# Introduction to Supersymmetry

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# Chapter 1

# Motivation to look beyond the SM

The standard model of particle physics (SM) is very successful and experimentally well confirmed. However, some questions can't be addressed within the SM.

## 1.1 Observations

### 1.1.1 Dark Matter

The energy budget of the universe is well known today:

| | | |
|---|---|---|
| Visible Matter | 0.03% | Heavy Elements |
| | 0.3% | Neutrinos |
| | 0.5% | Stars |
| | 4 % | Free hydrogen and helium |
| Dark Matter | 25 % | Weakly interacting new particle (WIMP)? |
| Dark Energy | 70% | ??? |

⇒ The SM can only explain 4.9% of the entire energy in the universe

### 1.1.2 Baryon Asymmetry

We don't see any anti-matter in the observable universe. However, the Big Bang should have produced equal amounts of matter and anti-matter, i.e. the asymmetry must have been introduced later.
In general: one needs interactions which violate CP (charge-parity) to break the symmetry between matter and anti-matter.

⇒ The amount of CP violation in the SM is too small to explain the observed matter–anti-matter asymmetry

## 1.2 Experimental deviations

Not all experiments are in perfect agreement with the SM. In some observables, a sizeable deviation was found

**Anomalous magnetic dipole moment**

The magnetic momentum of an elementary particle is given by

$$m_S = -\frac{g\mu_B S}{\hbar} \tag{1.1}$$

$\mu_B$: Bohr magneton; $S$: Spin
The $g$ factor is predicted to be **2** by Dirac's theory, but higher order effects change this.:

$$\text{Anomalous magnetic moment} \qquad a = \frac{g-2}{2} \tag{1.2}$$

The anomalous magnetic moments are among the best measured and most precisely calculated observables:

$$a_\mu^{\text{SM}} = 0.001\ 165\ 918\ 04\ (51) \tag{1.3}$$

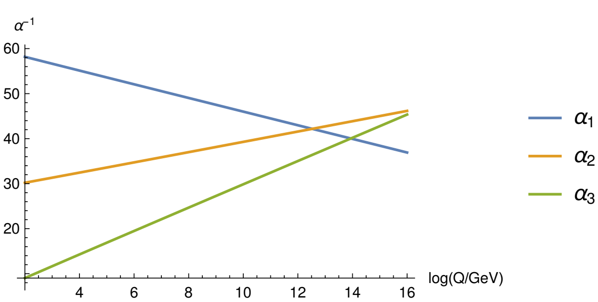
$$a_\mu^{\text{exp}} = 0.001\ 165\ 920\ 9(6) \tag{1.4}$$

⇒ There is a 3.5 $\sigma$ deviation between the measured anomalous magnetic moment of the myon and the SM prediction

## 1.3 Theoretical Issues

### 1.3.1 Gauge coupling unification

The coupling strength between particles is an energy dependent quantity. The energy dependence is described by the renormalisation group equations (RGEs). For the three gauge couplings of the SM one finds the following behaviour:




⇒ The gauge couplings in the SM don't unify. However, a grand unified theory (GUT) like $SO(10)$ or $SU(5)$ predict such an unification.

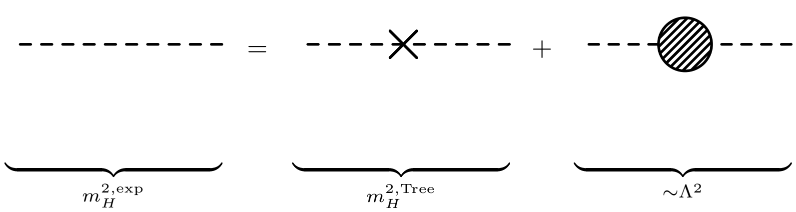
It's not possible to embed the SM in a GUT theory without introducing new matter. It's not clear at which scale the new particles come into play. However, the lighter they are, the bigger their impact is: less particles are needed in low-scale BSM models.

### 1.3.2 Hierarchy problem

The Higgs particle is the only fundamental scalar in the SM. While fermion and vector boson masses are protected by symmetries (chiral and gauge symmetries) against large radiative corrections, the masses of scalars don't have such a protection mechanism. Therefore, the observable mass is given by

$$m^{2,\text{obs}} = m^{2,\text{Tree}} + \delta m^2 \tag{1.5}$$
$$\simeq m^{2,\text{Tree}} + \Lambda^2 \tag{1.6}$$

where $m^{2,\text{Tree}}$ is the mass parameter in the Lagrangian and $\Lambda$ is the scale of new physics. We know that (at least) one scale exists at which new interactions come into play: the Planck scale ($M_P \sim 10^{18}$ GeV) at which gravity becomes important.



(1.7)

⇒ The SM has no natural explanation why the observed Higgs mass is $\sim 125$ GeV, but it demands a cancellation of 32 digits between unrelated parameters.

## 1.4 Why supersymmetry?

Supersymmetry (SUSY) provides possible explanations for all these questions:

- New Particles can form the DM
- New sources of CP violation to generate the Baryon asymmetry
- New loop contributions to $a_\mu$
- Changes the running of gauge couplings → Unification!
- The Higgs mass is protected by the new symmetry and naturally light

Because of these reasons, minimal supersymmetry was for a long time the top candidate for an extensions of the SM. However, with the negative searches at LHC the picture is changing: heavier SUSY masses introduce a new (small) hierarchy problem in the theory. Nevertheless:

- Other benefits of SUSY (dark matter, gauge coupling unification, CP violation) are hardly affected
- The corrections to the Higgs mass are only logarithmic dependent on the SUSY scale, not quadratic as in the SM alone

- There are still unexplored corners in which light SUSY particles are possible within minimal supersymmetry
- There is an increasing interest in non-minimal SUSY models which avoid the small hierarchy problem

