Introduction to Supersymmetry

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Florian Staub,
florian.staub@kit.edu

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Contents

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Chapter 1

Motivation to look beyond the SM

The standard model of particle physics (SM) is very successful and experimentally well confirmed. However, some questions can't be addressed within the SM.

1.1 Observations

1.1.1 Dark Matter

The energy budget of the universe is well known today:

Visible Matter	0.03%	Heavy Elements
	0.3%	Neutrinos
	0.5%	Stars
	4 %	Free hydrogen and helium
Dark Matter	25 %	Weakly interacting new particle (WIMP)?
Dark Energy	70%	???

⇒ The SM can only explain 4.9% of the entire energy in the universe

1.1.2 Baryon Asymmetry

We don't see any anti-matter in the observable universe. However, the Big Bang should have produced equal amounts of matter and anti-matter, i.e. the asymmetry must have been introduced later.
In general: one needs interactions which violate CP (charge-parity) to break the symmetry between matter and anti-matter.

⇒ The amount of CP violation in the SM is too small to explain the observed matter–anti-matter asymmetry

1.2 Experimental deviations

Not all experiments are in perfect agreement with the SM. In some observables, a sizeable deviation was found

Anomalous magnetic dipole moment

The magnetic momentum of an elementary particle is given by

$$m_S = -\frac{g\mu_B S}{\hbar} \tag{1.1}$$

μ_B: Bohr magneton; S: Spin
The g factor is predicted to be **2** by Dirac's theory, but higher order effects change this.:

$$\text{Anomalous magnetic moment} \qquad a = \frac{g-2}{2} \tag{1.2}$$

The anomalous magnetic moments are among the best measured and most precisely calculated observables:

$$a_\mu^{\mathrm{SM}} = 0.001\ 165\ 918\ 04\ (51) \tag{1.3}$$

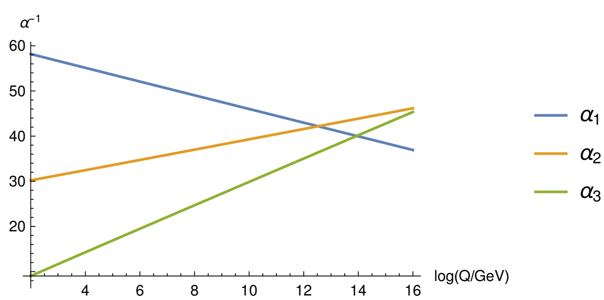
$$a_\mu^{\mathrm{exp}} = 0.001\ 165\ 920\ 9(6) \tag{1.4}$$

⇒ There is a 3.5 σ deviation between the measured anomalous magnetic moment of the myon and the SM prediction

1.3 Theoretical Issues

1.3.1 Gauge coupling unification

The coupling strength between particles is an energy dependent quantity. The energy dependence is described by the renormalisation group equations (RGEs). For the three gauge couplings of the SM one finds the following behaviour:



⇒ The gauge couplings in the SM don't unify. However, a grand unified theory (GUT) like $SO(10)$ or $SU(5)$ predict such an unification.

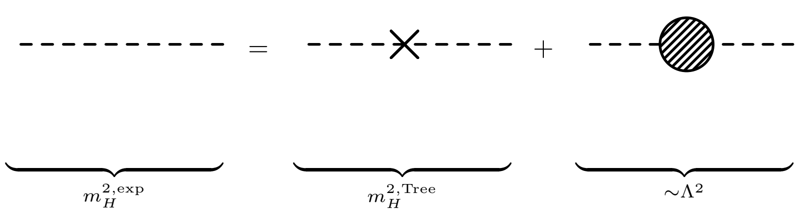
It's not possible to embed the SM in a GUT theory without introducing new matter. It's not clear at which scale the new particles come into play. However, the lighter they are, the bigger their impact is: less particles are needed in low-scale BSM models.

1.3.2 Hierarchy problem

The Higgs particle is the only fundamental scalar in the SM. While fermion and vector boson masses are protected by symmetries (chiral and gauge symmetries) against large radiative corrections, the masses of scalars don't have such a protection mechanism. Therefore, the observable mass is given by

$$m^{2,\text{obs}} = m^{2,\text{Tree}} + \delta m^2 \tag{1.5}$$
$$\simeq m^{2,\text{Tree}} + \Lambda^2 \tag{1.6}$$

where $m^{2,\text{Tree}}$ is the mass parameter in the Lagrangian and Λ is the scale of new physics. We know that (at least) one scale exists at which new interactions come into play: the Planck scale ($M_P \sim 10^{18}$ GeV) at which gravity becomes important.



(1.7)

⇒ The SM has no natural explanation why the observed Higgs mass is ~ 125 GeV, but it demands a cancellation of 32 digits between unrelated parameters.

1.4 Why supersymmetry?

Supersymmetry (SUSY) provides possible explanations for all these questions:

- New Particles can form the DM
- New sources of CP violation to generate the Baryon asymmetry
- New loop contributions to a_μ
- Changes the running of gauge couplings → Unification!
- The Higgs mass is protected by the new symmetry and naturally light

Because of these reasons, minimal supersymmetry was for a long time the top candidate for an extensions of the SM. However, with the negative searches at LHC the picture is changing: heavier SUSY masses introduce a new (small) hierarchy problem in the theory. Nevertheless:

- Other benefits of SUSY (dark matter, gauge coupling unification, CP violation) are hardly affected
- The corrections to the Higgs mass are only logarithmic dependent on the SUSY scale, not quadratic as in the SM alone

- There are still unexplored corners in which light SUSY particles are possible within minimal supersymmetry
- There is an increasing interest in non-minimal SUSY models which avoid the small hierarchy problem

Chapter 2

Basics

2.1 Notations and conventions

- Natural units (formally $\hbar = c = 1$) are used everywhere.
- Lorentz indices are always denoted by Greek characters, $\mu, \nu, .. = 0, 1, 2, 3$.
- Four-vectors for space–time coordinates and particle momenta are written as

$$x = (x^\mu) = (x^0, \vec{x}), \quad x^0 = t\,,$$
$$p = (p^\mu) = (p^0, \vec{p}), \quad p^0 = E = \sqrt{\vec{p}^2 + m^2}\,.$$

- Co- and Contravariant vectors are related by

$$a_\mu = g_{\mu\nu}\, a^\nu,$$

  with the metric tensor

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- The 4-dimensional scalar product is

$$a^2 = g_{\mu\nu}\, a^\mu a^\nu = a_\mu a^\mu, \quad a \cdot b = a_\mu b^\mu = a^0 b^0 - \vec{a} \cdot \vec{b}\,.$$

- Covariant and contravariant components of the derivatives are written as

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = g_{\mu\nu}\, \partial^\nu, \quad \partial^\nu = \frac{\partial}{\partial x_\nu} \qquad [\, \partial^0 = \partial_0, \ \ \partial^k = -\partial_k \,]\,,$$
$$\Box = \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \Delta\,.$$

2.2 Group Theory

2.2.1 Axioms

A collection of elements g_i form a group if the following conditions are fulfilled:

a) **Closure** under a multiplication operator; i.e., if g_i and g_j are members of the group, then $g_i \cdot g_j$ is also a member of the group

b) **Associativity** under multiplication; i.e.

$$g_i \cdot (g_j \cdot g_k) = (g_i \cdot g_j) \cdot g_k \tag{2.1}$$

c) **An identity element**; i.e., there exist an element $\mathbf{1}$ such that

$$\mathbf{1} \cdot g_i = g_i \cdot \mathbf{1} = g_i \tag{2.2}$$

d) **An inverse**; i.e. every element g_i has an element g_i^{-1} such that

$$g_i \cdot g_i^{-1} = \mathbf{1} \tag{2.3}$$

2.2.2 Lie Groups

2.2.2.1 Definition

Lie Groups are both groups and differentiable manifolds.

Any group element continuously connected to the identity can be written

$$U = e^{i\Theta_a T^a} \tag{2.4}$$

where the Θ_a is a real parameter and the T^a are the group generators, which live in the Lie Algebra.

The generators T^a , which generate infinitesimal group transformations, form the Lie Algebra.

The Lie algebra is defined by its commutation relations

$$[T^a, T^b] = i f^{abc} T_c \tag{2.5}$$

where f^{abc} are known as the structure constants.

By definition they are antitsymmetric

$$f^{abc} = -f^{acb} \tag{2.6}$$

We are interested in so called semi-simple Lie groups as $SU(N)$ and $SO(N)$. We focus in the following on $SU(N)$. These groups preserve a complex inner product. Finite dimensional representations of semi-simple Lie algebras are always Hermitian, so one can build quantum theories which are unitarity based on such algebras. The complex inner product is

$$U^\dagger U = 1 \tag{2.7}$$

defined on N dimensional complex vector spaces, for $U(N)$. Note that in all cases we can write $U(N) = SU(N) \times U(1)$ where the $U(1)$ represents an overall phase. There are $N^2 - 1$ generators for $SU(N)$. To see this, let us write the identity infinitesimally as

$$0 = 1 - e^0 \tag{2.8}$$
$$= 1 - e^{[i\Theta_a T_a + (i\Theta_a T_a)^\dagger]} \tag{2.9}$$
$$= 1 - (1 + i\Theta_a T_a)(1 - i\Theta_a T_a^\dagger) \tag{2.10}$$
$$= -i\Theta_a (T^\dagger)_a + i\Theta_a T_a \tag{2.11}$$
$$\Rightarrow T = T^\dagger \tag{2.12}$$

so we can count the generators by counting $N \times N$ Hermitian matrices. Such matrices have $\frac{1}{2}N(N-1)$ imaginary components and $\frac{1}{2}N(N+1)$ real components, but then we subtract the identity matrix, which just generates $U(1)$. Thus, we find for the number of generators

$$\#(T_a) = \frac{1}{2}N(N-1) + \frac{1}{2}N(N+1) - 1 = N^2 - 1 \tag{2.13}$$

2.2.2.2 Representations

The groups and algebras discussed above are abstract mathematical objects. We want to have these groups act on quantum states and fields, which are vectors, so we need to represent the groups as matrices. There are an infinite number of different representations for a given simple group. However, there are two obvious and most important representations, which occur most often in physics settings. They are

a) the fundamental representations

b) the adjoint representations

> The fundamental representation is the representation defining $SU(N)$ and $SO(N)$ as $N \times N$ matrices acting on N dimensional vectors. To write the fundamental formally, we say that N fields transform under it as
>
> $$\phi_i \to \phi_i + i\alpha_a (T_f^a)_i^j \phi_j \tag{2.14}$$
>
> where $i = 1, \ldots, N$, $a = 1, \ldots N^2 - 1$ and the α_a are real numbers. The complex conjugate fields transform in the anti-fundamental $\overline{f}$, which is just the conjugate of this
>
> $$\phi_i^* \to \phi_i^* - i\alpha_a (T_f^{a*})_i^j \phi_j^* \tag{2.15}$$

Since T_f^a are Hermitian, we have $T_{\bar{\mathrm{f}}} = (T_f)^*$.
The normalisation of generators is arbitrary and is usually chosen so that

$$\mathrm{Tr} T_f^a T_f^b = \frac{1}{2}\delta_{ab} \tag{2.16}$$

> The other important representation is the adjoint. The point is to think of the generators themselves as the vectors. Thus, the generators are
>
> $$(T_{\mathrm{adj}}^a)_c^b = -if^{abc} \tag{2.17}$$

How can we see that the $T_{\rm adj}$ actually satisfy the Lie algebra, and thus are really a representation? This is given immediately by the Jacobi identity

$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0 \tag{2.18}$$

written as

$$\begin{aligned}
0 &= [T^a, f^{bcd}T_d] + [T^b, f^{cad}T_d] + [T^a, f^{abd}T_d] && (2.19)\\
&= f^{bcd}[T^a, T_d] + f^{cad}[T^b, T_d] + f^{abd}[T^a, T_d] && (2.20)\\
&= f^{bcd}f^{ade}T_e + f^{cad}f^{bde}T_e + f^{abd}f^{ade}T_d && (2.21)\\
&\Rightarrow f^{cbd}f^{ade} - f^{abd}f^{cde} = f^{cad}f^{dbe} && (2.22)\\
&\Rightarrow [T^c_{\rm adj}, T^a_{\rm adj}] = i f^{cad}T^d_{\rm adj} && (2.23)
\end{aligned}$$

The dimension of the adjoint representation is N^2-1 for $SU(N)$.

