^\mu \,+ m^2)\,D(x-y) = -\delta^4(x-y)\,. \tag{2.65}$$

$D(x-y)$ is called Green function. The solution can easily be determined by a Fourier transformation

$$D(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\,D(k)\,e^{-ik(x-y)} \tag{2.66}$$

giving in momentum space

$$(k^2 - m^2)\,D(k) = 1\,. \tag{2.67}$$

The solution

$$i\,D(k) = \frac{i}{k^2 - m^2 + i\epsilon} \tag{2.68}$$

is the *causal Green function* or the *Feynman propagator* of the scalar field. The overall factor $i$ is by convention. The term $+i\epsilon$ in the denominator with an infinitesimal $\epsilon > 0$ is a prescription of how to treat the pole in the integral (2.66); it corresponds to the special boundary condition of causality for $D(x-y)$ in Minkowski space, which means

- propagation of a particle from $y$ to $x$ if $x^0 > y^0$,
- propagation of an antiparticle from $x$ to $y$ if $y^0 > x^0$.

In a Feynman diagram, the scalar propagator is drawn as dashed line.

Complex Scalar $\phi$  (2.69)

Real Scalar $\varphi$  (2.70)

For complex scalars the arrow shows the flow of the charge.

#### 2.3.2.2 Dirac fields

**Equation of motion** Spin-$\frac{1}{2}$ particles with mass $m$ are often described by 4-component spinor fields,

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix} . \tag{2.71}$$

and obey the *Dirac–Equation*

$$(i\gamma^\mu \partial_\mu - m)\,\psi = 0 . \tag{2.72}$$

This equation is obtained from the Lagrangian

$$\mathcal{L}_{\text{fermion}} = \overline{\psi}\,(i\gamma^\mu \partial_\mu - m)\,\psi , \tag{2.73}$$

involving the adjoint spinor

$$\overline{\psi} = \psi^\dagger\,\gamma^0 = (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*) . \tag{2.74}$$

The Dirac matrices $\gamma^\mu$ ($\mu = 0, 1, 2, 3$) are $4 \times 4$ matrices which fulfil the anti-commutator relations

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} . \tag{2.75}$$

One possible representation is to express the matrices in terms of the the Pauli matrices $\sigma_{1,2,3}$ as

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} . \tag{2.76}$$

Another matrix, $\gamma_5$, is often very useful:

$$\gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.77}$$

There are two types of solutions for the Dirac equation, corresponding to particle and anti-particle wave functions,

$$u(p)\,e^{-ipx} \quad \text{and} \quad v(p)\,e^{ipx} \tag{2.78}$$

which are used to write the Dirac field as

$$\psi(x) = \frac{1}{(2\pi)^{3/2}} \sum_\sigma \int \frac{\mathrm{d}^3k}{2k^0} \left[c_\sigma(k)\,u_\sigma(k)\,e^{-ikx} \,+\, d^\dagger_\sigma(k)\,v_\sigma(k)\,e^{ikx}\right], \tag{2.79}$$

with

- annihilation operators $c_\sigma$ for particles and $d_\sigma$ for anti-particles
- creation operators $c^\dagger_\sigma$ and $d^\dagger_\sigma$ for particles and antiparticles

We still have to determine the propagator of the Dirac field, which is the solution of the inhomogeneous Dirac equation with point-like source,

$$(i\gamma^\mu\partial_\mu - m)\,S(x-y) = \mathbf{1}\,\delta^4(x-y)\,. \tag{2.80}$$

Using a Fourier transformation as in the scalar case, we find

$$i\,S(k) = \frac{i}{\not{k} - m + i\epsilon} = \frac{i\,(\not{k} + m)}{k^2 - m^2 + i\epsilon}\,, \tag{2.81}$$

We introduce a graphical symbol for the propagator:

Dirac Fermion $\psi$  (2.82)

The arrow at the line denotes the flow of the *particle* charge.

#### 2.3.2.3 Weyl Fermions

We have so far used 4-component (Dirac) fermions. However, it will turn out that it is often more convenient to use actually 2-component notation: in any model which violates parity (as the SM or all extension of it), each Dirac fermion has left-handed and right-handed parts with completely different electroweak gauge interactions. If one used four-component spinor notation instead, then there would be clumsy left- and right-handed projection operators

$$P_L = (1-\gamma_5)/2, \qquad\qquad P_R = (1+\gamma_5)/2 \tag{2.83}$$

all over the place. The two-component Weyl fermion notation has the advantage of treating fermionic degrees of freedom with different gauge quantum numbers separately from the start. An even better reason for using two-component notation here is that in supersymmetric models the minimal building blocks of matter are chiral supermultiplets, each of which contains a single two-component Weyl fermion.

In this representation, a four-component Dirac spinor is written in terms of 2 two-component, complex anti-commuting objects $\xi_\alpha$ and $(\chi^\dagger)^{\dot\alpha} \equiv \chi^{\dagger\dot\alpha}$, with two distinct types of spinor indices $\alpha = 1,2$ and $\dot\alpha = 1,2$:

$$\Psi_D = \begin{pmatrix} \xi_\alpha \\ \chi^{\dagger\dot\alpha} \end{pmatrix}. \tag{2.84}$$

It follows that

$$\overline{\Psi}_D = \Psi_D^\dagger \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \chi^\alpha & \xi^\dagger_{\dot\alpha} \end{pmatrix} . \tag{2.85}$$

Undotted (dotted) indices from the beginning of the Greek alphabet are used for the first (last) two components of a Dirac spinor. The field $\xi$ is called a "left-handed Weyl spinor" and $\chi^\dagger$ is a "right-handed Weyl spinor". The names fit, because

$$P_L\Psi_D = \begin{pmatrix} \xi_\alpha \\ 0 \end{pmatrix}, \qquad P_R\Psi_D = \begin{pmatrix} 0 \\ \chi^{\dagger\dot\alpha} \end{pmatrix} . \tag{2.86}$$

The Hermitian conjugate of any left-handed Weyl spinor is a right-handed Weyl spinor:

$$\psi^\dagger_{\dot\alpha} \equiv (\psi_\alpha)^\dagger = (\psi^\dagger)_{\dot\alpha} , \tag{2.87}$$

and vice versa:

$$(\psi^{\dagger\dot\alpha})^\dagger = \psi^\alpha. \tag{2.88}$$

Therefore, any particular fermionic degrees of freedom can be described equally well using a left-handed Weyl spinor (with an undotted index) or by a right-handed one (with a dotted index). By convention, all names of fermion fields are chosen so that left-handed Weyl spinors do not carry daggers and right-handed Weyl spinors do carry daggers, as in eq. (2.84).