# Chapter 2

# Basics

## 2.1 Notations and conventions

- Natural units (formally $\hbar = c = 1$) are used everywhere.
- Lorentz indices are always denoted by Greek characters, $\mu, \nu, .. = 0, 1, 2, 3$.
- Four-vectors for space–time coordinates and particle momenta are written as

$$x = (x^\mu) = (x^0, \vec{x}), \quad x^0 = t\,,$$
$$p = (p^\mu) = (p^0, \vec{p}), \quad p^0 = E = \sqrt{\vec{p}^2 + m^2}\,.$$

- Co- and Contravariant vectors are related by

$$a_\mu = g_{\mu\nu}\, a^\nu,$$

  with the metric tensor

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- The 4-dimensional scalar product is

$$a^2 = g_{\mu\nu}\, a^\mu a^\nu = a_\mu a^\mu, \quad a \cdot b = a_\mu b^\mu = a^0 b^0 - \vec{a} \cdot \vec{b}\,.$$

- Covariant and contravariant components of the derivatives are written as

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = g_{\mu\nu}\, \partial^\nu, \quad \partial^\nu = \frac{\partial}{\partial x_\nu} \qquad [\, \partial^0 = \partial_0, \;\; \partial^k = -\partial_k \,]\,,$$
$$\Box = \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \Delta\,.$$

## 2.2 Group Theory

### 2.2.1 Axioms

A collection of elements $g_i$ form a group if the following conditions are fulfilled:

**a)** **Closure** under a multiplication operator; i.e., if $g_i$ and $g_j$ are members of the group, then $g_i \cdot g_j$ is also a member of the group

**b)** **Associativity** under multiplication; i.e.

$$g_i \cdot (g_j \cdot g_k) = (g_i \cdot g_j) \cdot g_k \tag{2.1}$$

**c)** **An identity element**; i.e., there exist an element $\mathbf{1}$ such that

$$\mathbf{1} \cdot g_i = g_i \cdot \mathbf{1} = g_i \tag{2.2}$$

**d)** **An inverse**; i.e. every element $g_i$ has an element $g_i^{-1}$ such that

$$g_i \cdot g_i^{-1} = \mathbf{1} \tag{2.3}$$

### 2.2.2 Lie Groups

#### 2.2.2.1 Definition

Lie Groups are both groups and differentiable manifolds.

Any group element continuously connected to the identity can be written

$$U = e^{i\Theta_a T^a} \tag{2.4}$$

where the $\Theta_a$ is a real parameter and the $T^a$ are the group generators, which live in the Lie Algebra.

The generators $T^a$ , which generate infinitesimal group transformations, form the Lie Algebra.

The Lie algebra is defined by its commutation relations

$$[T^a, T^b] = if^{abc}T_c \tag{2.5}$$

where $f^{abc}$ are known as the structure constants.

By definition they are antitsymmetric

$$f^{abc} = -f^{acb} \tag{2.6}$$

#### 2.2.2.2 Groups considered in the following

We are interested in so called semi-simple Lie groups as $SU(N)$ and $SO(N)$. We focus in the following on $SU(N)$. These groups preserve a complex inner product. Finite dimensional representations of semi-simple Lie algebras are always Hermitian, so one can build quantum theories which are unitarity based on such algebras. The complex inner product is

$$U^\dagger U = 1 \tag{2.7}$$

defined on $N$ dimensional complex vector spaces, for $U(N)$. Note that in all cases we can write $U(N) = SU(N) \times U(1)$ where the $U(1)$ represents an overall phase. There are $N^2 - 1$ generators for $SU(N)$. To see this, let us write the identity infinitesimally as

$$0 = 1 - e^0 \tag{2.8}$$
$$= 1 - e^{[i\Theta_a T_a + (i\Theta_a T_a)^\dagger]} \tag{2.9}$$
$$= 1 - (1 + i\Theta_a T_a)(1 - i\Theta_a T_a^\dagger) \tag{2.10}$$
$$= - i\Theta_a (T^\dagger)_a + i\Theta_a T_a \tag{2.11}$$
$$\Rightarrow T = T^\dagger \tag{2.12}$$

so we can count the generators by counting $N \times N$ Hermitian matrices. Such matrices have $\frac{1}{2}N(N-1)$ imaginary components and $\frac{1}{2}N(N+1)$ real components, but then we subtract the identity matrix, which just generates $U(1)$. Thus, we find for the number of generators

$$\#(T_a) = \frac{1}{2}N(N-1) + \frac{1}{2}N(N+1) - 1 = N^2 - 1 \tag{2.13}$$

#### 2.2.2.3 Representations

The groups and algebras discussed above are abstract mathematical objects. We want to have these groups act on quantum states and fields, which are vectors, so we need to represent the groups as matrices. There are an infinite number of different representations for a given simple group. However, there are two obvious and most important representations, which occur most often in physics settings. They are

**a)** the fundamental representations

**b)** the adjoint representations

The fundamental representation is the representation defining $SU(N)$ and $SO(N)$ as $N \times N$ matrices acting on $N$ dimensional vectors. To write the fundamental formally, we say that $N$ fields transform under it as

$$\phi_i \rightarrow \phi_i + i\alpha_a (T_f^a)_i^j \phi_j \tag{2.14}$$

where $i = 1, \ldots, N$, $a = 1, \ldots N^2 - 1$ and the $\alpha_a$ are real numbers. The complex conjugate fields transform in the anti-fundamental $\overline{f}$, which is just the conjugate of this

$$\phi_i^* \rightarrow \phi_i^* - i\alpha_a (T_f^{a*})_i^j \phi_j^* \tag{2.15}$$

Since $T_f^a$ are Hermitian, we have $T_{\bar{\mathrm{f}}} = (T_f)^*$.
The normalisation of generators is arbitrary and is usually chosen so that

$$\mathrm{Tr} T_f^a T_f^b = \frac{1}{2}\delta_{ab} \tag{2.16}$$

The other important representation is the adjoint. The point is to think of the generators themselves as the vectors. Thus, the generators are

$$(T_{\mathrm{adj}}^a)_c^b = -if^{abc} \tag{2.17}$$

How can we see that the $T_{\text{adj}}$ actually satisfy the Lie algebra, and thus are really a representation? This is given immediately by the Jacobi identity