2.2.2.3 Group constants

The quadratic Casimir is defined as

$$T^a_R T^a_R = C_2(R)\mathbf{1} \tag{2.24}$$

This must be proportional to the identity (when acting on a single given irreducible representation) because it commutes with all generators of the group, which follows from

$$\begin{aligned}
[T^a_R T^a_R, T^b_R] =& T^a_R T^a_R T^b_R - T^b_R T^a_R T^a_R && (2.25)\\
=& T^a_R([T^a_R, T^b_R] + T^b_R T^a_R]) - ([T^b_R, T^a_R] + T^a_R T^b_R) T^a_R && (2.26)\\
=& T^a_R(i f^{abc} T^c_R) - (i f^{bac} T^c_R T^a_R) && (2.27)\\
=& i f^{abc} T^a_R T^c_R + i f^{abc} T^c_R T^a_R && (2.28)\\
=& 0 && (2.29)
\end{aligned}$$

because of anti-symmetry of f^{abc}.
Another important quantity is the Dynkin index $I(R)$

$$\mathrm{Tr}[T^a_R T^b_R] = I(R)\delta_{ab} \tag{2.30}$$

The quantity $I(R)$ is the index of the representation. We have that

$$I(f) = \frac{1}{2} \tag{2.31}$$

and

$$I(G) = N \tag{2.32}$$

for $SU(N)$ and our normalisation. The Dynkin index and the quadratic Casimir are related

$$d(R)C_2(R) = I(R)d(G) \tag{2.33}$$

where $d(R)$ is the dimension of the representation, and $d(G)$ of the algebra, namely N^2-1 for $SU(N)$. Thus

$$\begin{aligned}
C_2(f) =& \frac{N^2-1}{N} && (2.34)\\
C_2(G) =& N && (2.35)
\end{aligned}$$

2.2.2.4 Examples

2.2.2.4.1 $SU(2)$

For $SU(2)$ the common generators for the fundamental representation T_f^a are related to the Pauli matrices σ^a $(i = 1, 2, 3)$ by

$$T_f^a = \frac{1}{2}\sigma^a \tag{2.36}$$

with

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.37}$$

For later, it is also helpful to introduce

$$\sigma^0 = \overline{\sigma}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.38}$$

and $\overline{\sigma}^i = -\sigma^i$. The Lie algebra

$$[\sigma^a, \sigma^b] = i f^{abc}\sigma_c \tag{2.39}$$

is fulfilled for

$$f^{abc} = \epsilon^{abc} \tag{2.40}$$

where ϵ^{abc} is the Levi-Civita tensor. And we have

$$d(f) = 2 \qquad d(a) = 3 \tag{2.41}$$

$$C_2(f) = \frac{3}{2} \qquad C_2(a) = 3 \tag{2.42}$$

$$I(f) = \frac{1}{2} \qquad I(a) = 2 \tag{2.43}$$

2.2.2.4.2 SU(3)

The common representation for $SU(3)$ are given by the Gell-Mann Matrices λ^a

$$T_f^a = \frac{1}{2}\lambda^a \tag{2.44}$$

With

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.45}$$

$$\lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{2.46}$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.47}$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{2.48}$$

And we have

$$d(f) = 3 \qquad d(a) = 8 \tag{2.49}$$

$$C_2(f) = \frac{4}{3} \qquad C_2(a) = 3 \tag{2.50}$$

$$I(f) = \frac{1}{2} \qquad I(a) = 3 \tag{2.51}$$

2.2.3 Other groups relevant in particle physics

a) Lorentz Group: the Lorentz group is the set of all 4×4 real matrices that leave the line element in Minkowski space invariant:

$$s^2 = (x^0)^2 - (x^i)^2 = x^\mu g_{\mu\nu} x^\nu \tag{2.52}$$

It is parametrised by

$$x'^\mu = \Lambda^\mu_\nu x^\nu \tag{2.53}$$

The Lorentz group has six generators:

- three generators J^i creating rotations
- three generators K^i creating boosts

b) Poincare Group: the Poincare group is the generalisation of the Lorentz group including translation:

$$x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu \tag{2.54}$$

The generator of the translation is the four momentum operator p_μ

2.3 Quantum Field Theory

2.3.1 Lagrangian formalism

We are working with the Lagrangian formalism of quantum field theory. The basic features are

- space–time symmetry in terms of Lorentz invariance, as well as internal symmetries like gauge symmetries
- causality
- local interactions

Particles are described by fields that are operators on the quantum mechanical Hilbert space of the particle states, acting as creation and annihilation operators for particles and antiparticles. We need in the following particles characterised by their spin:

- spin-0: complex or real scalar fields $\phi(x)$, $\varphi(x)$
- spin-$\frac{1}{2}$: fermions, described by two- or four component spinor fields $\psi_{L,R}$, $\psi(x)$.
- spin-1: vector bosons $A_\mu(x)$

The dynamics of the physical system involving a set of fields Φ is determined by the Lorentz-invariant Lagrangian $\mathcal{L}$. The action is given by

$$S[\Phi] = \int \mathrm{d}^4x\, \mathcal{L}\big(\Phi(x)\big)\,, \tag{2.55}$$

The equations of motions follow as Euler–Lagrange equations from Hamilton's principle,

$$\delta S = S[\Phi + \delta\Phi] - S[\Phi] = 0\,. \tag{2.56}$$

Let's go back to mechanics: for n generalised coordinates q_i and velocities $\dot{q}_i$ the Lagrangian reads: $L(q_1, \dots \dot{q}_1, \dots)$ The equations of motion are calculated from $(i = 1, \dots n)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0\,. \tag{2.57}$$

Going to field theory, one has to perform the replacement

$$q_i \to \Phi(x)\,, \quad \dot{q}_i \to \partial_\mu \Phi(x)\,, \quad L(q_1, \dots q_n, \dot{q}_1, \dots \dot{q}_n) \to \mathcal{L}(\Phi(x), \partial_\mu \Phi(x)) \tag{2.58}$$

The equations of motion become field equations which are calculated from

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} - \frac{\partial \mathcal{L}}{\partial \Phi} = 0\,, \tag{2.59}$$

2.3.2 Free quantum fields

2.3.2.1 Scalar fields

The equation of motion for a scalar field is known as K̂lein–Gordon equation:

$$(\partial_\mu \partial^\mu + m^2)\,\phi = 0\,. \tag{2.60}$$

The solution can be expanded in terms of the complete set of plane waves $e^{\pm ikx}$,

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3k}{2k^0} \left[a(k)\, e^{-ikx} \; + \; a^\dagger(k)\, e^{ikx}\right] \tag{2.61}$$

with $k^0 = \sqrt{\vec{k}^2 + m^2}$. Here, we used annihilate and creation operators $a^\dagger$, a:

$$\begin{aligned} a^\dagger(k)\,|0\rangle &= |k\rangle \\ a(k)\,|k'\rangle &= 2k^0\,\delta^3(\vec{k} - \vec{k}\,')\,|0\rangle\,. \end{aligned} \tag{2.62}$$

The Lagrangian for a free real or complex scalar field with mass m is

$$\mathcal{L}_{\text{real}} = \frac{1}{2}\,(\partial_\mu\phi)^2 - \frac{m^2}{2}\,\phi^2 \tag{2.63}$$

$$\mathcal{L}_{\text{complex}} = (\partial_\mu\phi)^\dagger(\partial^\mu\phi) - m^2\,\phi^\dagger\phi \tag{2.64}$$

One can easily check that they give us the Klein–Gordon Equation as equation of motion. A complex scalar field $\phi^\dagger \neq \phi$ has two degrees of freedom. It describes spin-less particles which carry a charge and can be interpreted as particles and antiparticles.
So far, we have considered particles without any space–time restrictions. Now, we want to consider the case that a particle propagates from a point-like source at a given space-time point. This is described by the inhomogeneous field equation

$$(\partial_\mu\partial^\mu \; + m^2)\,D(x-y) = -\delta^4(x-y)\,. \tag{2.65}$$

$D(x-y)$ is called Green function. The solution can easily be determined by a Fourier transformation

$$D(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\,D(k)\,e^{-ik(x-y)} \tag{2.66}$$

giving in momentum space

$$(k^2 - m^2)\,D(k) = 1\,. \tag{2.67}$$

The solution

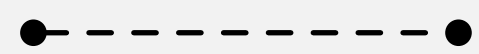
$$i\,D(k) = \frac{i}{k^2 - m^2 + i\epsilon} \tag{2.68}$$

is the *causal Green function* or the *Feynman propagator* of the scalar field. The overall factor i is by convention. The term $+i\epsilon$ in the denominator with an infinitesimal $\epsilon > 0$ is a prescription of how to treat the pole in the integral (2.66); it corresponds to the special boundary condition of causality for $D(x-y)$ in Minkowski space, which means

- propagation of a particle from y to x if $x^0 > y^0$,
- propagation of an antiparticle from x to y if $y^0 > x^0$.

In a Feynman diagram, the scalar propagator is drawn as dashed line.

Complex Scalar $\phi(k,m)$  $\frac{1}{k^2-m^2+i\epsilon}$ (2.69)

Real Scalar $\varphi(k,m)$  $\frac{1}{k^2-m^2+i\epsilon}$ (2.70)

For complex scalars the arrow shows the flow of the charge.

2.3.2.2 Dirac fields

Equation of motion Spin-$\frac{1}{2}$ particles with mass m are often described by 4-component spinor fields,

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}. \tag{2.71}$$

and obey the *Dirac–Equation*

$$(i\gamma^\mu \partial_\mu - m)\,\psi = 0\,. \tag{2.72}$$

This equation is obtained from the Lagrangian

$$\mathcal{L}_{\text{fermion}} = \overline{\psi}\,(i\gamma^\mu \partial_\mu - m)\,\psi\,, \tag{2.73}$$

involving the adjoint spinor

$$\overline{\psi} = \psi^\dagger\,\gamma^0 = (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*)\,. \tag{2.74}$$

The Dirac matrices γ^μ ($\mu = 0,1,2,3$) are 4×4 matrices which fulfil the anti-commutator relations

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\,. \tag{2.75}$$

One possible representation is to express the matrices in terms of the the Pauli matrices $\sigma_{1,2,3}$ as

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}. \tag{2.76}$$

Another matrix, γ_5, is often very useful:

$$\gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.77}$$

There are two types of solutions for the Dirac equation, corresponding to particle and anti-particle wave functions,

$$u(p)\,e^{-ipx} \quad \text{and} \quad v(p)\,e^{ipx} \tag{2.78}$$

which are used to write the Dirac field as

$$\psi(x) = \frac{1}{(2\pi)^{3/2}} \sum_\sigma \int \frac{\mathrm{d}^3 k}{2k^0} \left[c_\sigma(k)\, u_\sigma(k)\, e^{-ikx} \,+\, d_\sigma^\dagger(k)\, v_\sigma(k)\, e^{ikx} \right], \tag{2.79}$$

with

- annihilation operators c_σ for particles and d_σ for anti-particles
- creation operators $c_\sigma^\dagger$ and $d_\sigma^\dagger$ for particles and antiparticles

We still have to determine the propagator of the Dirac field, which is the solution of the inhomogeneous Dirac equation with point-like source,

$$(i\gamma^\mu \partial_\mu - m)\, S(x-y) = \mathbf{1}\, \delta^4(x-y)\,. \tag{2.80}$$

Using a Fourier transformation as in the scalar case, we find

$$i\,S(k) = \frac{i}{\not{k} - m + i\epsilon} = \frac{i\,(\not{k} + m)}{k^2 - m^2 + i\epsilon}\,, \tag{2.81}$$

We introduce a graphical symbol for the propagator:

Dirac Fermion $\psi(k,m)$  $i\frac{\not{k}-m}{k^2-m^2+i\epsilon}$ (2.82)

The arrow at the line denotes the flow of the *particle* charge.
External fermions are depicted as

incoming particle  $u(k)$ (2.83)

incoming anti-particle  $\overline{v}(k)$ (2.84)

outgoing anti-particle  $v(k)$ (2.85)

outgoing particle  $\overline{u}(k)$ (2.86)

2.3.2.3 Vector fields

A vector field $A_\mu(x)$ describes particles with spin 1. We concentrate here on the massless case with two degrees of freedom.
The Lagrangian of such a field is

$$\mathcal{L} = -\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g f^{abc} A^b_\mu A^c_\nu\,. \tag{2.87}$$

The last term is only present for non-Abelian gauge fields. The field equations are Maxwell's equations for the vector potential,

$$\left(\Box\, g^{\mu\nu} - \partial^\mu \partial^\nu\right) A_\nu = 0\,. \tag{2.88}$$

The propagator of the vector fields depends on the chosen gauge. In general R_ξ gauge it is given by

$$i\,D_{\rho\nu}(k) = \frac{i}{k^2 + i\epsilon}\left[-g_{\nu\rho} + (1-\xi)\,\frac{k_\nu k_\rho}{k^2}\right]. \tag{2.89}$$

which becomes very simple in Feynman gauge with $\xi = 1$.