**Playing with indices** The heights of the dotted and undotted spinor indices are important. The spinor indices are raised and lowered using the anti-symmetric symbol

$$\epsilon^{12} = -\epsilon^{21} = \epsilon_{21} = -\epsilon_{12} = 1, \qquad \epsilon_{11} = \epsilon_{22} = \epsilon^{11} = \epsilon^{22} = 0, \tag{2.89}$$

according to

$$\xi_\alpha = \epsilon_{\alpha\beta}\xi^\beta, \qquad \xi^\alpha = \epsilon^{\alpha\beta}\xi_\beta, \qquad \chi^\dagger_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\chi^{\dagger\dot\beta}, \qquad \chi^{\dagger\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\chi^\dagger_{\dot\beta} . \tag{2.90}$$

This is consistent since $\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \epsilon^{\gamma\beta}\epsilon_{\beta\alpha} = \delta^\gamma_\alpha$ and $\epsilon_{\dot\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma} = \epsilon^{\dot\gamma\dot\beta}\epsilon_{\dot\beta\dot\alpha} = \delta^{\dot\gamma}_{\dot\alpha}$.
As a convention, repeated spinor indices contracted like

$${}^\alpha{}_\alpha \qquad \text{or} \qquad {}_{\dot\alpha}{}^{\dot\alpha} \tag{2.91}$$

can be suppressed. In particular,

$$\xi\chi \equiv \xi^\alpha\chi_\alpha = \xi^\alpha\epsilon_{\alpha\beta}\chi^\beta = -\chi^\beta\epsilon_{\alpha\beta}\xi^\alpha = \chi^\beta\epsilon_{\beta\alpha}\xi^\alpha = \chi^\beta\xi_\beta \equiv \chi\xi \tag{2.92}$$

with, conveniently, no minus sign in the end. [A minus sign appeared in eq. (2.92) from exchanging the order of anti-commuting spinors, but it disappeared due to the anti-symmetry of the $\epsilon$ symbol.] Likewise,

$\xi^\dagger\chi^\dagger$ and $\chi^\dagger\xi^\dagger$ are equivalent abbreviations for $\chi^\dagger_{\dot\alpha}\xi^{\dagger\dot\alpha} = \xi^\dagger_{\dot\alpha}\chi^{\dagger\dot\alpha}$, and in fact this is the complex conjugate of $\xi\chi$:

$$(\xi\chi)^* = \chi^\dagger\xi^\dagger = \xi^\dagger\chi^\dagger. \tag{2.93}$$

In a similar way, one can check that

$$(\chi^\dagger\overline{\sigma}^\mu\xi)^* \;=\; \xi^\dagger\overline{\sigma}^\mu\chi \;=\; -\chi\sigma^\mu\xi^\dagger \;=\; -(\xi\sigma^\mu\chi^\dagger)^* \tag{2.94}$$

stands for $\xi^\dagger_{\dot\alpha}(\overline{\sigma}^\mu)^{\dot\alpha\alpha}\chi_\alpha$, etc. Note that when taking the complex conjugate of a spinor bilinear, one reverses the order. The spinors here are assumed to be classical fields; for quantum fields the complex conjugation operation in these equations would be replaced by Hermitian conjugation in the Hilbert space operator sense.

**Lagrangian for Weyl fermions** With these conventions, the Dirac Lagrangian can now be rewritten:

$$\mathcal{L}_{\text{Dirac}} \;=\; i\xi^\dagger\overline{\sigma}^\mu\partial_\mu\xi + i\chi^\dagger\overline{\sigma}^\mu\partial_\mu\chi - M(\xi\chi + \xi^\dagger\chi^\dagger) \tag{2.95}$$

where we have dropped a total derivative piece $-i\partial_\mu(\chi^\dagger\overline{\sigma}^\mu\chi)$, which does not affect the action.
A four-component Majorana spinor can be obtained from the Dirac spinor of eq. (2.85) by imposing the constraint $\chi = \xi$, so that

$$\Psi_{\text{M}} = \begin{pmatrix}\xi_\alpha\\ \xi^{\dagger\dot\alpha}\end{pmatrix}, \qquad\qquad \overline{\Psi}_{\text{M}} = \begin{pmatrix}\xi^\alpha & \xi^\dagger_{\dot\alpha}\end{pmatrix}. \tag{2.96}$$

The four-component spinor form of the Lagrangian for a Majorana fermion with mass $M$,

$$\mathcal{L}_{\text{Majorana}} \;=\; \frac{i}{2}\overline{\Psi}_{\text{M}}\gamma^\mu\partial_\mu\Psi_{\text{M}} - \frac{1}{2}M\overline{\Psi}_{\text{M}}\Psi_{\text{M}} \tag{2.97}$$

can therefore be rewritten as

$$\mathcal{L}_{\text{Majorana}} \;=\; i\xi^\dagger\overline{\sigma}^\mu\partial_\mu\xi - \frac{1}{2}M(\xi\xi + \xi^\dagger\xi^\dagger) \tag{2.98}$$

in the more economical two-component Weyl spinor representation. Note that even though $\xi_\alpha$ is anti-commuting, $\xi\xi$ and its complex conjugate $\xi^\dagger\xi^\dagger$ do not vanish, because of the suppressed $\epsilon$ symbol, see eq. (2.92). Explicitly, $\xi\xi = \epsilon^{\alpha\beta}\xi_\beta\xi_\alpha = \xi_2\xi_1 - \xi_1\xi_2 = 2\xi_2\xi_1$.