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0 \tag{2.18}$$

written as

$$0 = [T^a, f^{bcd}T_d] + [T^b, f^{cad}T_d] + [T^c, f^{abd}T_d] \tag{2.19}$$
$$= f^{bcd}[T^a, T_d] + f^{cad}[T^b, T_d] + f^{abd}[T^c, T_d] \tag{2.20}$$
$$= f^{bcd}f^{ade}T_e + f^{cad}f^{bde}T_e + f^{abd}f^{cde}T_e \tag{2.21}$$
$$\Rightarrow f^{cbd}f^{ade} - f^{abd}f^{cde} = f^{cad}f^{dbe} \tag{2.22}$$
$$\Rightarrow [T^c_{\text{adj}}, T^a_{\text{adj}}] = if^{cad}T^d_{\text{adj}} \tag{2.23}$$

The dimension of the adjoint representation is $N^2 - 1$ for $SU(N)$.

#### 2.2.2.4 Group constants

The quadratic Casimir is defined as

$$T^a_R T^a_R = C_2(R)\mathbf{1} \tag{2.24}$$

This must be proportional to the identity (when acting on a single given irreducible representation) because it commutes with all generators of the group, which follows from

$$[T^a_R T^a_R, T^b_R] = T^a_R T^a_R T^b_R - T^b_R T^a_R T^a_R \tag{2.25}$$
$$= T^a_R([T^a_R, T^b_R] + T^b_R T^a_R]) - ([T^b_R, T^a_R] + T^a_R T^b_R)T^a_R \tag{2.26}$$
$$= T^a_R(if^{abc}T^c_R) - (if^{bac}T^c_R T^a_R) \tag{2.27}$$
$$= if^{abc}T^a_R T^c_R + if^{abc}T^c_R T^a_R \tag{2.28}$$
$$= 0 \tag{2.29}$$

because of anti-symmetry of $f^{abc}$.
Another important quantity is the Dynkin index $I(R)$

$$\text{Tr}[T^a_R T^b_R] = I(R)\delta_{ab} \tag{2.30}$$

The quantity $I(R)$ is the index of the representation. We have that

$$I(f) = \frac{1}{2} \tag{2.31}$$

and

$$I(G) = N \tag{2.32}$$

for $SU(N)$ and our normalisation. The Dynkin index and the quadratic Casimir are related

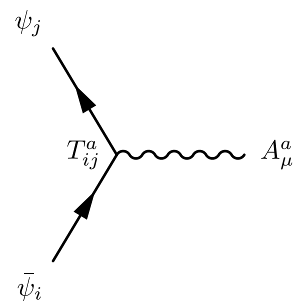
$$d(R)C_2(R) = I(R)d(G) \tag{2.33}$$

where $d(R)$ is the dimension of the representation, and $d(G)$ of the algebra, namely $N^2 - 1$ for $SU(N)$. Thus

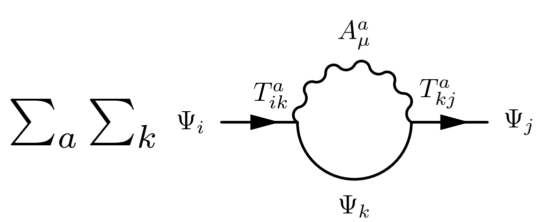
$$C_2(f) = \frac{N^2 - 1}{2N} \tag{2.34}$$
$$C_2(G) = N \tag{2.35}$$

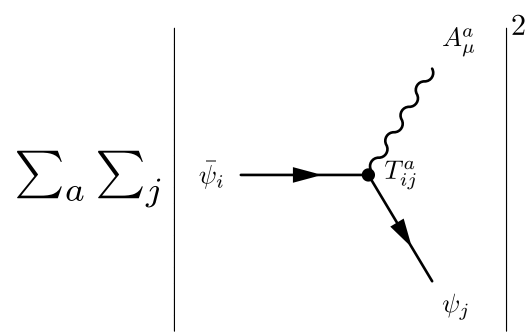
#### 2.2.2.5 Why do we need that?

**a)** The interactions between gauge bosons (and gauginos) are proportional to the generators of Lie group

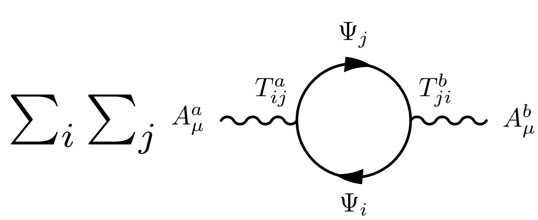


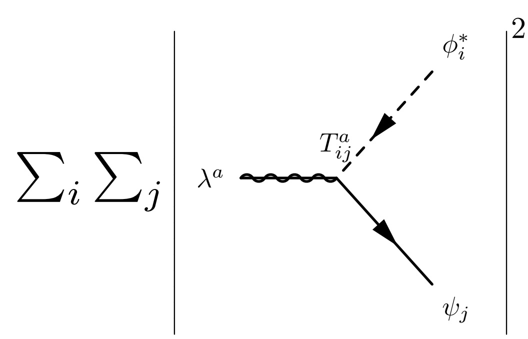

**b)** Loop corrections with gauge bosons/gauginos (or decays into them) are proportional to the quadratic Casimir $C_2$




**c)** Loop corrections to gauge bosons/gauginos (or decays of them) are proportional to the Dynkin index $I$




#### 2.2.2.6 Examples

##### 2.2.2.6.1 $SU(2)$

For $SU(2)$ the common generators for the fundamental representation $T_f^a$ are related to the Pauli matrices $\sigma^a$ $(i = 1, 2, 3)$ by

$$T_f^a = \frac{1}{2}\sigma^a \tag{2.36}$$

with

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.37}$$

For later, it is also helpful to introduce

$$\sigma^0 = \overline{\sigma}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.38}$$