The graphical symbol for the vector-field propagator (for both massive and massless) is a wavy line which carries the momentum k and two Lorentz indices

massless Vector boson $A_\mu(k)$  $-\,i\frac{g_{\mu\nu}}{k^2+i\epsilon}$ (2.90)

massive Vector boson $A_\mu(k,m)$  $-\,i\frac{g_{\mu\nu}-\frac{k_\nu k_\mu}{m^2}}{k^2-m^2+i\epsilon}$ (2.91)

(Possible) arrows at the lines denote the flow of the *particle* charge.
External vectors are depicted as

incoming particle  ϵ_μ (2.92)

outgoing particle  ϵ^*_μ (2.93)

2.3.3 Gauge invariance

So far, we have not considered any symmetry. We change that now and apply (local) gauge transformations to the fields.

$$\phi(x) \to e^{ig\Lambda(x)}\phi(x) \tag{2.94}$$
$$\phi(x)^* \to \phi(x)^* e^{-ig\Lambda(x)} \tag{2.95}$$
$$\Psi(x) \to e^{ig\Lambda(x)}\Psi(x) \tag{2.96}$$
$$\bar\Psi(x) \to \bar\Psi e^{-ig\Lambda(x)} \tag{2.97}$$

However, one can check that the Lagrangians for scalars and fermions are **not** invariant under these transformations. For instance, the fermionic part of the Lagrangian transforms as

$$\begin{aligned}
\mathcal{L}'_{\text{fermion}} =& i(\bar{\Psi})'\partial\!\!\!/(\Psi)' - m(\bar{\Psi})'\Psi' && (2.98)\\
=& i\bar{\Psi}e^{-ig\Lambda(x)}\partial\!\!\!/ e^{ig\Lambda(x)}\Psi - m\bar{\Psi}\underbrace{\left(e^{-ig\Lambda(x)}e^{ig\Lambda(x)}\right)}_{=1}\Psi && (2.99)\\
=& i\bar{\Psi}(ig\Lambda(x)\partial\!\!\!/\Lambda(x))(\partial\!\!\!/\Psi) - m\bar{\Psi}\Psi && (2.100)\\
\neq& \mathcal{L}_{\text{fermion}} && (2.101)
\end{aligned}$$

We need another ingredient to built kinetic terms for scalars and fermions which are gauge invariant: we introduce a massless gauge fields A_μ which transforms as

$$A_\mu \to A_\mu - \partial_\mu\Lambda(x) \tag{2.102}$$

In addition, we define the covariant derivative:

$$\partial_\mu \to D_\mu = \partial_\mu + igA_\mu \tag{2.103}$$

g is a free parameter which we call 'gauge coupling'. One finds that the covariant derivative transforms as

$$\begin{aligned}
(D_\mu\Psi)' =& D'_\mu\Psi' && (2.104)\\
=& (\partial_\mu + ig(A_\mu - \partial_\mu\Lambda))e^{ig\Lambda}\Psi && (2.105)\\
=& e^{ig\Lambda}(\partial_\mu + igA_\mu)\Psi - ig\partial_\mu\Lambda\Psi + (\partial_\mu e^{ig\Lambda})\Psi && (2.106)\\
=& e^{ig\Lambda}(\partial_\mu + igA_\mu)\Psi && (2.107)\\
=& e^{ig\Lambda}D_\mu\Psi && (2.108)
\end{aligned}$$

Thus, the Lagrangian with derivatives replaced by covariant derivatives are invariant.

$$\bar{\Psi}D_\mu\Psi \to (\bar{\Psi})'(D_\mu\Psi)' = \bar{\Psi}e^{-ig\Lambda}e^{ig\Lambda}D_\mu\Psi = \bar{\Psi}D_\mu\Psi \tag{2.109}$$

Similarly, one can show that for the scalar terms in the Lagrangian the identity

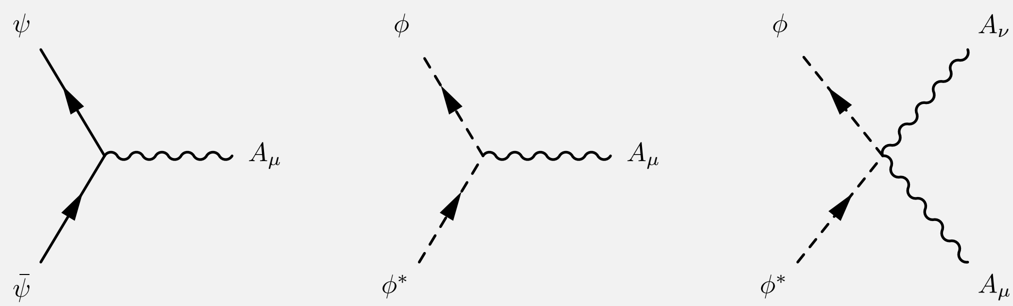
$$(D_\mu\phi D^\mu\phi^*)' = D_\mu\phi D^\mu\phi^* \tag{2.110}$$

holds.

In order to obtain a gauge theory, i.e. a theory in which the Lagrangian is invariant under a local transformation, the derivative must be replaced by the **covariant derivative** involving gauge fields:

$$\partial^\mu \to D_\mu = \partial^\mu + igTA_\mu \tag{2.111}$$

This introduces interaction terms between the fermions and scalars and the gauge fields which are represented by the following Feynman diagrams:



$ig\gamma_\mu$ $\quad ig(p^\nu_{\text{in}} - p^\mu_{\text{out}})$ $\quad ig^2 g_{\mu\nu}$

2.3.4 Spontaneous symmetry breaking

A mass term for gauge bosons would read

$$m_V^2 A_\mu A^\mu \tag{2.112}$$

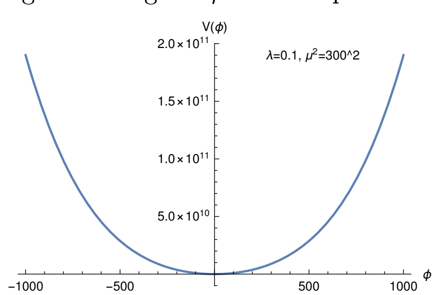
However, this is not gauge invariant:

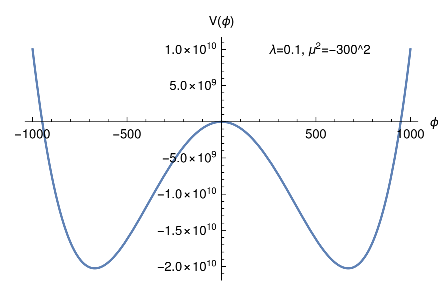
$$(m_V^2 A_\mu A^\mu)' = m_V^2 A_\mu A^\mu + m_V^2 (\partial_\mu \Lambda)(\partial^\mu \Lambda) \tag{2.113}$$

Thus, explicit mass terms are not possible and we must generated them via the so called *Higgs–mechanism*. Let's assume a real scalar φ and the following potential:

$$V(\varphi) = \frac{1}{2}\lambda\varphi^4 + \mu^2\varphi \tag{2.114}$$

Depending on the sign of μ^2 the shape of the potential is different





For

- $\mu^2 > 0$: $\varphi = 0$ is the correct vacuum
- $\mu^< 0$: the vacuum is at $\varphi \neq 0$

We shift φ in a way that we are for $\varphi = 0$ at the minimum of the potential:

$$\phi \to \phi + v \tag{2.115}$$

We find

$$V(\varphi = 0) = \frac{1}{2}\lambda v^4 + \mu^2 v^2 \tag{2.116}$$

$$\to \frac{\partial V}{\partial v} = 2\lambda v^3 + 2v\mu^2 \equiv 0 \tag{2.117}$$

Thus

$$v = \sqrt{-\mu^2/\lambda} \tag{2.118}$$

is the value of the VEV (vacuum expectation value).

Higgs mechanism We consider now a gauge theory with a complex field ϕ. We want to insert

$$\phi \to \frac{1}{\sqrt{2}}(\varphi + v + i\sigma) \tag{2.119}$$

in the general Lagrangian

$$\mathcal{L} = D_\mu \phi D^\mu \phi^* - m^2|\phi|^2 - \lambda|\phi|^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2.120}$$

We get

$$\begin{aligned}\mathcal{L} = & -\frac{1}{4}F_{\mu\nu}F^{\mu\mu} + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma \\ & + gvA_\mu\partial^\mu\sigma + \frac{1}{2}g^2v^2A_\mu A^\mu \\ & + \frac{1}{2}g^2(A_\mu)^2\varphi(2v+\varphi) - \frac{1}{2}\varphi^2(3\lambda v^2 + m^2) - \lambda v\varphi^3 - \frac{1}{4}\lambda\varphi^4\end{aligned} \tag{2.121}$$

The first line are just the ordinary kinetic terms. However, we see that an effective mass term $\frac{1}{2}g^2v^2$ for the vector bosons has been generated. There is also a term which mixes the field σ, which becomes massless, and A_μ.
A massive vector boson has three degrees of freedom, while a massless one has only two. Therefore, one says that σ is 'eaten' by the vector boson to form its longitudinal component. σ is called 'Goldstone' (or 'Nambu-Goldstone') boson.
It is common to introduce gauge fixing terms in a way that they cancel the mixing terms between field σ and A^μ.

$$\mathcal{L}_{GF} = -\frac{1}{2\xi}\left(\partial_\mu A^\mu - gv\xi\sigma\right)^2 \tag{2.122}$$

Thus, the Lagrangian becomes

$$\mathcal{L} + \mathcal{L}_{GF} = +\frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - g^2v^2\xi\sigma^2 + \frac{1}{2}g^2v^2A_\mu A^\mu + \dots \tag{2.123}$$

what gives a relation between the Goldstone mass and the mass of the vector boson

$$M_G^2 = \xi M_A^2 \tag{2.124}$$

In the unitarity gauge $\xi \to \infty$, the Goldstone disappears from the spectrum. The same could have been obtained by starting with the gauge transformation However, before we do this, we apply the following gauge transformation:

$$\phi \to \phi' = e^{-i\sigma/v}\phi = \frac{1}{\sqrt{2}}(v + \varphi) \tag{2.125}$$

$$A_\mu \to A'_\mu = A_\mu - \frac{1}{gv}\partial_\mu\sigma \tag{2.126}$$

The Higgs mechanism generates mass terms for vector-boson due to vacuum expectation values of a complex scalar field

$$\phi \to \frac{1}{\sqrt{2}}\left(\varphi + i\sigma + v\right) \tag{2.127}$$

While the real (CP-even) component φ of the scalar is a physical degree of freedom, the imaginary (CP-odd) component σ becomes the longitudinal mode of the massive vector boson. In general R_ξ gauge the Goldstone mass M_G is related to the mass M_A of the vector boson A^μ by

$$M_G^2 = \xi M_A^2 \tag{2.128}$$

2.3.5 Weyl Fermions

We have so far used 4-component (Dirac) fermions. However, it will turn out that it is often more convenient to use actually 2-component notation:

- in any model which violates parity (as the SM or all extension of it), each Dirac fermion has left-handed and right-handed parts with completely different electroweak gauge interactions:
  $\rightarrow$ The two-component Weyl fermion notation has the advantage of treating fermionic degrees of freedom with different gauge quantum numbers separately from the start.
- if one uses four-component spinor notation in the SM (or beyond), then there would be a sea of projection operators

  $$P_L = (1-\gamma_5)/2, \qquad P_R = (1+\gamma_5)/2 \tag{2.129}$$

- in supersymmetric models the minimal building blocks of matter are chiral supermultiplets, each of which contains a single two-component Weyl fermion

Since the two-component notation might be unfamiliar, we want to practice it a bit!

2.3.5.1 Two-component spinors

In this representation, a four-component Dirac spinor is written in terms of 2 two-component, complex anti-commuting objects ξ_α and $(\chi^\dagger)^{\dot\alpha} \equiv \chi^{\dagger\dot\alpha}$, with two distinct types of spinor indices $\alpha = 1,2$ and $\dot\alpha = 1,2$:

$$\Psi_D = \begin{pmatrix} \xi_\alpha \\ \chi^{\dagger\dot\alpha} \end{pmatrix}. \tag{2.130}$$

It follows that

$$\overline{\Psi}_D = \Psi_D^\dagger \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \chi^\alpha & \xi^\dagger_{\dot\alpha} \end{pmatrix}. \tag{2.131}$$

Undotted (dotted) indices from the beginning of the Greek alphabet are used for the first (last) two components of a Dirac spinor. The field ξ is called a "left-handed Weyl spinor" and $\chi^\dagger$ is a "right-handed Weyl spinor". The names fit, because

$$P_L\Psi_D = \begin{pmatrix} \xi_\alpha \\ 0 \end{pmatrix}, \qquad P_R\Psi_D = \begin{pmatrix} 0 \\ \chi^{\dagger\dot\alpha} \end{pmatrix}. \tag{2.132}$$

The Hermitian conjugate of any left-handed Weyl spinor is a right-handed Weyl spinor:

$$\psi^\dagger_{\dot\alpha} \equiv (\psi_\alpha)^\dagger = (\psi^\dagger)_{\dot\alpha}\,, \tag{2.133}$$

and vice versa:

$$(\psi^{\dagger\dot\alpha})^\dagger = \psi^\alpha. \tag{2.134}$$

Any particular fermionic degrees of freedom can be described equally well using a left-handed Weyl spinor (with an undotted index) or by a right-handed one (with a dotted index). By convention, all names of fermion fields are chosen so that left-handed Weyl spinors do not carry daggers and right-handed Weyl spinors do carry daggers.