Any theory involving spin-1/2 fermions can always be written in terms of a collection of left-handed Weyl spinors $\psi_i$ with

$$\mathcal{L} \;=\; i\psi^{\dagger i}\overline{\sigma}^\mu\partial_\mu\psi_i - M^{ij}(\psi^\dagger_i\psi^\dagger_j - \psi_i\psi_j) \tag{2.99}$$

For $i = j$ one has a Majorana mass term, and $i \neq j$ gives Dirac mass term.

There is a different $\psi_i$ for the left-handed piece and for the Hermitian conjugate of the right-handed piece of a Dirac fermion. Given any expression involving bilinears of four-component spinors

$$\Psi_i = \begin{pmatrix}\xi_i\\ \chi^\dagger_i\end{pmatrix}, \tag{2.100}$$

labelled by a flavor or gauge-representation index $i$, one can translate into two-component Weyl spinor language (or vice versa) using the dictionary:

$$\overline{\Psi}_i P_L\Psi_j = \chi_i\xi_j, \qquad\qquad \overline{\Psi}_i P_R\Psi_j = \xi^\dagger_i\chi^\dagger_j, \tag{2.101}$$

$$\overline{\Psi}_i\gamma^\mu P_L\Psi_j = \xi^\dagger_i\overline{\sigma}^\mu\xi_j, \qquad\qquad \overline{\Psi}_i\gamma^\mu P_R\Psi_j = \chi_i\sigma^\mu\chi^\dagger_j \tag{2.102}$$

#### 2.3.2.4 Vector fields

A vector field $A_\mu(x)$ describes particles with spin 1. We concentrate here on the massless case with two degrees of freedom.
The Lagrangian of such a field is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g f^{abc} A^b_\mu A^c_\nu \,. \tag{2.103}$$

The last term is only present for non-Abelian gauge fields. The field equations are Maxwell's equations for the vector potential,

$$\left( \Box \, g^{\mu\nu} - \partial^\mu \partial^\nu \right) A_\nu = 0 \,. \tag{2.104}$$

The propagator of the vector fields depends on the chosen gauge. In general $R_\xi$ gauge it is given by

$$i \, D_{\rho\nu}(k) = \frac{i}{k^2 + i\epsilon} \left[ -g_{\nu\rho} + (1-\xi) \frac{k_\nu k_\rho}{k^2} \right] . \tag{2.105}$$

which becomes very simple in Feynman gauge with $\xi = 1$.

The graphical symbol for the vector-field propagator (for both massive and massless) is a wavy line which carries the momentum $k$ and two Lorentz indices

massless or massive Vector boson $A_\mu$  (2.106)

The arrow at the line denotes the flow of the *particle* charge.

### 2.3.3 Gauge invariance

So far, we have not considered any symmetry. We change that now and apply (local) gauge transformations to the fields.

$$\phi(x) \to e^{ig\Lambda(x)} \phi(x) \tag{2.107}$$
$$\phi(x)^* \to \phi(x)^* e^{-ig\Lambda(x)} \tag{2.108}$$
$$\Psi(x) \to e^{ig\Lambda(x)} \Psi(x) \tag{2.109}$$
$$\bar{\Psi}(x) \to \bar{\Psi} e^{-ig\Lambda(x)} \tag{2.110}$$

However, one can check that the Lagrangians for scalars and fermions are **not** invariant under these transformations. For instance, the fermionic part of the Lagrangian transforms as

$$\begin{aligned}
\mathcal{L}'_{\text{fermion}} &= i(\bar{\Psi})' \partial\!\!\!/ (\Psi)' - m (\bar{\Psi})' \Psi' && (2.111)\\
&= i\bar{\Psi} e^{-ig\Lambda(x)} \partial\!\!\!/ e^{ig\Lambda(x)} \Psi - m\bar{\Psi} \underbrace{\left(e^{-ig\Lambda(x)} e^{ig\Lambda(x)}\right)}_{=1} \Psi && (2.112)\\
&= i\bar{\Psi} (ig\Lambda(x) \partial\!\!\!/ \Lambda(x)) (\partial\!\!\!/ \Psi) - m\bar{\Psi}\Psi && (2.113)\\
&\neq \mathcal{L}_{\text{fermion}} && (2.114)
\end{aligned}$$

We need another ingredient to built kinetic terms for scalars and fermions which are gauge invariant: we introduce a massless gauge fields $A_\mu$ which transforms as

$$A_\mu \to A_\mu - \partial_\mu \Lambda(x) \tag{2.115}$$

In addition, we define the covariant derivative:

$$\partial_\mu \to D_\mu = \partial_\mu + igA_\mu \tag{2.116}$$

$g$ is a free parameter which we call 'gauge coupling'. One finds that the covariant derivative transforms as

$$(D_\mu\Psi)' = D'_\mu\Psi' \tag{2.117}$$
$$= (\partial_\mu + ig(A_\mu - \partial_\mu\Lambda))e^{ig\Lambda}\Psi \tag{2.118}$$
$$= e^{ig\Lambda}(\partial_\mu + igA_\mu)\Psi - ig\partial_\mu\Lambda\Psi + (\partial_\mu e^{ig\Lambda})\Psi \tag{2.119}$$
$$= e^{ig\Lambda}(\partial_\mu + igA_\mu)\Psi \tag{2.120}$$
$$= e^{ig\Lambda}D_\mu\Psi \tag{2.121}$$

Thus, the Lagrangian with derivatives replaced by covariant derivatives are invariant.

$$\bar\Psi D_\mu \Psi \to (\bar\Psi)'(D_\mu\Psi)' = \bar\Psi e^{-ig\Lambda}e^{ig\Lambda}D_\mu\Psi = \bar\Psi D_\mu\Psi \tag{2.122}$$