and $\overline{\sigma}^i = -\sigma^i$. The Lie algebra

$$[\sigma^a, \sigma^b] = i f^{abc}\sigma_c \tag{2.39}$$

is fulfilled for

$$f^{abc} = \epsilon^{abc} \tag{2.40}$$

where $\epsilon^{abc}$ is the Levi-Civita tensor. And we have

$$d(f) = 2 \qquad d(a) = 3 \tag{2.41}$$

$$C_2(f) = \frac{3}{4} \qquad C_2(a) = 3 \tag{2.42}$$

$$I(f) = \frac{1}{2} \qquad I(a) = 2 \tag{2.43}$$

#### 2.2.2.6.2 SU(3)

The common representation for $SU(3)$ are given by the Gell-Mann Matrices $\lambda^a$

$$T_f^a = \frac{1}{2}\lambda^a \tag{2.44}$$

With

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.45}$$

$$\lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{2.46}$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.47}$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{2.48}$$

And we have

$$d(f) = 3 \qquad d(a) = 8 \tag{2.49}$$

$$C_2(f) = \frac{4}{3} \qquad C_2(a) = 3 \tag{2.50}$$

$$I(f) = \frac{1}{2} \qquad I(a) = 3 \tag{2.51}$$

### 2.2.3 Other groups relevant in particle physics

**a) Lorentz Group**: the Lorentz group is the set of all $4 \times 4$ real matrices that leave the line element in Minkowski space invariant:

$$s^2 = (x^0)^2 - (x^i)^2 = x^\mu g_{\mu\nu} x^\nu \tag{2.52}$$

It is parametrised by

$$x'^\mu = \Lambda^\mu_\nu x^\nu \tag{2.53}$$

The Lorentz group has six generators:

- three generators $J^i$ creating rotations
- three generators $K^i$ creating boosts

**b) Poincare Group**: the Poincare group is the generalisation of the Lorentz group including translation:

$$x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu \tag{2.54}$$

The generator of the translation is the four momentum operator $p_\mu$

## 2.3 Quantum Field Theory

### 2.3.1 Lagrangian formalism

We are working with the Lagrangian formalism of quantum field theory. The basic features are

- space–time symmetry in terms of Lorentz invariance, as well as internal symmetries like gauge symmetries
- causality
- local interactions

Particles are described by fields that are operators on the quantum mechanical Hilbert space of the particle states, acting as creation and annihilation operators for particles and antiparticles. We need in the following particles characterised by their spin:

- spin-0: complex or real scalar fields $\phi(x)$, $\varphi(x)$
- spin-$\frac{1}{2}$: fermions, described by two- or four component spinor fields $\psi_{L,R}$, $\psi(x)$.
- spin-1: vector bosons $A_\mu(x)$

The dynamics of the physical system involving a set of fields $\Phi$ is determined by the Lorentz-invariant Lagrangian $\mathcal{L}$. The action is given by

$$S[\Phi] = \int \mathrm{d}^4x \, \mathcal{L}\big(\Phi(x)\big) \,, \tag{2.55}$$

The equations of motions follow as Euler–Lagrange equations from Hamilton's principle,

$$\delta S = S[\Phi + \delta\Phi] - S[\Phi] = 0 \,. \tag{2.56}$$

Let's go back to mechanics: for $n$ generalised coordinates $q_i$ and velocities $\dot{q}_i$ the Lagrangian reads: $L(q_1, \ldots \dot{q}_1, \ldots)$ The equations of motion are calculated from $(i = 1, \ldots n)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \,. \tag{2.57}$$

Going to field theory, one has to perform the replacement

$$q_i \to \Phi(x) \,, \quad \dot{q}_i \to \partial_\mu \Phi(x) \,, \quad L(q_1, \ldots q_n, \dot{q}_1, \ldots \dot{q}_n) \to \mathcal{L}(\Phi(x), \partial_\mu \Phi(x)) \tag{2.58}$$

The equations of motion become field equations which are calculated from

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} - \frac{\partial \mathcal{L}}{\partial \Phi} = 0 \,, \tag{2.59}$$

### 2.3.2 Free quantum fields

#### 2.3.2.1 Scalar fields

The equation of motion for a scalar field is known as Klein–Gordon equation:

$$(\partial_\mu \partial^\mu + m^2) \, \phi = 0 \,. \tag{2.60}$$

The solution can be expanded in terms of the complete set of plane waves $e^{\pm ikx}$,

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3k}{2k^0} \left[a(k)\, e^{-ikx} \,+\, a^\dagger(k)\, e^{ikx}\,\right] \tag{2.61}$$

with $k^0 = \sqrt{\vec{k}^{\,2} + m^2}$. Here, we used annihilate and creation operators $a^\dagger$, $a$:

$$\begin{aligned} a^\dagger(k)\,|0\rangle &= |k\rangle \\ a(k)\,|k'\rangle &= 2k^0\,\delta^3(\vec{k} - \vec{k}\,')\,|0\rangle\,. \end{aligned} \tag{2.62}$$

The Lagrangian for a free real or complex scalar field with mass $m$ is

$$\mathcal{L}_{\text{real}} = \frac{1}{2}\,(\partial_\mu\phi)^2 - \frac{m^2}{2}\,\phi^2 \tag{2.63}$$

$$\mathcal{L}_{\text{complex}} = (\partial_\mu\phi)^\dagger(\partial^\mu\phi) - m^2\,\phi^\dagger\phi \tag{2.64}$$