2.3.5.2 Index operations

The heights of the dotted and undotted spinor indices are important. The spinor indices are raised and lowered using the anti-symmetric symbol

$$\epsilon^{12} = -\epsilon^{21} = \epsilon_{21} = -\epsilon_{12} = 1, \qquad \epsilon_{11} = \epsilon_{22} = \epsilon^{11} = \epsilon^{22} = 0, \tag{2.135}$$

according to

$$\xi_\alpha = \epsilon_{\alpha\beta}\xi^\beta, \qquad \xi^\alpha = \epsilon^{\alpha\beta}\xi_\beta, \qquad \chi^\dagger_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\chi^{\dagger\dot\beta}, \qquad \chi^{\dagger\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\chi^\dagger_{\dot\beta}\,. \tag{2.136}$$

This is consistent since $\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \epsilon^{\gamma\beta}\epsilon_{\beta\alpha} = \delta_\alpha^\gamma$ and $\epsilon_{\dot\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma} = \epsilon^{\dot\gamma\dot\beta}\epsilon_{\dot\beta\dot\alpha} = \delta_{\dot\alpha}^{\dot\gamma}$.
As a convention, repeated spinor indices contracted like

$$^\alpha{}_\alpha \qquad \text{or} \qquad {}_{\dot\alpha}{}^{\dot\alpha} \tag{2.137}$$

can be suppressed. In particular,

$$\xi\chi \equiv \xi^\alpha \chi_\alpha = \xi^\alpha \epsilon_{\alpha\beta}\chi^\beta = -\chi^\beta \epsilon_{\alpha\beta}\xi^\alpha = \chi^\beta \epsilon_{\beta\alpha}\xi^\alpha = \chi^\beta \xi_\beta \equiv \chi\xi \tag{2.138}$$

with, conveniently, no minus sign in the end. [A minus sign appeared in eq. (2.138) from exchanging the order of anti-commuting spinors, but it disappeared due to the anti-symmetry of the ϵ symbol.] Likewise, $\xi^\dagger\chi^\dagger$ and $\chi^\dagger\xi^\dagger$ are equivalent abbreviations for $\chi^\dagger_{\dot\alpha}\xi^{\dagger\dot\alpha} = \xi^\dagger_{\dot\alpha}\chi^{\dagger\dot\alpha}$, and in fact this is the complex conjugate of $\xi\chi$:

$$(\xi\chi)^* = \chi^\dagger\xi^\dagger = \xi^\dagger\chi^\dagger. \tag{2.139}$$

In a similar way, one can check that

$$(\chi^\dagger\overline{\sigma}^\mu\xi)^* \;=\; \xi^\dagger\overline{\sigma}^\mu\chi \;=\; -\chi\sigma^\mu\xi^\dagger \;=\; -(\xi\sigma^\mu\chi^\dagger)^* \tag{2.140}$$

stands for $\xi^\dagger_{\dot\alpha}(\overline{\sigma}^\mu)^{\dot\alpha\alpha}\chi_\alpha$, etc. Note that when taking the complex conjugate of a spinor bilinear, one reverses the order. The spinors here are assumed to be classical fields; for quantum fields the complex conjugation operation in these equations would be replaced by Hermitian conjugation in the Hilbert space operator sense.

2.3.5.3 Lagrangian for Weyl fermions

With these conventions, the Dirac Lagrangian can now be rewritten:

$$\mathcal{L}_{\text{Dirac}} \;=\; i\xi^\dagger\overline{\sigma}^\mu\partial_\mu\xi + i\chi^\dagger\overline{\sigma}^\mu\partial_\mu\chi - M(\xi\chi + \xi^\dagger\chi^\dagger) \tag{2.141}$$

where we have dropped a total derivative piece $-i\partial_\mu(\chi^\dagger\overline{\sigma}^\mu\chi)$, which does not affect the action.
A four-component Majorana spinor can be obtained from the Dirac spinor of eq. (2.131) by imposing the constraint $\chi = \xi$, so that

$$\Psi_{\text{M}} = \begin{pmatrix} \xi_\alpha \\ \xi^{\dagger\dot\alpha} \end{pmatrix}, \qquad\qquad \overline{\Psi}_{\text{M}} = \begin{pmatrix} \xi^\alpha & \xi^\dagger_{\dot\alpha} \end{pmatrix}. \tag{2.142}$$

The four-component spinor form of the Lagrangian for a Majorana fermion with mass M,

$$\mathcal{L}_{\text{Majorana}} \;=\; \frac{i}{2}\overline{\Psi}_{\text{M}}\gamma^\mu\partial_\mu\Psi_{\text{M}} - \frac{1}{2}M\overline{\Psi}_{\text{M}}\Psi_{\text{M}} \tag{2.143}$$

can therefore be rewritten as

$$\mathcal{L}_{\text{Majorana}} \;=\; i\xi^\dagger\overline{\sigma}^\mu\partial_\mu\xi - \frac{1}{2}M(\xi\xi + \xi^\dagger\xi^\dagger) \tag{2.144}$$

in the more economical two-component Weyl spinor representation. Note that even though ξ_α is anti-commuting, $\xi\xi$ and its complex conjugate $\xi^\dagger\xi^\dagger$ do not vanish, because of the suppressed ϵ symbol, see eq. (2.138). Explicitly, $\xi\xi = \epsilon^{\alpha\beta}\xi_\beta\xi_\alpha = \xi_2\xi_1 - \xi_1\xi_2 = 2\xi_2\xi_1$.

> Any theory involving spin-1/2 fermions can always be written in terms of a collection of left-handed Weyl spinors ψ_i with
>
> $$\mathcal{L} \;=\; i\psi^{\dagger i}\overline{\sigma}^\mu\partial_\mu\psi_i - M^{ij}(\psi^\dagger_i\psi^\dagger_j - \psi_i\psi_j) \tag{2.145}$$
>
> For $i = j$ one has a Majorana mass term, and $i \neq j$ gives Dirac mass term.

There is a different ψ_i for the left-handed piece and for the Hermitian conjugate of the right-handed piece of a Dirac fermion. Given any expression involving bilinears of four-component spinors

$$\Psi_i = \begin{pmatrix} \xi_i \\ \chi_i^\dagger \end{pmatrix}, \tag{2.146}$$

labelled by a flavor or gauge-representation index i, one can translate into two-component Weyl spinor language (or vice versa) using the dictionary:

$$\overline{\Psi}_i P_L \Psi_j = \chi_i \xi_j, \qquad \overline{\Psi}_i P_R \Psi_j = \xi_i^\dagger \chi_j^\dagger, \tag{2.147}$$

$$\overline{\Psi}_i \gamma^\mu P_L \Psi_j = \xi_i^\dagger \overline{\sigma}^\mu \xi_j, \qquad \overline{\Psi}_i \gamma^\mu P_R \Psi_j = \chi_i \sigma^\mu \chi_j^\dagger \tag{2.148}$$

2.4 The Standard Model of Particle Physics

2.4.1 Gauge Symmetries

The so called standard model of particle physics (SM) is a gauge theory.

The gauge symmetry of the SM is

$$\mathcal{G} = SU(3)_C \times SU(2)_L \times U(1)_Y \tag{2.149}$$

with

- C: Colour
- L: Left
- Y: Hypercharge

2.4.2 Particle Content

Before symmetry breaking, the particle content of the SM is

Vector Bosons	B	$(\mathbf{1},\mathbf{1})_0$
	W	$(\mathbf{1},\mathbf{2})_0$
	g	$(\mathbf{8},\mathbf{0})_0$
Fermions	e_R	$(\mathbf{1},\mathbf{1})_1$
(3 Generations)	l	$(\mathbf{1},\mathbf{2})_{-1/2}$
	u_R	$(\bar{\mathbf{3}},\mathbf{1})_{-2/3}$
	d_R	$(\bar{\mathbf{3}},\mathbf{1})_{1/3}$
	q	$(\mathbf{3},\mathbf{2})_{1/6}$
Scalar	H	$(\mathbf{1},\mathbf{2})_{1/2}$

The last column shows the quantum numbers with respect to $\mathcal{G}$. These quantum numbers are not as random as it might look. Special conditions must be fulfilled to avoid anomalies, e.g.

- Gauge anomalies

$$\sum_f Y(f)^3 \equiv 0 \tag{2.150}$$

- Gauge $\times$ gravity anomalies

$$\sum_f Y(f) \equiv 0 \tag{2.151}$$

- Witten anomaly: even number of $SU(2)$ doublets

Check:

$$\sum_f Y(f) = \underbrace{3}_{\text{generations}} \times \left(Y(e) + \underbrace{2}_{\text{isospin}} \times Y(l) + \underbrace{3}_{\text{color}} \times Y(u_R) + 3 \times Y(d_R) + 2 \times 3 \times Y(q) \right) \tag{2.152}$$

$$=3 \times \left(1 + 2\left(-\frac{1}{2}\right) + 3\left(-\frac{2}{3}\right) + 3\left(\frac{1}{3}\right) + 6\left(\frac{1}{6}\right)\right) \tag{2.153}$$

$$=3 \times (1 - 1 - 2 + 1 + 1) \tag{2.154}$$

$$=0 \tag{2.155}$$

$$\sum_f Y(f)^3 = 3 \times \left(1 + 2\left(-\frac{1}{8}\right) + 3\left(-\frac{8}{27}\right) + 3\left(\frac{1}{27}\right) + 6\left(\frac{1}{216}\right)\right) \tag{2.156}$$

$$=3 \times \left(1 - \frac{1}{4} - \frac{8}{9} + \frac{1}{9} + \frac{1}{36}\right) \tag{2.157}$$

$$=0 \tag{2.158}$$

$\Rightarrow$ One needs to be careful when adding new fermions in order not to introduce anomalies

2.4.3 Gauge part of the Lagrangian

The gauge part of the Lagrangian before symmetry breaking reads

$$L = D_\mu H D^\mu H^* + i \sum_f f^\dagger \sigma^\mu D_\mu f + \sum_V V_{\mu\nu} V^{\mu\nu} \tag{2.159}$$

with $f = \{l, e_R, q, d_R, u_R\}$ and $V = \{B, W^a, G^a\}$. Let's be more explicit at some examples. Note, we consider only one generation of fermions because gauge couplings are always flavour diagonal.

- Right leptons

$$e_R^\dagger \sigma^\mu D_\mu e_R = e_R^\dagger \sigma^\mu (\partial_\mu + i g_1 B_\mu) e_R \tag{2.160}$$

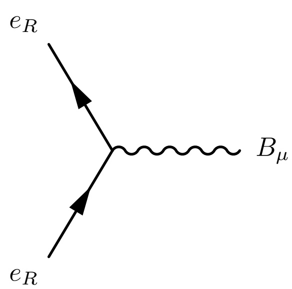
- Left leptons carry one isospin index, i.e. l_i with $i = 1, 2$

$$l^\dagger \sigma^\mu D_\mu l = l_i^\dagger \sigma^\mu (\partial_\mu \delta_{ij} - i\frac{1}{2} g_1 B_\mu \delta_{ij} + i g_2 \frac{\sigma^a}{2} W_\mu^a) l_j \tag{2.161}$$

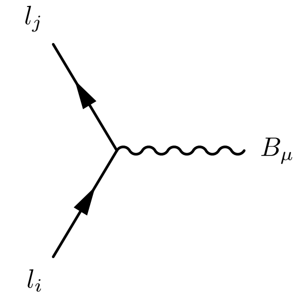
- Right up-quarks carry one colour index, i.e. $u_{R,\alpha}$ with $\alpha = 1,2,3$

$$u_R^\dagger \sigma^\mu D_\mu u_R = u_{R,\alpha}^\dagger \sigma^\mu (\partial_\mu \delta_{\alpha\beta} - i\frac{2}{3} g_1 B_\mu \delta_{\alpha\beta} + i g_3 \frac{\lambda^a}{2} G^a) u_{R\beta} \tag{2.162}$$

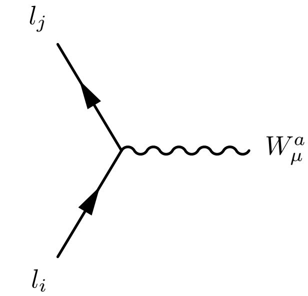
From these expressions the vertices are derived:



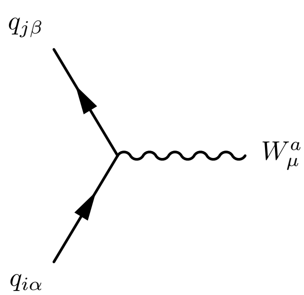
$\sigma_\mu g_1$



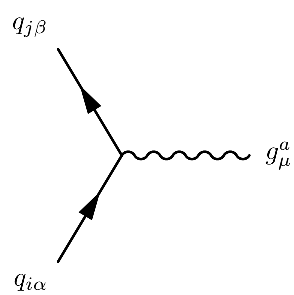
$-\sigma_\mu \delta_{ij} \frac{g_1}{2}$



$\sigma_\mu \sigma_{ij}^a \frac{g_2}{2}$



$\sigma_\mu \delta_{\alpha\beta} \sigma_{ij}^a \frac{g_2}{2}$



$\sigma_\mu \delta_{ij} \lambda_{ij}^a \frac{g_3}{2}$

2.4.4 Electroweak symmetry breaking

2.4.4.1 The Higgs potential

The Higgs potential in the SM is given by

$$V(H) = \frac{1}{2}\lambda |H|^4 + \mu^2 |H|^2 \tag{2.163}$$

Note, different conventions for the normalisation of the quartic coupling exist in literature. $\mu^2 < 0$ causes a spontaneous breaking of the electroweak symmetry (EWSB). The Higgs field becomes

$$\begin{pmatrix} H^+ \\ H^0 \end{pmatrix} \to \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}\left(h + iG^0 + v\right) \end{pmatrix} \tag{2.164}$$

The Higgs potential becomes

$$V = \frac{1}{8}\lambda((G^0)^2 + (h+v)^2 + 2G^+G^-)^2 + \frac{1}{2}\mu^2((G^0)^2 + (h+v)^2 + 2G^+G^-) \tag{2.165}$$

We can calculate the Higgs coupling and masses form this potential

a) Tadpole conditions: The condition for being at the minimum of the potential is

$$\frac{\partial V(h=0)}{\partial v} \equiv 0 = \frac{\partial}{\partial v}\left(\frac{1}{8}\lambda v^4 + \frac{1}{2}\mu^2 v^2\right) \tag{2.166}$$

$$= \frac{1}{2}\lambda v^3 + \mu^2 v \tag{2.167}$$

$$\rightarrow \mu^2 = -\frac{1}{2}v^2\lambda \tag{2.168}$$

Thus, one can eliminate μ^2 from all following expressions.

b) CP-even mass: the Higgs mass is given by

$$m_h^2 = \frac{\partial^2 V}{\partial h^2}|_{h=G^0=G^+=0} \tag{2.169}$$

$$= \frac{3}{2}\lambda v^2 + \mu^2 \tag{2.170}$$

$$= \frac{3}{2}\lambda v^2 - \frac{1}{2}\lambda v^2 \tag{2.171}$$

$$= \lambda v^2 \tag{2.172}$$

c) Goldstone masses: the mass of G^0 becomes

$$m_{G^0}^2 = \frac{\partial^2 V}{\partial {G^0}^2}|_{h=G^0=G^+=0} \tag{2.173}$$

$$= \mu^2 + \frac{1}{2}\lambda v^2 \qquad = 0 \tag{2.174}$$

Since we are working here in Landau gauge, the Goldstone mass vanishes as expected. Similarly, one can show $m_{G^+}^2 = 0$

d) Cubic Higgs coupling: the cubic Higgs self-interaction is

$$c_{hhh} = \frac{\partial^3 L}{\partial h^3}|_{h=G^0=G^+=0} \tag{2.175}$$

$$= -3v\lambda \tag{2.176}$$

$$= -3\frac{m_h^2}{v} \tag{2.177}$$

e) Quartic Higgs coupling: the quartic Higgs self-interaction is

$$c_{hhhh} = \frac{\partial^4 L}{\partial h^4}|_{h=G^0=G^+=0} \tag{2.178}$$

$$= -3\lambda \tag{2.179}$$

The entire Higgs sector of the SM can be parametrised after EWSB by just two parameters: λ and v.