Similarly, one can show that for the scalar terms in the Lagrangian the identity

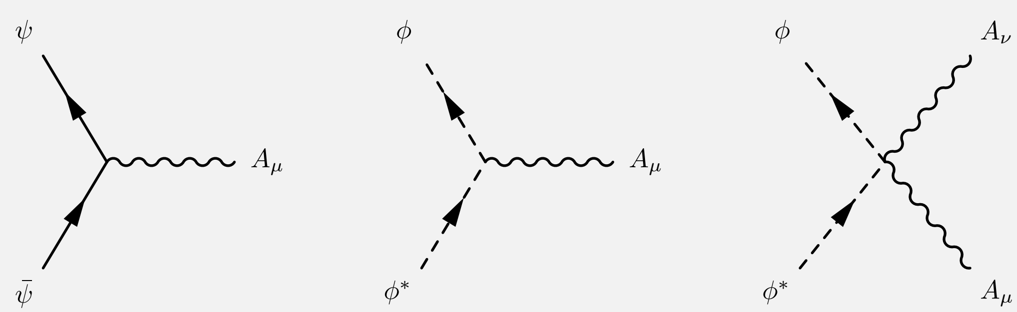
$$(D_\mu\phi D^\mu\phi^*)' = D_\mu\phi D^\mu\phi^* \tag{2.123}$$

holds.

In order to obtain a gauge theory, i.e. a theory in which the Lagrangian is invariant under a local transformation, the derivative must be replaced by the **covariant derivative** involving gauge fields:

$$\partial^\mu \to D_\mu = \partial^\mu + igTA_\mu \tag{2.124}$$

This introduces interaction terms between the fermions and scalars and the gauge fields which are represented by the following Feynman diagrams:




$ig\gamma_\mu$ $\quad ig(p^\nu_{\text{in}} - p^\mu_{\text{out}})$ $\quad ig^2 g_{\mu\nu}$

### 2.3.4 Spontaneous symmetry breaking

A mass term for gauge bosons would read

$$m_V^2 A_\mu A^\mu \tag{2.125}$$

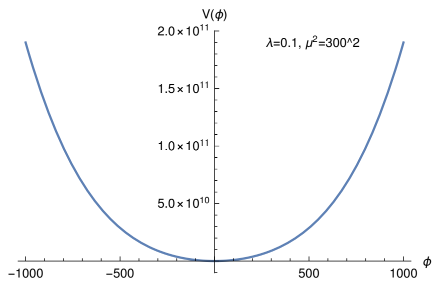
However, this is not gauge invariant:

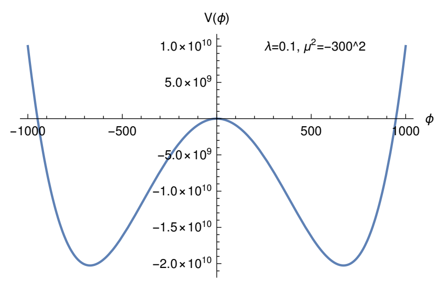
$$(m_V^2 A_\mu A^\mu)' = m_V^2 A_\mu A^\mu + m_V^2 (\partial_\mu \Lambda)(\partial^\mu \Lambda) \tag{2.126}$$

Thus, explicit mass terms are not possible and we must generated them via the so called *Higgs–mechanism*. Let's assume a real scalar $\varphi$ and the following potential:

$$V(\varphi) = \frac{1}{2}\lambda\varphi^4 + \mu^2\varphi \tag{2.127}$$

Depending on the sign of $\mu^2$ the shape of the potential is different




For

- $\mu^2 > 0$: $\varphi = 0$ is the correct vacuum
- $\mu^< 0$: the vacuum is at $\varphi \neq 0$

We shift $\varphi$ in a way that we are for $\varphi = 0$ at the minimum of the potential:

$$\phi \to \phi + v \tag{2.128}$$

We find

$$V(\varphi = 0) = \frac{1}{2}\lambda v^4 + \mu^2 v^2 \tag{2.129}$$

$$\to \frac{\partial V}{\partial v} = 2\lambda v^3 + 2v\mu^2 \equiv 0 \tag{2.130}$$

Thus

$$v = \sqrt{-\mu^2/\lambda} \tag{2.131}$$

is the value of the VEV (vacuum expectation value).

**Higgs mechanism** We consider now a gauge theory with a complex field $\phi$. We want to insert

$$\phi \rightarrow \frac{1}{\sqrt{2}}(\varphi + v + i\sigma) \tag{2.132}$$

in the general Lagrangian

$$\mathcal{L} = D_\mu \phi D^\mu \phi^* - m^2|\phi|^2 - \lambda|\phi|^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2.133}$$

We get

$$\begin{aligned}\mathcal{L} =& -\frac{1}{4}F_{\mu\nu}F^{\mu\mu} + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma \\ &+ gvA_\mu\partial^\mu\sigma + \frac{1}{2}g^2v^2A_\mu A^\mu \\ &+ \frac{1}{2}g^2(A_\mu)^2\varphi(2v+\varphi) - \frac{1}{2}\varphi^2(3\lambda v^2 + m^2) - \lambda v\varphi^3 - \frac{1}{4}\lambda\varphi^4\end{aligned} \tag{2.134}$$

The first line are just the ordinary kinetic terms. However, we see that an effective mass term $\frac{1}{2}g^2v^2$ for the vector bosons has been generated. There is also a term which mixes the field $\sigma$, which becomes massless, and $A_\mu$.
A massive vector boson has three degrees of freedom, while a massless one has only two. Therefore, one says that $\sigma$ is 'eaten' by the vector boson to form its longitudinal component. $\sigma$ is called 'Goldstone' (or 'Nambu-Goldstone') boson.
It is common to introduce gauge fixing terms in a way that they cancel the mixing terms between field $\sigma$ and $A^\mu$.