One can easily check that they give us the Klein–Gordon Equation as equation of motion. A complex scalar field $\phi^\dagger \neq \phi$ has two degrees of freedom. It describes spin-less particles which carry a charge and can be interpreted as particles and antiparticles.
So far, we have considered particles without any space–time restrictions. Now, we want to consider the case that a particle propagates from a point-like source at a given space-time point. This is described by the inhomogeneous field equation

$$(\partial_\mu\partial^\mu \,+ m^2)\,D(x-y) = -\delta^4(x-y)\,. \tag{2.65}$$

$D(x-y)$ is called Green function. The solution can easily be determined by a Fourier transformation

$$D(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\,D(k)\,e^{-ik(x-y)} \tag{2.66}$$

giving in momentum space

$$(k^2 - m^2)\,D(k) = 1\,. \tag{2.67}$$

The solution

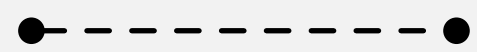
$$i\,D(k) = \frac{i}{k^2 - m^2 + i\epsilon} \tag{2.68}$$

is the *causal Green function* or the *Feynman propagator* of the scalar field. The overall factor $i$ is by convention. The term $+i\epsilon$ in the denominator with an infinitesimal $\epsilon > 0$ is a prescription of how to treat the pole in the integral (2.66); it corresponds to the special boundary condition of causality for $D(x-y)$ in Minkowski space, which means

- propagation of a particle from $y$ to $x$ if $x^0 > y^0$,
- propagation of an antiparticle from $x$ to $y$ if $y^0 > x^0$.

In a Feynman diagram, the scalar propagator is drawn as dashed line.

Complex Scalar $\phi(k,m)$  $\frac{1}{k^2-m^2+i\epsilon}$ (2.69)

Real Scalar $\varphi(k,m)$  $\frac{1}{k^2-m^2+i\epsilon}$ (2.70)

For complex scalars the arrow shows the flow of the charge.

#### 2.3.2.2 Dirac fields

**Equation of motion** Spin-$\frac{1}{2}$ particles with mass $m$ are often described by 4-component spinor fields,

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix} . \tag{2.71}$$

and obey the *Dirac–Equation*

$$(i\gamma^\mu \partial_\mu - m)\, \psi = 0 \, . \tag{2.72}$$

This equation is obtained from the Lagrangian

$$\mathcal{L}_{\text{fermion}} = \overline{\psi}\, (i\gamma^\mu \partial_\mu - m)\, \psi \, , \tag{2.73}$$

involving the adjoint spinor

$$\overline{\psi} = \psi^\dagger \gamma^0 = (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*) \, . \tag{2.74}$$

The Dirac matrices $\gamma^\mu$ ($\mu = 0,1,2,3$) are $4 \times 4$ matrices which fulfil the anti-commutator relations

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \, . \tag{2.75}$$

One possible representation is to express the matrices in terms of the the Pauli matrices $\sigma_{1,2,3}$ as

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} . \tag{2.76}$$

Another matrix, $\gamma_5$, is often very useful:

$$\gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.77}$$

There are two types of solutions for the Dirac equation, corresponding to particle and anti-particle wave functions,

$$u(p)\, e^{-ipx} \quad \text{and} \quad v(p)\, e^{ipx} \tag{2.78}$$

which are used to write the Dirac field as

$$\psi(x) = \frac{1}{(2\pi)^{3/2}} \sum_\sigma \int \frac{\mathrm{d}^3 k}{2k^0} \left[ c_\sigma(k)\, u_\sigma(k)\, e^{-ikx} \,+\, d_\sigma^\dagger(k)\, v_\sigma(k)\, e^{ikx} \right], \tag{2.79}$$

with

- annihilation operators $c_\sigma$ for particles and $d_\sigma$ for anti-particles
- creation operators $c_\sigma^\dagger$ and $d_\sigma^\dagger$ for particles and antiparticles

We still have to determine the propagator of the Dirac field, which is the solution of the inhomogeneous Dirac equation with point-like source,

$$(i\gamma^\mu \partial_\mu - m)\, S(x-y) = \mathbf{1}\, \delta^4(x-y)\,. \tag{2.80}$$

Using a Fourier transformation as in the scalar case, we find

$$i\, S(k) = \frac{i}{\not{k} - m + i\epsilon} = \frac{i\,(\not{k} + m)}{k^2 - m^2 + i\epsilon}\,, \tag{2.81}$$

We introduce a graphical symbol for the propagator:

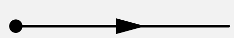
Dirac Fermion $\psi(k,m)$  $i\,\frac{\not{k}-m}{k^2-m^2+i\epsilon}$ (2.82)

The arrow at the line denotes the flow of the *particle* charge.
External fermions are depicted as

incoming particle  $u(k)$ (2.83)

incoming anti-particle  $\overline{v}(k)$ (2.84)

outgoing anti-particle  $v(k)$ (2.85)

outgoing particle  $\overline{u}(k)$ (2.86)

#### 2.3.2.3 Vector fields

A vector field $A_\mu(x)$ describes particles with spin 1. We concentrate here on the massless case with two degrees of freedom.
The Lagrangian of such a field is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g f^{abc} A_\mu^b A_\nu^c\,. \tag{2.87}$$

The last term is only present for non-Abelian gauge fields. The field equations are Maxwell's equations for the vector potential,

$$\left(\Box\, g^{\mu\nu} - \partial^\mu \partial^\nu\right) A_\nu = 0\,. \tag{2.88}$$

The propagator of the vector fields depends on the chosen gauge. In general $R_\xi$ gauge it is given by

$$i\, D_{\rho\nu}(k) = \frac{i}{k^2 + i\epsilon} \left[ -g_{\nu\rho} + (1-\xi)\, \frac{k_\nu k_\rho}{k^2} \right]. \tag{2.89}$$

which becomes very simple in Feynman gauge with $\xi = 1$.