2.4.4.2 Electroweak gauge bosons

The gauge interactions of the Higgs field become after EWSB:

$$D_\mu H D^\mu H^* = \left(\partial_\mu \delta_{ik} + i(\frac{1}{2} g_1 B_\mu \delta_{ik} + g_2 \frac{\sigma^a_{ik}}{2} W^a) H_i\right) \left(\partial_\mu \delta_{jk} - i(\frac{1}{2} g_1 B_\mu \delta_{jk} + g_2 \frac{\sigma^a_{jk}}{2} W^a) H^*_j\right) \tag{2.180}$$

$$\begin{aligned} =&\frac{1}{2}\partial_\mu h \partial^\mu h + \frac{1}{2}\partial_\mu G^0 \partial^\mu G^0 + \partial_\mu G^+ \partial^\mu G^- \\ &+ \frac{1}{4}\left((h+v)^2 + (G^0)^2\right)\left(g_1^2 B^2 - 2 g_1 g_2 B W^3 + g_2^2 (W_1^2 + W_2^2 + W_3^2)\right) \\ &+ \ldots \end{aligned} \tag{2.181}$$

On can see in the second line that not only mass terms for the vector bosons are generated, but also a mixing between B and W^3 occurs. The neutral mass matrix M_V reads

$$M_V^2 = (B W_3) \begin{pmatrix} \frac{1}{4} v^2 g_1^2 & -\frac{1}{4} g_1 g_2 v^2 \\ -\frac{1}{4} g_1 g_2 v^2 & \frac{1}{4} g_2^2 v^2 \end{pmatrix} \begin{pmatrix} B \\ W_3 \end{pmatrix} \tag{2.182}$$

The mixed particles, which appear after diagonalisation, are called photon (γ) and Z-Boson (Z). Their masses are the eigenvalues which are given by

$$m_\gamma = 0 \tag{2.183}$$

$$m_Z^2 = \frac{1}{4}(g_1^2 + g_2^2) v^2 \tag{2.184}$$

The rotation matrix which diagonalises $M_V^2 4$ is

$$\begin{pmatrix} \gamma \\ Z \end{pmatrix} = \begin{pmatrix} \cos\Theta_W & \sin\Theta_W \\ -\sin\Theta_W & \cos\Theta_W \end{pmatrix} \begin{pmatrix} B \\ W^3 \end{pmatrix} \tag{2.185}$$

with the Weinberg angle Θ_W. This defines the electric charge as:

$$e = g_1 \cos\Theta_W = g_2 \sin\Theta_W \tag{2.186}$$

One remaining massless gauge boson corresponds to one unbroken symmetry. Therefore, the remaining symmetry of the SM is

$$\mathcal{G} \to SU(3)_C \times U(1)_{em} \tag{2.187}$$

Since W_1 and W_2 are not electromagnet eigenstates, they are combined to new eigenstate of $U(1)_{em}$

$$W^\pm = \frac{1}{\sqrt{2}}(W_1 \pm i W_2) \tag{2.188}$$

The mass of $W^\pm$ is given by

$$M_W^2 = \frac{1}{4} g^2 v^2 \tag{2.189}$$

The massless states G^0 and $G^\pm$ are the Goldstone bosons of Z and $W^\pm$ and form their longitudinal components.

Let's count the (real) degrees of freedom

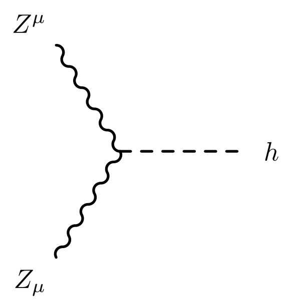
Before EWSB			After EWSB		
massless vectors:	B, W^a	4	massless vectors:	γ	1
massive vectors:	-	0	massive vectors:	Z, W^+	3
complex scalars:	H^0, $H^\pm$	4	complex scalars:	$G^\pm$	2
real scalars:	-	0	real scalars:	h, G^0	2

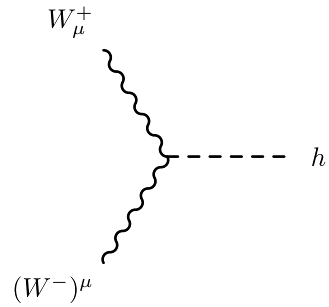
The kinetic term for the mass eigenstates h, the SM Higgs boson, becomes after applying all rotations:

$$
\begin{aligned}
\mathcal{L} = &\left(\partial_\mu \delta_{ij} + i\frac{1}{2}g_1\delta_{ij}B_\mu + i\frac{1}{2}g_2\sigma^a_{ij}W^a_\mu\right) H_i \left(\partial^\mu \delta_{ij} - i\frac{1}{2}g_1\delta_{ij}B^\mu - i\frac{1}{2}g_2\sigma^a_{ji}(W^a)^\mu\right) H^*_j \\
=&\ldots \\
=&\frac{1}{4}(h+v)^2\Big[2g_2^2W^+_\mu(W^-)^\mu + \gamma_\mu Z^\mu\left((g_1^2-g_2^2)\sin 2\Theta_W + 2g_1g_2\cos 2\Theta_W\right) + \\
&\qquad \gamma_\mu\gamma^\mu(g_1\cos\Theta_W - g_2\sin\Theta_W)^2 + Z_\mu Z^\mu(g_1\sin\Theta_W + g_2\cos\Theta)^2\Big] \\
&+ \left(\partial_\mu + i\gamma_\mu(g_1\cos\Theta - g_2\sin\Theta) + iZ_\mu(g_1\sin\Theta_W + g_2\cos\Theta)\right)h \\
&\quad \left(\partial^\mu - i\gamma^\mu(g_1\cos\Theta - g_2\sin\Theta) - iZ^\mu(g_1\sin\Theta_W + g_2\cos\Theta)\right)h \qquad (2.190) \\
&+ \mathcal{L}(G^0, G^+, h) \\
=&\frac{1}{4}\frac{e^2}{\sin^2\Theta_W}(h+v)^2\left(2W^+_\mu(W^-)^\mu + \frac{1}{\cos^2\Theta_W}Z_\mu Z^\mu\right) + \frac{1}{2}\partial_\mu h\partial^\mu h + \mathcal{L}(G^0, G^+, h) \qquad (2.191)
\end{aligned}
$$

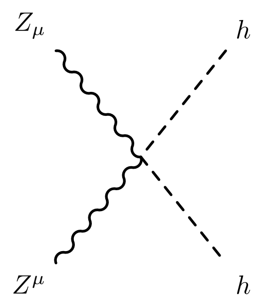
Thus, the couplings between the Higgs to the photon drop out after performing all replacements correctly[1]. There is also no $h-h-Z$ interaction (which is forbidden by CP), but only $h-G^0-Z$. The vertices for the Higgs to the gauge bosons are given by

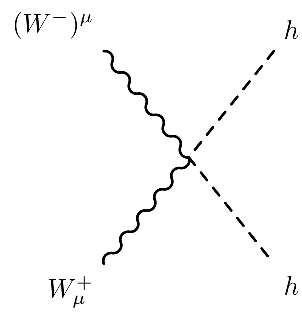
[1]'At tree-level, the photon couples only to charged particles and the Higgs only to massive ones'





$2iv\frac{e^2}{\sin^2 2\Theta_W}$ $\frac{1}{2}iv\frac{e^2}{\sin^2\Theta_W}$





$2i\frac{e^2}{\sin^2 2\Theta_W}$ $\frac{1}{2}i\frac{e^2}{\sin^2\Theta_W}$

2.4.5 Fermion masses and Yukawa sector

It is not possible in the SM to write down mass terms for fermions because of the quantum numbers for left and right fields.

$\Rightarrow$ Fermion masses are spontaneously generated after EWSB via interactions with the Higgs field

The interactions between the Higgs and the SM fermions are called 'Yukawa' interactions.

$$\mathcal{L}_Y = Y_u q^\dagger u_R H + Y_d q^\dagger d_R H^* + Y_e l^\dagger e_R H^* \quad + \text{h.c.} \tag{2.192}$$

In the general case, Y_f are (complex) 3×3 matrices. Thus, in the most general form the Lagrangian reads with all indices written explicitly

$$Y_u q^\dagger u_R H \equiv \delta_{\alpha\beta} Y_{u,ab} q^\dagger_{ai\alpha} u_{R,b\beta} \epsilon_{ij} H_j \tag{2.193}$$

If we neglect flavour mixing for the moment, one can write

$$\mathcal{L}_{Y_u} = Y_u q^\dagger_{i\alpha} u_{R,\beta} \epsilon_{ij} H_j \tag{2.194}$$
$$= Y_u (u^\dagger_{L,\alpha} H_0 - d^\dagger_{L,\alpha} H^+) u_{R,\beta} \delta_{\alpha\beta} \tag{2.195}$$

what becomes after EWSB

$$\mathcal{L}_{Y_u} = \frac{1}{\sqrt{2}}(v+h) Y_u u \,{}^\dagger{}_L \, u_R + \dots \tag{2.196}$$

i.e. the fermion mass is given by

$$m_u = \frac{1}{\sqrt{2}} v Y_u \tag{2.197}$$

If we include flavour mixing, the mass terms for the quarks after EWSB read

$$\mathcal{L}_q = (d_L^\dagger s_L^\dagger b_L^\dagger) \begin{pmatrix} vY_{d,11} & vY_{d,12} & vY_{d,13} \\ vY_{d,21} & vY_{d,22} & vY_{d,33} \\ vY_{d,31} & vY_{d,32} & vY_{d,33} \end{pmatrix} \begin{pmatrix} d_R \\ u_R \\ b_R \end{pmatrix} + (u_L^\dagger c_L^\dagger t_L^\dagger) \begin{pmatrix} vY_{u,11} & vY_{u,12} & vY_{u,13} \\ vY_{u,21} & vY_{u,22} & vY_{u,33} \\ vY_{u,31} & vY_{u,32} & vY_{u,33} \end{pmatrix} \begin{pmatrix} u_R \\ c_R \\ t_R \end{pmatrix} \tag{2.198}$$

The six quark masses are the eigenvalues of the matrices vY_d and vY_u. These matrices are diagonalised by four unitary matrices:

$$u_R \rightarrow U_R = U_u^* u_R \tag{2.199}$$
$$d_R \rightarrow D_R = U_d^* d_R \tag{2.200}$$
$$u_L \rightarrow U_L = V_u u_L \tag{2.201}$$
$$d_L \rightarrow D_L = V_d u_L \tag{2.202}$$

Only one combination of these matrices is physically relevant and defines the CKM (Cabibbo-Kobayashi-Maskawa) matrix

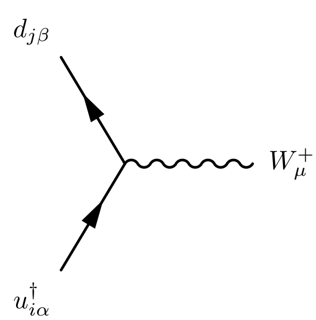
$$V_{\text{CKM}} = V_u^\dagger V_d \tag{2.203}$$

The entire flavour structure of the SM quark sector is encoded in the CKM matrix which can be parametrised by three angles Θ_{12}, Θ_{23}, Θ_{13} and one phase δ

$$V_{\text{CKM}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.204}$$

δ is the only source of CP violation in the SM and highly restricted by experiments

The CKM matrix shows up explicitly in vertices involving the W-boson



$$-\frac{i}{\sqrt{2}} g_2 \sigma_\mu V_{CKM}^{ij} \delta_{\alpha\beta}$$

Chapter 3

Supersymmetric Formalities

3.1 Basics

3.1.1 SUSY transformations

A supersymmetry transformation turns a bosonic state into a fermionic state, and vice versa.

$$Q|\text{Boson}\rangle = |\text{Fermion}\rangle, \qquad Q|\text{Fermion}\rangle = |\text{Boson}\rangle. \tag{3.1}$$

The properties of the operator Q are:

- Q is an anti-commuting spinor
- $Q^\dagger$ is also a symmetry generator
- Q, $Q^\dagger$ are carry spin $1/2 \rightarrow$ SUSY is a space–time symmetry.
- Q and $Q^\dagger$ satisfy the following algebra (schematically):

$$\{Q, Q^\dagger\} = P^\mu, \tag{3.2}$$
$$\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0, \tag{3.3}$$
$$[P^\mu, Q] = [P^\mu, Q^\dagger] = 0, \tag{3.4}$$

  where P^μ is the four-momentum generator of spacetime translations. Note, we skipped here the spinor indices on Q, $Q^\dagger$. (The accurate expressions could be given once we have developed the necessary formalism.)