$$\mathcal{L}_{GF} = -\frac{1}{2\xi}\left(\partial_\mu A^\mu - gv\xi\sigma\right)^2 \tag{2.135}$$

Thus, the Lagrangian becomes

$$\mathcal{L} + \mathcal{L}_{GF} = +\frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - g^2v^2\xi\sigma^2 + \frac{1}{2}g^2v^2A_\mu A^\mu + \dots \tag{2.136}$$

what gives a relation between the Goldstone mass and the mass of the vector boson

$$M_G^2 = \xi M_A^2 \tag{2.137}$$

In the unitarity gauge $\xi \rightarrow \infty$, the Goldstone disappears from the spectrum. The same could have been obtained by starting with the gauge transformation However, before we do this, we apply the following gauge transformation:

$$\phi \rightarrow \phi' = e^{-i\sigma/v}\phi = \frac{1}{\sqrt{2}}(v+\varphi) \tag{2.138}$$

$$A_\mu \rightarrow A'_\mu = A_\mu - \frac{1}{gv}\partial_\mu\sigma \tag{2.139}$$

The Higgs mechanism generates mass terms for vector-boson due to vacuum expectation values of a complex scalar field

$$\phi \to \frac{1}{\sqrt{2}} (\varphi + i\sigma + v) \tag{2.140}$$

While the real (CP-even) component $\varphi$ of the scalar is a physical degree of freedom, the imaginary (CP-odd) component $\sigma$ becomes the longitudinal mode of the massive vector boson. In general $R_\xi$ gauge the Goldstone mass $M_G$ is related to the mass $M_A$ of the vector boson $A^\mu$ by

$$M_G^2 = \xi M_A^2 \tag{2.141}$$

## 2.4 The Standard Model of Particle Physics

### 2.4.1 Gauge Symmetries

The so called standard model of particle physics (SM) is a gauge theory.

The gauge symmetry of the SM is

$$\mathcal{G} = SU(3)_C \times SU(2)_L \times U(1)_Y \tag{2.142}$$

with

- $C$: Colour
- $L$: Left
- $Y$: Hypercharge

### 2.4.2 Particle Content

Before symmetry breaking, the particle content of the SM is

| | | |
|---|---|---|
| Vector Bosons | $B$ | $(\mathbf{1}, \mathbf{1})_0$ |
| | $W$ | $(\mathbf{1}, \mathbf{2})_0$ |
| | $g$ | $(\mathbf{8}, \mathbf{0})_0$ |
| Fermions | $e_R$ | $(\mathbf{1}, \mathbf{1})_1$ |
| (3 Generations) | $l$ | $(\mathbf{1}, \mathbf{2})_{-1/2}$ |
| | $u_R$ | $(\bar{\mathbf{3}}, \mathbf{1})_{-2/3}$ |
| | $d_R$ | $(\bar{\mathbf{3}}, \mathbf{1})_{1/3}$ |
| | $q$ | $(\mathbf{3}, \mathbf{2})_{1/6}$ |
| Scalar | $H$ | $(\mathbf{1}, \mathbf{2})_{1/2}$ |

The last column shows the quantum numbers with respect to $\mathcal{G}$. These quantum numbers are not as random as it might look. Special conditions must be fulfilled to avoid anomalies, e.g.

- Gauge anomalies

$$\sum_f Y(f)^3 \equiv 0 \tag{2.143}$$

- Gauge × gravity anomalies

$$\sum_f Y(f) \equiv 0 \tag{2.144}$$

- Witten anomaly: even number of $SU(2)$ doublets

**Check**:

$$\sum_f Y(f) = \underbrace{3}_{\text{generations}} \times \left( Y(e) + \underbrace{2}_{\text{isospin}} \times Y(l) + \underbrace{3}_{\text{color}} \times Y(u_R) + 3 \times Y(d_R) + 2 \times 3 \times Y(q) \right) \tag{2.145}$$

$$= 3 \times \left(1 + 2\left(-\frac{1}{2}\right) + 3\left(-\frac{2}{3}\right) + 3\left(\frac{1}{3}\right) + 6\left(\frac{1}{6}\right)\right) \tag{2.146}$$

$$= 3 \times (1 - 1 - 2 + 1 + 1) \tag{2.147}$$

$$= 0 \tag{2.148}$$

$$\sum_f Y(f)^3 = 3 \times \left(1 + 2\left(-\frac{1}{8}\right) + 3\left(-\frac{8}{27}\right) + 3\left(\frac{1}{27}\right) + 6\left(\frac{1}{216}\right)\right) \tag{2.149}$$

$$= 3 \times \left(1 - \frac{1}{4} - \frac{8}{9} + \frac{1}{9} + \frac{1}{36}\right) \tag{2.150}$$

$$= 0 \tag{2.151}$$

⇒ One needs to be careful when adding new fermions in order not to introduce anomalies

### 2.4.3 Gauge part of the Lagrangian

The gauge part of the Lagrangian before symmetry breaking reads

$$L = D_\mu H D^\mu H^* + i \sum_f f^\dagger \sigma^\mu D_\mu f + \sum_V V_{\mu\nu} V^{\mu\nu} \tag{2.152}$$

with $f = \{l, e_R, q, d_R, u_R\}$ and $V = \{B, W^a, G^a\}$. Let's be more explicit at some examples. Note, we consider only one generation of fermions because gauge couplings are always flavour diagonal.