The graphical symbol for the vector-field propagator (for both massive and massless) is a wavy line which carries the momentum $k$ and two Lorentz indices

massless Vector boson $A_\mu(k)$

$$-\, i \frac{g_{\mu\nu}}{k^2 + i\epsilon} \tag{2.90}$$

massive Vector boson $A_\mu(k, m)$

$$-\, i \frac{g_{\mu\nu} - \frac{k_\nu k_\mu}{m^2}}{k^2 - m^2 + i\epsilon} \tag{2.91}$$

(Possible) arrows at the lines denote the flow of the *particle* charge.
External vectors are depicted as

incoming particle

$$\epsilon_\mu \tag{2.92}$$

outgoing particle

$$\epsilon_\mu^* \tag{2.93}$$

### 2.3.3 Gauge invariance

So far, we have not considered any symmetry. We change that now and apply (local) gauge transformations to the fields.

$$\phi(x) \to e^{ig\Lambda(x)} \phi(x) \tag{2.94}$$
$$\phi(x)^* \to \phi(x)^* e^{-ig\Lambda(x)} \tag{2.95}$$
$$\Psi(x) \to e^{ig\Lambda(x)} \Psi(x) \tag{2.96}$$
$$\bar{\Psi}(x) \to \bar{\Psi} e^{-ig\Lambda(x)} \tag{2.97}$$

However, one can check that the Lagrangians for scalars and fermions are **not** invariant under these transformations. For instance, the fermionic part of the Lagrangian transforms as

$$\mathcal{L}'_{\text{fermion}} = i(\bar{\Psi})'\partial\!\!\!/(\Psi)' - m(\bar{\Psi})'\Psi' \tag{2.98}$$

$$= i\bar{\Psi}e^{-ig\Lambda(x)}\partial\!\!\!/ e^{ig\Lambda(x)}\Psi - m\bar{\Psi}\underbrace{(e^{-ig\Lambda(x)}e^{ig\Lambda(x)})}_{=1}\Psi \tag{2.99}$$

$$= i\bar{\Psi}(ig\partial\!\!\!/\Lambda(x))(\partial\!\!\!/\Psi) - m\bar{\Psi}\Psi \tag{2.100}$$

$$\neq \mathcal{L}_{\text{fermion}} \tag{2.101}$$

We need another ingredient to built kinetic terms for scalars and fermions which are gauge invariant: we introduce a massless gauge fields $A_\mu$ which transforms as

$$A_\mu \to A_\mu - \partial_\mu\Lambda(x) \tag{2.102}$$

In addition, we define the covariant derivative:

$$\partial_\mu \to D_\mu = \partial_\mu + igA_\mu \tag{2.103}$$

$g$ is a free parameter which we call 'gauge coupling'. One finds that the covariant derivative transforms as

$$(D_\mu\Psi)' = D'_\mu\Psi' \tag{2.104}$$

$$= (\partial_\mu + ig(A_\mu - \partial_\mu\Lambda))e^{ig\Lambda}\Psi \tag{2.105}$$

$$= e^{ig\Lambda}(\partial_\mu + igA_\mu)\Psi - e^{ig\Lambda}ig\partial_\mu\Lambda\Psi + (\partial_\mu e^{ig\Lambda})\Psi \tag{2.106}$$

$$= e^{ig\Lambda}(\partial_\mu + igA_\mu)\Psi \tag{2.107}$$

$$= e^{ig\Lambda}D_\mu\Psi \tag{2.108}$$

Thus, the Lagrangian with derivatives replaced by covariant derivatives are invariant.

$$\bar{\Psi}D_\mu\Psi \to (\bar{\Psi})'(D_\mu\Psi)' = \bar{\Psi}e^{-ig\Lambda}e^{ig\Lambda}D_\mu\Psi = \bar{\Psi}D_\mu\Psi \tag{2.109}$$

Similarly, one can show that for the scalar terms in the Lagrangian the identity

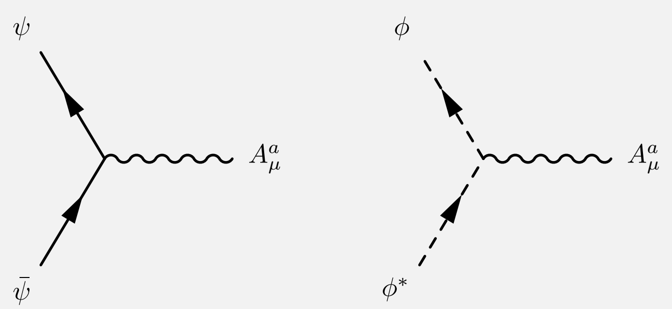
$$(D_\mu\phi D^\mu\phi^*)' = D_\mu\phi D^\mu\phi^* \tag{2.110}$$

holds.

In order to obtain a gauge theory, i.e. a theory in which the Lagrangian is invariant under a local transformation, the derivative must be replaced by the **covariant derivative** involving gauge fields:

$$\partial^\mu \to D_\mu = \partial^\mu + igT^a A^a_\mu \tag{2.111}$$

Here, $T^a$ are the generators of the gauge croup ($SU(N)$) and $a = 1, \ldots, N^2 - 1$.
This introduces interaction terms between the fermions and scalars and the gauge fields which are represented by the following Feynman diagrams:




$igT^a\gamma_\mu$ $igT^a(p^\mu_{\text{in}} - p^\mu_{\text{out}})$ $ig^2\{T^a, T^b\}g_{\mu\nu}$

### 2.3.4 Spontaneous symmetry breaking

A mass term for gauge bosons would read

$$m_V^2 A_\mu A^\mu \tag{2.112}$$

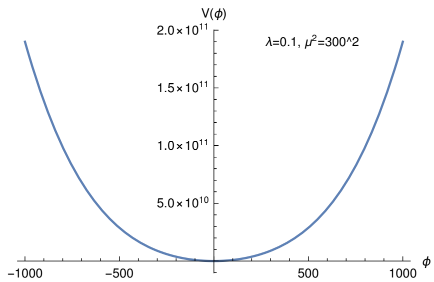
However, this is not gauge invariant:

$$(m_V^2 A_\mu A^\mu)' = m_V^2 A_\mu A^\mu - A_\mu \partial^\mu \Lambda - \partial_\mu \Lambda A^\mu + m_V^2 (\partial_\mu \Lambda)(\partial^\mu \Lambda) \tag{2.113}$$