- Q and $Q^\dagger$ commute with P^2
- Q and $Q^\dagger$ commute with all generators of gauge transformations

A non-trivial connection between internal and external symmetries was forbidden by the no-go theorem of Coleman-Mandula. However, this doesn't apply to spinor operators.

We consider only the case of a single set of generators Q, $Q^\dagger$, what is also called $N = 1$ supersymmetry. $N = 2$ or $N = 4$ theories are mathematically interesting, but phenomenologically not relevant in four space–time dimensions. One would need extra dimensions to get chiral fermions or parity violation.

3.1.2 Representations

A supersymmetric theory must consist of states which are irreducible representations of the SUSY algebra. These states are called ”supermultiplets”. The properties of supermultiplets are:

- Each supermultiplet consists of both fermionic and bosonic states. Those are called ”superpartners”
- If $|\Omega\rangle$ and $|\Omega'\rangle$ are members of the same supermultiplet, then $|\Omega'\rangle$ is proportional to some combination of Q and $Q^\dagger$ operators acting on $|\Omega\rangle$ (up to space–time translation or rotation)
- particles within the same supermultiplet must have equal eigenvalues of P^2, i.e. equal masses
- particles within the same supermultiplet must sit in the same representation of the gauge groups
- Each supermultiplet contains an equal number of fermionic and bosonic degrees of freedom

$$n_B = n_F \tag{3.5}$$

We are mainly interested in the following two kinds of supermultiplets:

a) Chiral supermultiplet: the simplest possibility for a supermultiplet consistent with eq. (3.5) has a single Weyl fermion (with two spin helicity states, so $n_F = 2$) and two real scalars (each with $n_B = 1$). It is convenient to arrange the real scalars as one complex field.

b) Vector supermultiplet: the simplest possibility of a supermultiplet containing gauge fields contains a spin-1 vector boson. We are only interested in renormalizable gauge theories, i.e. the vector boson must be massless (before spontaneous symmetry breaking) and has therefore two degrees of freedom: $n_B = 2$. Its superpartner is therefore a massless spin-1/2 Weyl fermion, again with two helicity states, so $n_F = 2$.

If we include gravity, then the spin-2 graviton (with 2 helicity states, so $n_B = 2$) has a spin-3/2 superpartner called the gravitino. The gravitino would be massless if supersymmetry were unbroken, and so it has $n_F = 2$ helicity states.

One can check that other possible combinations of particles which satisfy $n_B = n_F$ are always reducible. *For example*: If a gauge symmetry could be broken without SUSY breaking then a massless vector supermultiplet would “eat” a chiral supermultiplet. The degrees of freedom of the massive vector supermultiplet are:

$$\begin{aligned} \text{massive vector :} \quad & n_B = 3 \\ \text{massive Dirac fermion :} \quad & n_F = 4 \\ \text{a real scalar :} \quad & n_B = 1 \end{aligned}$$

3.2 SUSY Lagrangian

3.2.1 A free chiral supermultiplet

We have already seen that the easiest supersymmetric object is a chiral supermultiplet with a single left-handed two-component Weyl fermion ψ and a complex scalar ϕ. We forget for the moment about all possible interaction or mass terms. Under this assumption, the action of a single supermultiplet can be written in terms of its component fields as:

$$S = \int d^4x \; (\mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{fermion}}) , \tag{3.6}$$

with

$$\mathcal{L}_{\text{scalar}} = \partial^\mu \phi^* \partial_\mu \phi \tag{3.7}$$

$$\mathcal{L}_{\text{fermion}} = i\psi^\dagger \overline{\sigma}^\mu \partial_\mu \psi . \tag{3.8}$$

This is called the massless, non-interacting *Wess-Zumino model.*

3.2.1.1 SUSY invariance

A SUSY transformation should turn the scalar boson field ϕ into something involving the fermion field ψ_α. The simplest possibility is

$$\delta\phi = \epsilon\psi, \qquad \delta\phi^* = \epsilon^\dagger \psi^\dagger , \tag{3.9}$$

where ϵ^α parameterizes the supersymmetry transformation. ϵ^α is an infinitesimal, anti-commuting, two-component Weyl fermion which we assume for now to be constant, i.e.

$$\partial_\mu \epsilon^\alpha = 0 \tag{3.10}$$

The mass dimension os

$$[\epsilon] = [\phi] - [\psi] = 1 - \frac{3}{2} = -\frac{1}{2} \tag{3.11}$$

Applying the transformation, we find that the scalar part of the Lagrangian transforms as

$$\delta\mathcal{L}_{\text{scalar}} \;=\; \epsilon \partial^\mu \psi \, \partial_\mu \phi^* + \epsilon^\dagger \partial^\mu \psi^\dagger \, \partial_\mu \phi . \tag{3.12}$$

This must be canceled by $\delta\mathcal{L}_{\text{fermion}}$ (up to a total derivative). We can guess now how the transformation of the fermion must look like. There is only one chance (up to overall constants) that a cancellation can happen, namely

$$\delta\psi_\alpha \;=\; -i(\sigma^\mu \epsilon^\dagger)_\alpha \, \partial_\mu \phi, \qquad \delta\psi^\dagger_{\dot\alpha} \;=\; i(\epsilon\sigma^\mu)_{\dot\alpha} \, \partial_\mu \phi^* . \tag{3.13}$$

With this guess, one immediately obtains

$$\delta\mathcal{L}_{\text{fermion}} = -\epsilon\sigma^\mu \overline{\sigma}^\nu \partial_\nu \psi \, \partial_\mu \phi^* + \psi^\dagger \overline{\sigma}^\nu \sigma^\mu \epsilon^\dagger \, \partial_\mu \partial_\nu \phi \, . \tag{3.14}$$

This can be simplified by employing the Pauli matrix identities

$$\left[\sigma^\mu \overline{\sigma}^\nu + \sigma^\nu \overline{\sigma}^\mu\right]_\alpha{}^\beta = 2\eta^{\mu\nu}\delta^\beta_\alpha \tag{3.15}$$

$$\left[\overline{\sigma}^\mu \sigma^\nu + \overline{\sigma}^\nu \sigma^\mu\right]^{\dot\beta}{}_{\dot\alpha} = 2\eta^{\mu\nu}\delta^{\dot\beta}_{\dot\alpha} \tag{3.16}$$

Using the fact that partial derivatives commute ($\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$), we get

$$\begin{aligned} \delta\mathcal{L}_{\text{fermion}} \;=\; & -\epsilon \partial^\mu \psi \, \partial_\mu \phi^* - \epsilon^\dagger \partial^\mu \psi^\dagger \, \partial_\mu \phi \\ & -\partial_\mu \left(\epsilon \sigma^\nu \overline{\sigma}^\mu \psi \, \partial_\nu \phi^* - \epsilon\psi \, \partial^\mu \phi^* - \epsilon^\dagger \psi^\dagger \, \partial^\mu \phi \right) . \end{aligned} \tag{3.17}$$

The first two terms here just cancel against $\delta\mathcal{L}_{\text{scalar}}$, while the remaining contribution is a total derivative. So we arrive at

$$\delta S = \int d^4x \;\; (\delta\mathcal{L}_{\text{scalar}} + \delta\mathcal{L}_{\text{fermion}}) = 0, \tag{3.18}$$

justifying our guess of the numerical multiplicative factor made in eq. (3.13).

3.2.1.2 Closure of the SUSY algebra

We have shown so far that the Wess–Zumino Lagrangian is invariant under a SUSY transformation. However, me must also show that the SUSY algebra closes: the commutator of two SUSY transformations is another symmetry of the theory.

$$
\begin{aligned}
[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\phi &= (\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})\phi && (3.19)\\
&= \delta_{\epsilon_2}(\delta_{\epsilon_1}\phi) - \delta_{\epsilon_1}(\delta_{\epsilon_2}\phi) && (3.20)\\
&= \delta_{\epsilon_2}(\epsilon_1\psi) - \delta_{\epsilon_1}(\epsilon_\psi\phi) && (3.21)\\
&= i(-\epsilon_1\sigma^\mu\epsilon_2^\dagger + \epsilon_2\sigma^\mu\epsilon_1^\dagger)\,\partial_\mu\phi. && (3.22)
\end{aligned}
$$

Here, we used $\delta\phi = \epsilon\psi$ and $\delta\psi_\alpha = -i(\sigma^\mu\epsilon^\dagger)_\alpha\,\partial_\mu\phi$.
We have found that the commutator of two supersymmetry transformations gives us back the derivative of the original field. In the Heisenberg picture of quantum mechanics $i\partial_\mu$ corresponds to the generator of spacetime translations P_μ. Thus, this result agrees with our expectations from the SUSY algebra.

We must repeat the exercise for the fermionic case.

$$
\begin{aligned}
[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\psi_\alpha &= (\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})\psi_\alpha && (3.23)\\
&= \delta_{\epsilon_2}(-i(\sigma^\mu\epsilon_1^\dagger)_\alpha\,\partial_\mu\phi) - \delta_{\epsilon_1}(-i(\sigma^\mu\epsilon_2^\dagger)_\alpha\,\partial_\mu\phi)) && (3.24)\\
&= -i\underbrace{(\sigma^\mu\epsilon_1^\dagger)_\alpha}_{\chi_\alpha}\,\underbrace{\epsilon_2}_{\xi}\,\underbrace{\partial_\mu\psi}_{\eta} + i(\sigma^\mu\epsilon_2^\dagger)_\alpha\,\epsilon_1\partial_\mu\psi \quad \text{using} \quad \chi_\alpha\,(\xi\eta) = -\xi_\alpha\,(\eta\chi) - \eta_\alpha\,(\chi\xi) && (3.25)\\
&= i(\epsilon_{2\alpha}((\partial_\mu\psi)(\sigma^\mu\epsilon_1^\dagger)) + (\partial_\mu\psi)_\alpha((\sigma^\mu\epsilon_1^\dagger)\epsilon_2)) - (\epsilon_1 \leftrightarrow \epsilon_2) && (3.26)\\
&= i(\epsilon_{2\alpha}(\sigma^\mu\epsilon_1^\dagger\partial_\mu\psi) + (\epsilon_2)(\sigma^\mu\epsilon_1^\dagger))(\partial_\mu\psi)_\alpha) - (\epsilon_1 \leftrightarrow \epsilon_2) && (3.27)\\
&= i(-\epsilon_1\sigma^\mu\epsilon_2^\dagger + \epsilon_2\sigma^\mu\epsilon_1^\dagger)\,\partial_\mu\psi_\alpha + i\epsilon_{1\alpha}\,\epsilon_2^\dagger\overline{\sigma}^\mu\partial_\mu\psi - i\epsilon_{2\alpha}\,\epsilon_1^\dagger\overline{\sigma}^\mu\partial_\mu\psi && (3.28)
\end{aligned}
$$

In the last line we used

$$(\chi^\dagger\overline{\sigma}^\mu\xi)^* \;=\; \xi^\dagger\overline{\sigma}^\mu\chi \;=\; -\chi\sigma^\mu\xi^\dagger \;=\; -(\xi\sigma^\mu\chi^\dagger)^* \tag{3.29}$$

Thus, if we apply the Dirac equation

$$\overline{\sigma}^\mu\partial_\mu\psi = 0 \tag{3.30}$$

we find

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\psi_\alpha = i(-\epsilon_1\sigma^\mu\epsilon_2^\dagger + \epsilon_2\sigma^\mu\epsilon_1^\dagger)\,\partial_\mu\psi_\alpha \tag{3.31}$$

which is very similar to the scalar case.