- Right leptons

$$e_R^\dagger \sigma^\mu D_\mu e_R = e_R^\dagger \sigma^\mu (\partial_\mu + i g_1 B_\mu) e_R \tag{2.153}$$

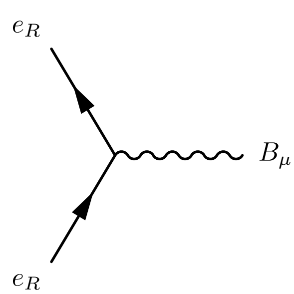
- Left leptons carry one isospin index, i.e. $l_i$ with $i = 1, 2$

$$l^\dagger \sigma^\mu D_\mu l = l_i^\dagger \sigma^\mu (\partial_\mu \delta_{ij} - i \frac{1}{2} g_1 B_\mu \delta_{ij} + i g_2 \frac{\sigma^a}{2} W_\mu^a) l_j \tag{2.154}$$

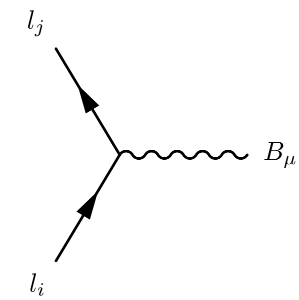
- Right up-quarks carry one colour index, i.e. $u_{R,\alpha}$ with $\alpha = 1, 2, 3$

$$u_R^\dagger \sigma^\mu D_\mu u_R = u_{R,\alpha}^\dagger \sigma^\mu (\partial_\mu \delta_{\alpha\beta} - i\frac{2}{3} g_1 B_\mu \delta_{\alpha\beta} + i g_3 \frac{\lambda^a}{2} G^a) u_{R\beta} \tag{2.155}$$

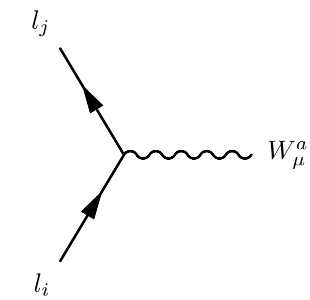
From these expressions the vertices are derived:



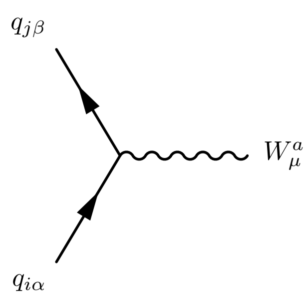

$\sigma_\mu g_1$



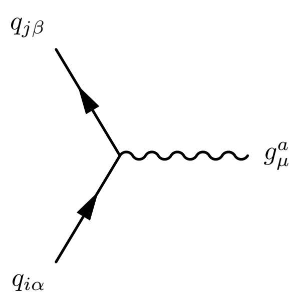

$-\sigma_\mu \delta_{ij} \frac{g_1}{2}$




$\sigma_\mu \sigma^a_{ij} \frac{g_2}{2}$




$\sigma_\mu \delta_{\alpha\beta} \sigma^a_{ij} \frac{g_2}{2}$




$\sigma_\mu \delta_{ij} \lambda^a_{ij} \frac{g_3}{2}$

## 2.4.4 Electroweak symmetry breaking

### 2.4.4.1 The Higgs potential

The Higgs potential in the SM is given by

$$V(H) = \frac{1}{2} \lambda |H|^4 + \mu^2 |H|^2 \tag{2.156}$$

Note, different conventions for the normalisation of the quartic coupling exist in literature. $\mu^2 < 0$ causes a spontaneous breaking of the electroweak symmetry (EWSB). The Higgs field becomes

$$\begin{pmatrix} H^+ \\ H^0 \end{pmatrix} \to \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}} \left( h + i G^0 + v \right) \end{pmatrix} \tag{2.157}$$

The Higgs potential becomes

$$V = \frac{1}{8}\lambda((G^0)^2 + (h+v)^2 + 2G^+G^-)^2 + \frac{1}{2}\mu^2((G^0)^2 + (h+v)^2 + 2G^+G^-) \tag{2.158}$$

We can calculate the Higgs coupling and masses form this potential

**a) Tadpole conditions**: The condition for being at the minimum of the potential is

$$\frac{\partial V(h=0)}{\partial v} \equiv 0 = \frac{\partial}{\partial v}\left(\frac{1}{8}\lambda v^4 + \frac{1}{2}\mu^2 v^2\right) \tag{2.159}$$

$$= \frac{1}{2}\lambda v^3 + \mu^2 v \tag{2.160}$$

$$\rightarrow \mu^2 = -\frac{1}{2}v^2\lambda \tag{2.161}$$

Thus, one can eliminate $\mu^2$ from all following expressions.

**b) CP-even mass**: the Higgs mass is given by

$$m_h^2 = \frac{\partial^2 V}{\partial h^2}|_{h=G^0=G^+=0} \tag{2.162}$$

$$= \frac{3}{2}\lambda v^2 + \mu^2 \tag{2.163}$$

$$= \frac{3}{2}\lambda v^2 - \frac{1}{2}\lambda v^2 \tag{2.164}$$

$$= \lambda v^2 \tag{2.165}$$

**c) Goldstone masses**: the mass of $G^0$ becomes

$$m_{G^0}^2 = \frac{\partial^2 V}{\partial {G^0}^2}|_{h=G^0=G^+=0} \tag{2.166}$$

$$= \mu^2 + \frac{1}{2}\lambda v^2 \qquad = 0 \tag{2.167}$$

Since we are working here in Landau gauge, the Goldstone mass vanishes as expected. Similarly, one can show $m_{G^+}^2 = 0$

**d) Cubic Higgs coupling**: the cubic Higgs self-interaction is

$$c_{hhh} = \frac{\partial^3 L}{\partial h^3}|_{h=G^0=G^+=0} \tag{2.168}$$

$$= -3v\lambda \tag{2.169}$$

$$= -3\frac{m_h^2}{v} \tag{2.170}$$

**e) Quartic Higgs coupling**: the quartic Higgs self-interaction is

$$c_{hhhh} = \frac{\partial^4 L}{\partial h^4}|_{h=G^0=G^+=0} \tag{2.171}$$

$$= -3\lambda \tag{2.172}$$

The entire Higgs sector of the SM can be parametrised after EWSB by just two parameters: $\lambda$ and $v$.