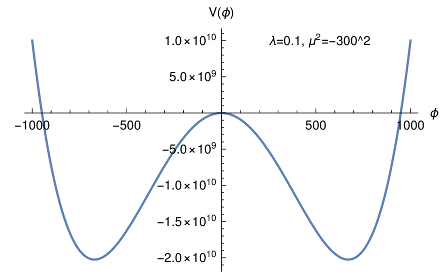
Thus, explicit mass terms are not possible and we must generated them Spontaneously. This idea is the famous *Higgs–mechanism*. For that, let's assume a real scalar $\varphi$ and the following potential:

$$V(\varphi) = \frac{1}{2}\lambda\varphi^4 + \mu^2\varphi^2 \tag{2.114}$$

Depending on the sign of $\mu^2$ the shape of the potential is different




$\mu^2 > 0$




$\mu^2 < 0$

For

- $\mu^2 > 0$: $\varphi = 0$ is the correct vacuum
- $\mu^2 < 0$: $\varphi = 0$ corresponds not to the bottoum of the potential, i.e. the correct vacuum is at $\varphi \neq 0$

We shift $\varphi$ in a way that we are for $\varphi = 0$ at the minimum of the potential:

$$\varphi \to \varphi + v \tag{2.115}$$

We find

$$V(\varphi = 0) = \frac{1}{2}\lambda v^4 + \mu^2 v^2 \tag{2.116}$$

$$\to \frac{\partial V}{\partial v} = 2\lambda v^3 + 2v\mu^2 \equiv 0 \tag{2.117}$$

Thus

$$v = \sqrt{-\mu^2/\lambda} \tag{2.118}$$

is the value of the VEV (vacuum expectation value).

**Higgs mechanism** We consider now a gauge theory with a complex field $\phi$. This field is decomposed in it's real components as well as a VEV as

$$\phi \to \frac{1}{\sqrt{2}}(\varphi + v + i\sigma) \tag{2.119}$$

When we insert this in the general Lagrangian

$$\mathcal{L} = D_\mu \phi D^\mu \phi^* - m^2|\phi|^2 - \lambda|\phi|^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2.120}$$

we get

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\mu} + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma \\ &+ gvA_\mu\partial^\mu\sigma - \frac{1}{2}g^2v^2A_\mu A^\mu \\ &+ \frac{1}{2}g^2(A_\mu)^2\varphi(2v+\varphi) - \frac{1}{2}\varphi^2(3\lambda v^2 + m^2) - \lambda v\varphi^3 - \frac{1}{4}\lambda\varphi^4 \end{aligned} \tag{2.121}$$

The first line show the ordinary kinetic terms. However, we see that an effective mass term $\frac{1}{2}g^2v^2$ for the vector bosons has been generated (last term in second line). On the other side, $\sigma$ is massless, but there is also a term which mixes the field $\sigma$ and $A_\mu$.

**Interpretation** A massive vector boson has three degrees of freedom, while a massless one has only two. Therefore, one says that $\sigma$ is 'eaten' by the vector boson to form its longitudinal component. $\sigma$ is called 'Goldstone' (or 'Nambu-Goldstone') boson.
It is common to introduce gauge fixing terms in a way that they cancel the mixing terms between field $\sigma$ and $A^\mu$.

$$\mathcal{L}_{GF} = -\frac{1}{2\xi}\left(\partial_\mu A^\mu - gv\xi\sigma\right)^2 \tag{2.122}$$

Thus, the Lagrangian becomes

$$\mathcal{L} + \mathcal{L}_{GF} = +\frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{2}g^2v^2\xi\sigma^2 - \frac{1}{2}g^2v^2A_\mu A^\mu + \dots \tag{2.123}$$

what gives a relation between the Goldstone mass and the mass of the vector boson

$$M_G^2 = \xi M_A^2 \tag{2.124}$$

In the unitarity gauge $\xi \to \infty$, the Goldstone disappears from the spectrum.
The same could have been obtained by starting with a gauge transformation. Using

$$(\varphi + i\sigma + v) \to e^{i\sigma/v}(v + \varphi) \tag{2.125}$$

together with

$$\phi \to \phi' = e^{-i\sigma/v}\phi = \frac{1}{\sqrt{2}}(v + \varphi) \tag{2.126}$$

$$A_\mu \to A'_\mu = A_\mu - \frac{1}{gv}\partial_\mu\sigma \tag{2.127}$$

which leads to the Lagrangian

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F'_{\mu\nu}F'^{\mu\mu} + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}g^2v^2A'_\mu A'^\mu \\ &+ \frac{1}{2}g^2(A'_\mu)^2\varphi(2v + \varphi) - \frac{1}{2}\varphi^2(3\lambda v^2 + m^2) - \lambda v\varphi^3 - \frac{1}{4}\lambda\varphi^4\end{aligned} \tag{2.128}$$

The Higgs mechanism generates mass terms for vector-boson due to vacuum expectation values of a complex scalar field

$$\phi \to \frac{1}{\sqrt{2}}\left(\varphi + i\sigma + v\right) \tag{2.129}$$

While the real (CP-even) component $\varphi$ of the scalar is a physical degree of freedom, the imaginary (CP-odd) component $\sigma$ becomes the longitudinal mode of the massive vector boson. In general $R_\xi$ gauge the Goldstone mass $M_G$ is related to the mass $M_A$ of the vector boson $A^\mu$ by

$$M_G^2 = \xi M_A^2 \tag{2.130}$$

### 2.3.5 Weyl Fermions

We have so far used 4-component (Dirac) fermions. However, it will turn out that it is often more convenient to use 2-component notation:

- in any model which violates parity (as the SM or all extension of it), each Dirac fermion has left-handed and right-handed parts with completely different electroweak gauge interactions:
  $\to$ The two-component Weyl fermion notation has the advantage of treating fermionic degrees of freedom with different gauge quantum numbers separately from the start.
- if one uses four-component spinor notation in the SM (or beyond), then there would be a sea of projection operators

$$P_L = (1 - \gamma_5)/2, \qquad P_R = (1 + \gamma_5)/2 \tag{2.131}$$

- in supersymmetric models the minimal building blocks of matter are chiral supermultiplets, each of which contains a single two-component Weyl fermion

Since the two-component notation might less familiar, we want to discuss it a bit.