We found so far that the SUSY algebra closes only on-shell. In order to consider the off-shell case, we play a trick and introduce another ingredient, so called *auxiliary fields F*. F are complex scalar fields which don't propagate. Their Lagrangian is just

$$\mathcal{L}_{\text{auxiliary}} = F^*F\,. \tag{3.32}$$

Note, the mass dimension of F is 2. One can easily check that the equation of motion from $\mathcal{L}_{\text{auxiliary}}$ is

$$F = F^* = 0 \tag{3.33}$$

We impose the following property of F under a SUSY transformation:

$$\delta F = -i\epsilon^\dagger\overline{\sigma}^\mu\partial_\mu\psi \tag{3.34}$$
$$\delta F^* = i\partial_\mu\psi^\dagger\overline{\sigma}^\mu\epsilon. \tag{3.35}$$

Now the auxiliary part of the Lagrangian density transforms as

$$\delta\mathcal{L}_{\text{auxiliary}} = -i\epsilon^\dagger\overline{\sigma}^\mu\partial_\mu\psi\, F^* + i\partial_\mu\psi^\dagger\overline{\sigma}^\mu\epsilon\, F, \tag{3.36}$$

which vanishes on-shell, but not for arbitrary off-shell field configurations. We also modify the transformation properties of our fermions:

$$\delta\psi_\alpha = -i(\sigma^\mu\epsilon^\dagger)_\alpha\,\partial_\mu\phi + \epsilon_\alpha F, \tag{3.37}$$
$$\delta\psi^\dagger_{\dot\alpha} = i(\epsilon\sigma^\mu)_{\dot\alpha}\,\partial_\mu\phi^* + \epsilon^\dagger_{\dot\alpha}F^*, \tag{3.38}$$

One can check that the additional contribution to $\delta\mathcal{L}_{\text{fermion}}$ cancels the ones from $\delta\mathcal{L}_{\text{auxiliary}}$, up to a total derivative term. Thus

$$\delta\mathcal{L} = \delta\mathcal{L}_{\text{scalar}} + \delta\mathcal{L}_{\text{fermion}} + \delta\mathcal{L}_{\text{auxiliary}} = 0 \tag{3.39}$$

If we now repeat the calculation from before, one finds for all fields $X = \phi, \phi^*, \psi, \psi^\dagger, F, F^*$

$$(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})X = i(-\epsilon_1\sigma^\mu\epsilon_2^\dagger + \epsilon_2\sigma^\mu\epsilon_1^\dagger)\,\partial_\mu X \tag{3.40}$$

also without applying the equations of motion. So, we found that the SUSY algebra closes all off-shell once we include the auxiliary fields.

What is the interpreation of all that? On-shell, the complex scalar field ϕ has two real propagating degrees of freedom, matching the two spin polarization states of ψ. Off-shell, however, the Weyl fermion ψ is a complex two-component object, so it has four real degrees of freedom. (Going on-shell eliminates half of the propagating degrees of freedom for ψ, because the Lagrangian is linear in time derivatives, so that the canonical momenta can be re-expressed in terms of the configuration variables without time derivatives and are not independent phase space coordinates.) To make the numbers of bosonic and fermionic degrees of freedom match off-shell as well as on-shell, we had to introduce two more real scalar degrees of freedom in the complex field F, which are eliminated when one goes on-shell. This counting is by

	ϕ	ψ	F
on-shell ($n_B = n_F = 2$)	2	2	0
off-shell ($n_B = n_F = 4$)	2	4	2

We can summarize the main outcame as follows:

A **chiral superfield** consists of

- A complex scalar ϕ
- A Weyl fermion ψ
- An auxiliary field F

and the free Lagrangian is given by

$$\mathcal{L}_{\text{free chiral}} = \partial^\mu\phi^*\partial_\mu\phi + i\psi^\dagger\overline{\sigma}^\mu\partial_\mu\psi + F^*F \tag{3.41}$$

3.2.2 Interactions of chiral supermultiplets

We go now one step further and consider several chiral supermultiplets which can interact among each other. We won't introduce gauge interactions, yet.

We start with the Lagrangina density of several free chiral supermultiplets labelled by an index i. We can easily generalise the result for one fields by writing

$$\mathcal{L}_{\text{free}} \quad = \quad \partial^\mu \phi^{*i} \partial_\mu \phi_i + i\psi^{\dagger i} \overline{\sigma}^\mu \partial_\mu \psi_i + F^{*i} F_i, \tag{3.42}$$

where we sum over repeated indices i. This Lagrangian is invariant under the individual supersymmetry transformation

$$\delta\phi_i = \epsilon\psi_i, \qquad \delta\phi^{*i} = \epsilon^\dagger \psi^{\dagger i}, \tag{3.43}$$

$$\delta(\psi_i)_\alpha = -i(\sigma^\mu \epsilon^\dagger)_\alpha \, \partial_\mu \phi_i + \epsilon_\alpha F_i, \qquad \delta(\psi^{\dagger i})_{\dot\alpha} = i(\epsilon\sigma^\mu)_{\dot\alpha} \, \partial_\mu \phi^{*i} + \epsilon^\dagger_{\dot\alpha} F^{*i}, \tag{3.44}$$

$$\delta F_i = -i\epsilon^\dagger \overline{\sigma}^\mu \partial_\mu \psi_i, \qquad \delta F^{*i} = i\partial_\mu \psi^{\dagger i} \overline{\sigma}^\mu \epsilon \,. \tag{3.45}$$

We want to find the most general set of renormalizable interactions that respects SUSY invariance. We start by writing down:

$$\mathcal{L}_{\text{int}} = \left(-\frac{1}{2} W^{ij} \psi_i \psi_j + W^i F_i + x^{ij} F_i F_j \right) + \text{c.c.} - U, \tag{3.46}$$

where the different coefficients are polynomials in the scalar fields ϕ_i, ϕ^{*i} of the schematic form:

- $W^{ij} \sim \phi_a$
- $W^i \sim \phi_a \phi_b$
- $x^{ij} \sim$ const
- $U \sim \phi_a \phi_b \phi_c \phi_d$

We must now require that $\mathcal{L}_{\text{int}}$ is invariant under the supersymmetry transformations, since $\mathcal{L}_{\text{free}}$ was already invariant by itself. The very schematic transformation properties of the different terms are

- $\delta W^{ij} \psi_i \psi_j \sim \psi^3 + \phi(\partial_\mu \phi + F)\psi$
- $\delta W^i F_i \sim \psi\phi F + \phi(\partial_\mu \psi)$
- $\delta U \sim \phi^3 \psi$
- $\delta x^{ij} F_i F_j \sim \partial \psi_i F_j$

There is obviously no possibility that to cancel the terms arising from U and x^{ij} against something else. So, we are left with

$$\mathcal{L}_{\text{int}} = \left(-\frac{1}{2} W^{ij} \psi_i \psi_j + W^i F_i \right) + \text{c.c.} \tag{3.47}$$

At this point, we are not assuming that W^{ij} and W^i are related, but we will see that thye are. From

$$\xi\chi \equiv \xi^\alpha \chi_\alpha = \xi^\alpha \epsilon_{\alpha\beta} \chi^\beta = -\chi^\beta \epsilon_{\alpha\beta} \xi^\alpha = \chi^\beta \epsilon_{\beta\alpha} \xi^\alpha = \chi^\beta \xi_\beta \equiv \chi\xi \tag{3.48}$$

we see that W^{ij} is symmetric under $i \leftrightarrow j$.

We want to find the most general which W^{ij} and W^i can have which is in agreement with the SUSY transformations. For this purpose we can check different pieces which must cancel separately.

a) We start with the part that contains four spinors.

$$\delta\mathcal{L}_{\text{int}}|_{4-\text{spinor}} = \left[-\frac{1}{2}\frac{\delta W^{ij}}{\delta\phi_k}(\epsilon\psi_k)(\psi_i\psi_j) - \frac{1}{2}\frac{\delta W^{ij}}{\delta\phi^{*k}}(\epsilon^\dagger\psi^{\dagger k})(\psi_i\psi_j)\right] + \text{c.c.} \tag{3.49}$$

The term proportional to $(\epsilon\psi_k)(\psi_i\psi_j)$ cannot cancel against any other term. However, the Fierz identity

$$\chi_\alpha\,(\xi\eta) = -\xi_\alpha\,(\eta\chi) - \eta_\alpha\,(\chi\xi) \tag{3.50}$$

implies

$$(\epsilon\psi_i)(\psi_j\psi_k) + (\epsilon\psi_j)(\psi_k\psi_i) + (\epsilon\psi_k)(\psi_i\psi_j) = 0, \tag{3.51}$$

Thus, in order to get $\delta\mathcal{L}_{\text{int}} = 0$, the term $\delta W^{ij}/\delta\phi_k$ must be totally symmetric under interchange of i, j, k. Consequently, W can only involve ϕ but not ϕ^*, **i.e. W^{ij} is a holomorphic function of the complex fields ϕ.**

Combining what we have learned so far, we can write

$$W^{ij} = M^{ij} + y^{ijk}\phi_k \tag{3.52}$$

Because of this form, we can write W^{ij} as

$$W^{ij} = \frac{\delta^2}{\delta\phi_i\delta\phi_j}W \tag{3.53}$$

where we have introduced a useful object

$$W = \frac{1}{2}M^{ij}\phi_i\phi_j + \frac{1}{6}y^{ijk}\phi_i\phi_j\phi_k, \tag{3.54}$$

called the *superpotential*.

b) We turn to the parts of $\delta\mathcal{L}_{\text{int}}$ that contain a spacetime derivative:

$$\delta\mathcal{L}_{\text{int}}|_\partial \;=\; \left(iW^{ij}\partial_\mu\phi_j\,\psi_i\sigma^\mu\epsilon^\dagger + iW^i\,\partial_\mu\psi_i\sigma^\mu\epsilon^\dagger\right) + \text{c.c.} \tag{3.55}$$

Here we have used the again the identity

$$(\chi^\dagger\overline{\sigma}^\mu\xi)^* \;=\; \xi^\dagger\overline{\sigma}^\mu\chi \;=\; -\chi\sigma^\mu\xi^\dagger \;=\; -(\xi\sigma^\mu\chi^\dagger)^* \tag{3.56}$$

on the second term, which came from $(\delta F_i)W^i$. Now we can use eq. (3.53) to observe that

$$W^{ij}\partial_\mu\phi_j = \partial_\mu\left(\frac{\delta W}{\delta\phi_i}\right). \tag{3.57}$$

Therefore, eq. (3.55) will be a total derivative if

$$W^i = \frac{\delta W}{\delta\phi_i} = M^{ij}\phi_j + \frac{1}{2}y^{ijk}\phi_j\phi_k\,, \tag{3.58}$$

which explains why we chose its name as we did.

c) The remaining terms in $\delta\mathcal{L}_{\text{int}}$ are all linear in F_i or F^{*i}, and it is easy to show that they cancel, given the results for W^i and W^{ij} that we have already found.

We have found that the most general non-gauge interactions for chiral supermultiplets are determined by a single holomorphic function of the complex scalar fields, the **superpotential** W. The general form of the superpotential in terms of scalar fields is

$$W(\phi) = L^i\phi_i + \frac{1}{2}M^{ij}\phi_i\phi_j + \frac{1}{6}y^{ijk}\phi_i\phi_j\phi_k \tag{3.59}$$

With

- M^{ij} is a symmetric mass matrix for the fermion fields
- y^{ijk} is a (Yukawa) coupling of a scalar ϕ_k and two fermions $\psi_i\psi_j$ that must be totally symmetric under interchange of i, j, k
- L^i a linear (tadpole) term which is only possible for pure gauge singlets

The auxiliary fields F_i and F^{*i} can be eliminated using their classical equations of motion.

$$\mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}} = F_iF^{*i} + W^iF_i + W_i^*F^{*i} + \dots \tag{3.60}$$

where the dots represent all terms independent of F, F^*. The equations of motion are

$$F_i = -W_i^*, \qquad F^{*i} = -W^i\,. \tag{3.61}$$

Thus the auxiliary fields can be expressed in terms of the scalar fields. Therefore, the Lagrangian can be written as

$$\mathcal{L} = \partial^\mu\phi^{*i}\partial_\mu\phi_i + i\psi^{\dagger i}\overline{\sigma}^\mu\partial_\mu\psi_i - \frac{1}{2}\left(W^{ij}\psi_i\psi_j + W_{ij}^*\psi^{\dagger i}\psi^{\dagger j}\right) - W^iW_i^*. \tag{3.62}$$

The scalar potential of the theory *without gauge interactions and unbroken supersymmetry* is completely fixed by the superpotential:

$$V(\phi,\phi^*) = W^kW_k^* = F^{*k}F_k \tag{3.63}$$

$$= M_{ik}^*M^{kj}\phi^{*i}\phi_j + \tfrac{1}{2}M^{in}y_{jkn}^*\phi_i\phi^{*j}\phi^{*k} + \tfrac{1}{2}M_{in}^*y^{jkn}\phi^{*i}\phi_j\phi_k + \tfrac{1}{4}y^{ijn}y_{kln}^*\phi_i\phi_j\phi^{*k}\phi^{*l} \tag{3.64}$$

This part is also called the F-term potential which has the following properties:

- This F-term potential is automatically bounded from below
- it is even always on-negative

We have finally found the most general form of the full interacting Lagrangian stemming from chiral superfields:

$$\begin{aligned}\mathcal{L}_{\text{chiral}} \;=\;& \partial^\mu\phi^{*i}\partial_\mu\phi_i - V(\phi,\phi^*) + i\psi^{\dagger i}\overline{\sigma}^\mu\partial_\mu\psi_i - \frac{1}{2}M^{ij}\psi_i\psi_j - \frac{1}{2}M_{ij}^*\psi^{\dagger i}\psi^{\dagger j}\\ & -\frac{1}{2}y^{ijk}\phi_i\psi_j\psi_k - \frac{1}{2}y_{ijk}^*\phi^{*i}\psi^{\dagger j}\psi^{\dagger k}.\end{aligned} \tag{3.65}$$

3.2.3 Lagrangians for vector supermultiplets

We want to include now gauge interactions. As we already mentioned, the gauge fields A^a_μ are part of vector supermultiplets. The other (propagating!) degrees of freedom are those of a a two-component Weyl fermion λ^a which we will call 'gaugino'. The index a here runs over the adjoint representation of the gauge group. The gauge transformations of the vector supermultiplet fields are

$$A^a_\mu \rightarrow A^a_\mu - \partial_\mu \Lambda^a + g f^{abc} A^b_\mu \Lambda^c, \tag{3.66}$$

$$\lambda^a \rightarrow \lambda^a + g f^{abc} \lambda^b \Lambda^c, \tag{3.67}$$

Before we start to check the SUSY properties, we count this time first the degrees of freedom in the on- and off-shell case:

- The on-shell degrees of freedom for A^a_μ and λ^a_α amount to two bosonic and two fermionic helicity states (for each a), as required by supersymmetry.
- Off-shell λ^a_α consists of two complex, or four real, fermionic degrees of freedom, while A^a_μ only has three real bosonic degrees of freedom one degree of freedom is removed by the inhomogeneous gauge transformation eq. (3.66).