#### 2.4.4.2 Electroweak gauge bosons

The gauge interactions of the Higgs field become after EWSB:

$$D_\mu H D^\mu H^* = \left(\partial_\mu \delta_{ik} + i(\frac{1}{2} g_1 B_\mu \delta_{ik} + g_2 \frac{\sigma^a_{ik}}{2} W^a) H_i\right) \left(\partial_\mu \delta_{jk} - i(\frac{1}{2} g_1 B_\mu \delta_{jk} + g_2 \frac{\sigma^a_{jk}}{2} W^a) H_j^*\right) \tag{2.173}$$

$$\begin{aligned} =&\frac{1}{2}\partial_\mu h \partial^\mu h + \frac{1}{2}\partial_\mu G^0 \partial^\mu G^0 + \partial_\mu G^+ \partial^\mu G^- \\ &+ \frac{1}{4}\left((h+v)^2 + (G^0)^2\right)\left(g_1^2 B^2 - 2 g_1 g_2 B W^3 + g_2^2 (W_1^2 + W_2^2 + W_3^2)\right) \\ &+ \dots \end{aligned} \tag{2.174}$$

On can see in the second line that not only mass terms for the vector bosons are generated, but also a mixing between $B$ and $W^3$ occurs. The neutral mass matrix $M_V$ reads

$$M_V^2 = (B W_3) \begin{pmatrix} \frac{1}{4} v^2 g_1^2 & -\frac{1}{4} g_1 g_2 v^2 \\ -\frac{1}{4} g_1 g_2 v^2 & \frac{1}{4} g_2^2 v^2 \end{pmatrix} \begin{pmatrix} B \\ W_3 \end{pmatrix} \tag{2.175}$$

The mixed particles, which appear after diagonalisation, are called photon ($\gamma$) and Z-Boson ($Z$). Their masses are the eigenvalues which are given by

$$m_\gamma = 0 \tag{2.176}$$

$$m_Z^2 = \frac{1}{4}(g_1^2 + g_2^2) v^2 \tag{2.177}$$

The rotation matrix which diagonalises $M_V^2 4$ is

$$\begin{pmatrix} \gamma \\ Z \end{pmatrix} = \begin{pmatrix} \cos\Theta_W & \sin\Theta_W \\ -\sin\Theta_W & \cos\Theta_W \end{pmatrix} \begin{pmatrix} B \\ W^3 \end{pmatrix} \tag{2.178}$$

with the Weinberg angle $\Theta_W$. This defines the electric charge as:

$$e = g_1 \cos\Theta_W = g_2 \sin\Theta_W \tag{2.179}$$

One remaining massless gauge boson corresponds to one unbroken symmetry. Therefore, the remaining symmetry of the SM is

$$\mathcal{G} \to SU(3)_C \times U(1)_{em} \tag{2.180}$$

Since $W_1$ and $W_2$ are not electromagnet eigenstates, they are combined to new eigenstate of $U(1)_{em}$

$$W^\pm = \frac{1}{\sqrt{2}}(W_1 \pm i W_2) \tag{2.181}$$

The mass of $W^\pm$ is given by

$$M_W^2 = \frac{1}{4} g^2 v^2 \tag{2.182}$$

The massless states $G^0$ and $G^\pm$ are the Goldstone bosons of $Z$ and $W^\pm$ and form their longitudinal components.

Let's count the (real) degrees of freedom

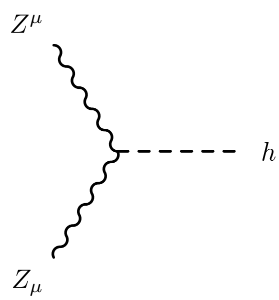
| Before EWSB | | | After EWSB | | |
|---|---|---|---|---|---|
| massless vectors: | $B$, $W^a$ | 4 | massless vectors: | $\gamma$ | 1 |
| massive vectors: | - | 0 | massive vectors: | $Z$, $W^+$ | 3 |
| complex scalars: | $H^0$, $H^\pm$ | 4 | complex scalars: | $G^\pm$ | 2 |
| real scalars: | - | 0 | real scalars: | $h$, $G^0$ | 2 |

The kinetic term for the mass eigenstates $h$, the SM Higgs boson, becomes after applying all rotations:

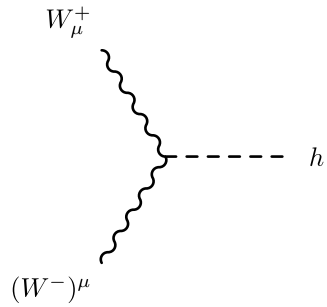
$$\begin{aligned}
\mathcal{L} =& \left(\partial_\mu \delta_{ij} + i\frac{1}{2} g_1 \delta_{ij} B_\mu + i\frac{1}{2} g_2 \sigma^a_{ij} W^a_\mu\right) H_i \left(\partial^\mu \delta_{ij} - i\frac{1}{2} g_1 \delta_{ij} B^\mu - i\frac{1}{2} g_2 \sigma^a_{ji} (W^a)^\mu\right) H^*_j \\
=& \ldots \\
=& \frac{1}{4}(h+v)^2 \Big[2g_2^2 W^+_\mu (W^-)^\mu + \gamma_\mu Z^\mu \left((g_1^2 - g_2^2)\sin 2\Theta_W + 2g_1 g_2 \cos 2\Theta_W\right) + \\
& \qquad \gamma_\mu \gamma^\mu (g_1 \cos\Theta_W - g_2 \sin\Theta_W)^2 + Z_\mu Z^\mu (g_1 \sin\Theta_W + g_2 \cos\Theta)^2\Big] \\
& + \left(\partial_\mu + i\gamma_\mu (g_1 \cos\Theta - g_2 \sin\Theta) + iZ_\mu (g_1 \sin\Theta_W + g_2 \cos\Theta)\right) h \\
& \qquad \left(\partial^\mu - i\gamma^\mu (g_1 \cos\Theta - g_2 \sin\Theta) - iZ^\mu (g_1 \sin\Theta_W + g_2 \cos\Theta)\right) h \qquad (2.183) \\
& + \mathcal{L}(G^0, G^+, h) \\
=& \frac{1}{4}\frac{e^2}{\sin^2\Theta_W}(h+v)^2 \left(2W^+_\mu (W^-)^\mu + \frac{1}{\cos^2\Theta_W} Z_\mu Z^\mu\right) + \frac{1}{2}\partial_\mu h \partial^\mu h + \mathcal{L}(G^0, G^+, h) \qquad (2.184)
\end{aligned}$$

Thus, the couplings between the Higgs to the photon drop out after performing all replacements correctly[1]. There is also no $h-h-Z$ interaction (which is forbidden by CP), but only $h-G^0-Z$. The vertices for the Higgs to the gauge bosons are given by

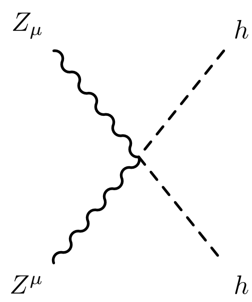
[1] 'At tree-level, the photon couples only to charged particles and the Higgs only to massive ones'



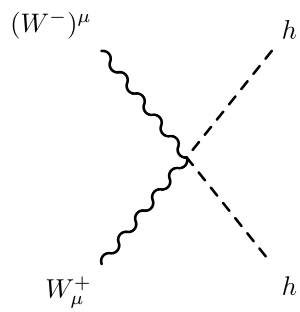

$2iv\frac{e^2}{\sin^2 2\Theta_W}$




$\frac{1}{2}iv\frac{e^2}{\sin^2 \Theta_W}$




$2i\frac{e^2}{\sin^2 2\Theta_W}$




$\frac{1}{2}i\frac{e^2}{\sin^2 \Theta_W}$

### 2.4.5 Fermion masses and Yukawa sector

It is not possible in the SM to write down mass terms for fermions because of the quantum numbers for left and right fields.

⇒ Fermion masses are spontaneously generated after EWSB via interactions with the Higgs field

The interactions between the Higgs and the SM fermions are called 'Yukawa' interactions.

$$\mathcal{L}_Y = Y_u q^\dagger u_R H + Y_d q^\dagger d_R H^* + Y_e l^\dagger e_R H^* \quad + \text{h.c.} \tag{2.185}$$

In the general case, $Y_f$ are (complex) $3 \times 3$ matrices. Thus, in the most general form the Lagrangian reads with all indices written explicitly

$$Y_u q^\dagger u_R H \equiv \delta_{\alpha\beta} Y_{u,ab} q^\dagger_{ai\alpha} u_{R,b\beta} \epsilon_{ij} H_j \tag{2.186}$$

If we neglect flavour mixing for the moment, one can write

$$\mathcal{L}_{Y_u} = Y_u q^\dagger_{i\alpha} u_{R,\beta} \epsilon_{ij} H_j \tag{2.187}$$
$$= Y_u (u^\dagger_{L,\alpha} H_0 - d^\dagger_{L,\alpha} H^+) u_{R,\beta} \delta_{\alpha\beta} \tag{2.188}$$

what becomes after EWSB

$$\mathcal{L}_{Y_u} = \frac{1}{\sqrt{2}}(v+h) Y_u u^\dagger_L u_R + \ldots \tag{2.189}$$

i.e. the fermion mass is given by

$$m_u = \frac{1}{\sqrt{2}} v Y_u \tag{2.190}$$

If we include flavour mixing, the mass terms for the quarks after EWSB read

$$\mathcal{L}_q = (d_L^\dagger s_L^\dagger b_L^\dagger) \begin{pmatrix} vY_{d,11} & vY_{d,12} & vY_{d,13} \\ vY_{d,21} & vY_{d,22} & vY_{d,33} \\ vY_{d,31} & vY_{d,32} & vY_{d,33} \end{pmatrix} \begin{pmatrix} d_R \\ u_R \\ b_R \end{pmatrix} + (u_L^\dagger c_L^\dagger t_L^\dagger) \begin{pmatrix} vY_{u,11} & vY_{u,12} & vY_{u,13} \\ vY_{u,21} & vY_{u,22} & vY_{u,33} \\ vY_{u,31} & vY_{u,32} & vY_{u,33} \end{pmatrix} \begin{pmatrix} u_R \\ c_R \\ t_R \end{pmatrix} \tag{2.191}$$

The six quark masses are the eigenvalues of the matrices $vY_d$ and $vY_u$. These matrices are diagonalised by four unitary matrices:

$$u_R \to U_R = U_u^* u_R \tag{2.192}$$
$$d_R \to D_R = U_d^* d_R \tag{2.193}$$
$$u_L \to U_L = V_u u_L \tag{2.194}$$
$$d_L \to D_L = V_d u_L \tag{2.195}$$

Only one combination of these matrices is physically relevant and defines the CKM (Cabibbo-Kobayashi-Maskawa) matrix

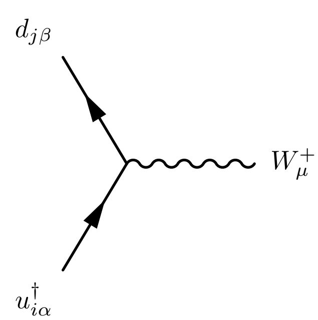
$$V_{\text{CKM}} = V_u^\dagger V_d \tag{2.196}$$

The entire flavour structure of the SM quark sector is encoded in the CKM matrix which can be parametrised by three angles $\Theta_{12}$, $\Theta_{23}$, $\Theta_{13}$ and one phase $\delta$

$$V_{\text{CKM}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13} e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.197}$$

$\delta$ is the only source of CP violation in the SM and highly restricted by experiments

The CKM matrix shows up explicitly in vertices involving the $W$-boson




$$-\frac{i}{\sqrt{2}} g_2 \sigma_\mu V_{CKM}^{ij} \delta_{\alpha\beta}$$

Chapter 3

# Supersymmetric Formalities