We will see that we need one real bosonic auxiliary field D^a to balance the degrees of freedom (and to close the SUSY algebra off-shell). The counting of degrees of freedom is summarized as

	A_μ	λ	D
on-shell ($n_B = n_F = 2$)	2	2	0
off-shell ($n_B = n_F = 4$)	3	4	1

The properties of the D field are:

- D transforms in the adjoint representation of the gauge group
- $(D^a)^* = D^a$ holds
- D fields have mass dimension of 2 as F fields
- D fields don't propagate, i.e.their Lagrangian is

$$\mathcal{L}_{\text{auxiliary}} = \frac{1}{2} D^a D^a \tag{3.68}$$

Therefore, the Lagrangian density for the components of a vector supermultiplet are

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + i \lambda^{\dagger a} \overline{\sigma}^\mu D_\mu \lambda^a + \frac{1}{2} D^a D^a, \tag{3.69}$$

where

$$D_\mu \lambda^a = \partial_\mu \lambda^a - g f^{abc} A^b_\mu \lambda^c \tag{3.70}$$

is the covariant derivative of the gaugino field.

Of course, $\mathcal{L}_{\text{gauge}}$ must form a supersymmetric theory. That means that is must be invariant under SUSY transformations and that the SUSY algebra closes. One can guess how SUSY transformations might look like which fulfill these properties. They must have the following properties:

- they should be linear in the infinitesimal parameters $\epsilon, \epsilon^\dagger$ which have mass dimension $\frac{1}{2}$
- δA_μ^a is real
- δD^a should be real and proportional to the field equations for the gaugino (in analogy with the role of the auxiliary field F)

Up to multiplicative constants, this results in

$$\delta A_\mu^a = -\frac{1}{\sqrt{2}}\left(\epsilon^\dagger \overline{\sigma}_\mu \lambda^a + \lambda^{\dagger a} \overline{\sigma}_\mu \epsilon\right), \tag{3.71}$$

$$\delta \lambda_\alpha^a = -\frac{i}{2\sqrt{2}}(\sigma^\mu \overline{\sigma}^\nu \epsilon)_\alpha \, F_{\mu\nu}^a + \frac{1}{\sqrt{2}}\epsilon_\alpha \, D^a, \tag{3.72}$$

$$\delta D^a = \frac{i}{\sqrt{2}}\left(-\epsilon^\dagger \overline{\sigma}^\mu D_\mu \lambda^a + D_\mu \lambda^{\dagger a} \overline{\sigma}^\mu \epsilon\right). \tag{3.73}$$

The factors of $\sqrt{2}$ are chosen so that the action obtained by integrating $\mathcal{L}_{\text{gauge}}$ is indeed invariant. After some (tedious) work, which we skip here, one finds that

- $\delta \mathcal{L}_{\text{gauge}} = 0$
- $(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})X = i(-\epsilon_1 \sigma^\mu \epsilon_2^\dagger + \epsilon_2 \sigma^\mu \epsilon_1^\dagger) D_\mu X$ for $X = \{F_{\mu\nu}^a, \lambda^a, \lambda^{\dagger a}, D^a\}$

If we had not included the auxiliary field D^a, then the supersymmetry algebra would hold only after using the equations of motion for λ^a and $\lambda^{\dagger a}$. The auxiliary fields satisfies a trivial equation of motion $D^a = 0$, but this is modified if one couples the gauge supermultiplets to chiral supermultiplets, as we now do.

3.2.4 Supersymmetric gauge interactions

The final step to obtain the full Lagrangian for a supersymmetric theory is to add gauge interactions between vector and chiral supermultiplets. As we already mentioned, supersymmetric and gauge transformations commute, i.e. the scalar, fermion, and auxiliary component of a chiral superfield is in the same representation of the gauge group, so

$$X_i \quad \to \quad X_i + ig\Lambda^a (T^a X)_i \tag{3.74}$$

for $X_i = \phi_i, \psi_i, F_i$. Exactly as for non-supersymmetric models, we obtain a supersymmetric gauge theory by replacing ordinary derivatives ∂_μ, by covariant derivatives:

$$D_\mu \phi_i \quad = \quad \partial_\mu \phi_i + igA_\mu^a (T^a \phi)_i \tag{3.75}$$

$$D_\mu \phi^{*i} \quad = \quad \partial_\mu \phi^{*i} - igA_\mu^a (\phi^* T^a)^i \tag{3.76}$$

$$D_\mu \psi_i \quad = \quad \partial_\mu \psi_i + igA_\mu^a (T^a \psi)_i. \tag{3.77}$$

In that way, we couple the vector bosons to the matter fields. Note, we have not yet checked that this replacement is in agreement with SUSY invariance! Moreover, the difference compared to non-supersymmetric models is that the vector superfields includes also gauginos and auxiliary fields. Thus, for full generality, we need to check if those can also couple to the components of the chiral superfield. If we restrict ourselves to renormalizable couplings, there are only three possibilities which one can write donwn

a) $(\phi^* T^a \psi)\lambda^a$

b) $\lambda^{\dagger a}(\psi^\dagger T^a \phi)$

c) $(\phi^* T^a \phi) D^a$

We must now check if these terms can – or even must – be included to obtain a supersymmetric theory. And if, what their overall coefficients are. To that end, we need to change our SUSY transformations as follows:

- Normal derivatives must be replaced by covariant derivatives
- δF_i must include a new term involving gauginos

The full SUSY transformations for matter fields become:

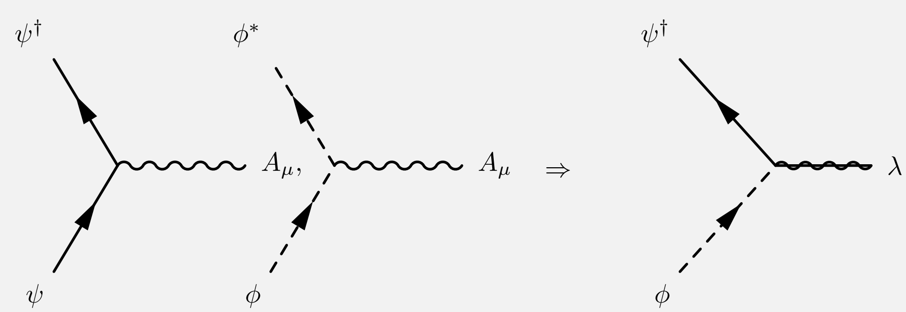
$$\delta\phi_i = \epsilon\psi_i \tag{3.78}$$
$$\delta\psi_{i\alpha} = -i(\sigma^\mu \epsilon^\dagger)_\alpha \, D_\mu \phi_i + \epsilon_\alpha F_i \tag{3.79}$$
$$\delta F_i = -i\epsilon^\dagger \overline{\sigma}^\mu D_\mu \psi_i \; + \; \sqrt{2} g (T^a \phi)_i \, \epsilon^\dagger \lambda^{\dagger a}. \tag{3.80}$$

which result in a supersymmetric theory if the additional terms in the Lagrangian are

$$\begin{aligned} \mathcal{L} \quad &= \quad \mathcal{L}_{\text{chiral}} + \mathcal{L}_{\text{gauge}} \\ & \quad\quad -\sqrt{2} g (\phi^* T^a \psi)\lambda^a - \sqrt{2} g \lambda^{\dagger a}(\psi^\dagger T^a \phi) + g(\phi^* T^a \phi) D^a. \end{aligned} \tag{3.81}$$

There is actually a 'naive' explanatio for the first two terms in the second line: one takes the usual interaction between a vector boson and two fermions and replaces two particles by their superpartners.

In a supersymmetric gauge theory, the supersymmetrized version of a coupling of a gauge boson to a pair of scalars or fermions becomes the interaction of a gaugino to a fermion/scalar which are superpartners:



$$\mathcal{L}_\lambda \quad = \quad -\sqrt{2} g (\phi^* T^a \psi)\lambda^a - \sqrt{2} g \lambda^{\dagger a}(\psi^\dagger T^a \phi) \tag{3.82}$$

With the last term in eq. (3.82), the Lagrangian for the D fields becomes

$$\mathcal{L}_D = \frac{1}{2} D^a D^a + g(\phi^* T^a \phi) D^a \tag{3.83}$$

which results in the equation of motion

$$D^a = -g(\phi^* T^a \phi). \tag{3.84}$$

Thus, like the auxiliary fields F_i and F^{*i}, the D^a can be expressed by a pair of the scalar fields. Consequently, $D^a D^a$ corresponds to a ϕ^4 term which is part of the scalar potential.

The full scalar potential of the theory is a sum of D- and F-term contributions

$$V(\phi,\phi^*) = F^{*i}F_i + \frac{1}{2}\sum_a D^a D^a = W_i^* W^i + \frac{1}{2}\sum_a g_a^2(\phi^* T^a \phi)^2. \tag{3.85}$$

Here, we have explicitly written $\sum_a$ which is the sum over all gauge groups of the theory. In contrast to non-supersymmetric models, the scalar potential has no free parameters (quartic couplings) but is completely fixed by gauge and Yukawa interactions.

3.2.5 Superfields and superspace

All the results which we have derived so far could also be obtained using so called 'superfield methods'. This approach is mathematically more elegant but also more involved. Therefore, we give here only the basic idea.
The so called superspace extends the four space–time coordinates by four additional coordinates Points in superspace are labeled by coordinates:

$$x^\mu,\ \theta^\alpha,\ \theta^\dagger_{\dot\alpha}. \tag{3.86}$$

Here θ^α and $\theta^\dagger_{\dot\alpha}$ are constant complex anti-commuting two-component spinors (Grassmann coordinates). Considering a single Grassmann variable η with

$$\eta^2 = 0 \tag{3.87}$$

one can express any function $f(\eta)$ as

$$f(\eta) = f_0 + \eta f_1 \tag{3.88}$$

Integration and derivation with respect to Grassmann variables are defined as:

$$\frac{df}{d\eta} = f_0 \tag{3.89}$$

$$\left.\begin{array}{l}\int d\eta = 0\\ \int d\eta\eta = 1\end{array}\right\} \int d\eta f = f_1 \tag{3.90}$$

One can write a superfields as function of Grassmann coordinates:

$$\hat\Phi \ = \ \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y), \tag{3.91}$$

The superpotential can be written in terms of superfields

$$W(\hat\Phi) = L_i\hat\Phi^i + \frac{1}{2}M_{ij}\hat\Phi^i\hat\Phi^j + \frac{1}{6}y_{ijk}\hat\Phi^i\hat\Phi^j\hat\Phi^k \tag{3.92}$$

from which the Lagrangian can be calculted as

$$\mathcal{L} = \int d^2\theta\theta(W(\hat\Phi) + \text{c.c.}) \tag{3.93}$$

We find that a product of three superfields becomes

$$\begin{aligned}\hat{\Phi}_i\Phi_j\Phi_k &= \phi_i\phi_j\phi_k + \sqrt{2}\theta(\psi_i\phi_j\phi_k + \psi_j\phi_i\phi_k + \psi_k\phi_i\phi_j) \\ &\quad + \theta\theta(\phi_i\phi_j F_k + \phi_i\phi_k F_j + \phi_j\phi_k F_i - \psi_i\psi_j\phi_k - \psi_i\psi_k\phi_j - \psi_j\psi_k\phi_i)\end{aligned} \tag{3.94}$$

Thus,

$$\mathcal{L} = \int d^2\theta\theta\hat{\Phi}_i\Phi_j\Phi_k \tag{3.95}$$

$$=(\phi_i\phi_j F_k + \phi_i\phi_k F_j + \phi_j\phi_k F_i - \psi_i\psi_j\phi_k - \psi_i\psi_k\phi_j - \psi_j\psi_k\phi_i) \tag{3.96}$$

Where we recovered the Yukawa-like interactions ($\psi\psi\phi$) and F-terms ($F\phi\phi$).

In order to define a supersymmetric theory, often the superpotential in terms of superfields is given:

$$W(\hat{\Phi}) = L_i\hat{\Phi}^i + \frac{1}{2}M_{ij}\hat{\Phi}^i\hat{\Phi}^j + \frac{1}{6}y_{ijk}\hat{\Phi}^i\hat{\Phi}^j\hat{\Phi}^k \tag{3.97}$$

The obtained Lagrangian from

$$\mathcal{L} = \int d^2\theta\theta(W(\hat{\Phi}) + \text{c.c.}) \tag{3.98}$$

is identical to the one which one gets from

$$W(\phi) = L^i\phi_i + \frac{1}{2}M^{ij}\phi_i\phi_j + \frac{1}{6}y^{ijk}\phi_i\phi_j\phi_k \tag{3.99}$$

and

$$\mathcal{L} = \left(-\frac{1}{2}\frac{\delta^2 W}{\delta\phi_i\delta\phi_j}\psi_i\psi_j + \frac{\delta W}{\delta\phi_i}F_i\right) + \text{c.c.} \tag{3.100}$$

3.3 SUSY breaking