#### 2.3.5.1 Two-component spinors

In this representation, a four-component Dirac spinor is written in terms of 2 two-component, complex anti-commuting objects $\xi_\alpha$ and $(\chi^\dagger)^{\dot\alpha} \equiv \chi^{\dagger\dot\alpha}$, with two distinct types of spinor indices $\alpha = 1, 2$ and $\dot\alpha = 1, 2$:

$$\Psi_D = \begin{pmatrix} \xi_\alpha \\ \chi^{\dagger\dot\alpha} \end{pmatrix}. \tag{2.132}$$

It follows that

$$\overline{\Psi}_D = \Psi_D^\dagger \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \chi^\alpha & \xi^\dagger_{\dot\alpha} \end{pmatrix} . \tag{2.133}$$

Undotted (dotted) indices from the beginning of the Greek alphabet are used for the first (last) two components of a Dirac spinor. The field $\xi$ is called a "left-handed Weyl spinor" and $\chi^\dagger$ is a "right-handed Weyl spinor". The names fit, because

$$P_L \Psi_D = \begin{pmatrix} \xi_\alpha \\ 0 \end{pmatrix}, \qquad P_R \Psi_D = \begin{pmatrix} 0 \\ \chi^{\dagger\dot\alpha} \end{pmatrix} . \tag{2.134}$$

The Hermitian conjugate of any left-handed Weyl spinor is a right-handed Weyl spinor:

$$\psi^\dagger_{\dot\alpha} \equiv (\psi_\alpha)^\dagger = (\psi^\dagger)_{\dot\alpha} \, , \tag{2.135}$$

and vice versa:

$$(\psi^{\dagger\dot\alpha})^\dagger = \psi^\alpha. \tag{2.136}$$

Any particular fermionic degrees of freedom can be described equally well using a left-handed Weyl spinor (with an undotted index) or by a right-handed one (with a dotted index). By convention, all names of fermion fields are chosen so that left-handed Weyl spinors do not carry daggers and right-handed Weyl spinors do carry daggers.

#### 2.3.5.2 Index operations

The heights of the dotted and undotted spinor indices are important. The spinor indices are raised and lowered using the anti-symmetric symbol

$$\epsilon^{12} = -\epsilon^{21} = \epsilon_{21} = -\epsilon_{12} = 1, \qquad \epsilon_{11} = \epsilon_{22} = \epsilon^{11} = \epsilon^{22} = 0, \tag{2.137}$$

according to

$$\xi_\alpha = \epsilon_{\alpha\beta}\xi^\beta, \qquad \xi^\alpha = \epsilon^{\alpha\beta}\xi_\beta, \qquad \chi^\dagger_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\chi^{\dagger\dot\beta}, \qquad \chi^{\dagger\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\chi^\dagger_{\dot\beta}\,. \tag{2.138}$$

This is consistent since $\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \epsilon^{\gamma\beta}\epsilon_{\beta\alpha} = \delta^\gamma_\alpha$ and $\epsilon_{\dot\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma} = \epsilon^{\dot\gamma\dot\beta}\epsilon_{\dot\beta\dot\alpha} = \delta^{\dot\gamma}_{\dot\alpha}$.
As a convention, repeated spinor indices contracted like

$$^\alpha{}_\alpha \qquad \text{or} \qquad {}_{\dot\alpha}{}^{\dot\alpha} \tag{2.139}$$

can be suppressed. In particular,

$$\xi\chi \equiv \xi^\alpha\chi_\alpha = \xi^\alpha\epsilon_{\alpha\beta}\chi^\beta = -\chi^\beta\epsilon_{\alpha\beta}\xi^\alpha = \chi^\beta\epsilon_{\beta\alpha}\xi^\alpha = \chi^\beta\xi_\beta \equiv \chi\xi \tag{2.140}$$

with, conveniently, no minus sign in the end. [A minus sign appeared first from exchanging the order of anti-commuting spinors, but it disappeared due to the anti-symmetry of the $\epsilon$ symbol.] Likewise, $\xi^\dagger\chi^\dagger$ and $\chi^\dagger\xi^\dagger$ are equivalent abbreviations for $\chi^\dagger_{\dot\alpha}\xi^{\dagger\dot\alpha} = \xi^\dagger_{\dot\alpha}\chi^{\dagger\dot\alpha}$, and in fact this is the complex conjugate of $\xi\chi$:

$$(\xi\chi)^* = \chi^\dagger\xi^\dagger = \xi^\dagger\chi^\dagger. \tag{2.141}$$

The explicit relation between $\sigma$ and $\overline{\sigma}$ is

$$\sigma^\mu_{\alpha\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\epsilon_{\alpha\beta}\overline{\sigma}^{\mu,\dot\beta\beta} \tag{2.142}$$

Using that, one can check that

$$\begin{aligned}
(\chi^\dagger\overline{\sigma}^\mu\xi)^* &= \xi^\dagger\overline{\sigma}^{\mu,\dagger}\chi \\
&= \xi^\dagger\overline{\sigma}^\mu\chi \\
&= \xi^\dagger_{\dot\alpha}\overline{\sigma}^{\mu,\dot\alpha\alpha}\chi_\alpha \\
&= \xi^\dagger_{\dot\alpha}\epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}\sigma^\mu_{\beta\dot\beta}\chi_\alpha \\
&= \xi^{\dagger,\dot\beta}\sigma^\mu_{\beta\dot\beta}\chi^\beta \\
&= -\chi^\beta\sigma^\mu_{\beta\dot\beta}\xi^{\dagger,\dot\beta} \\
&= -\chi\sigma^\mu\xi^\dagger \\
&= -(\xi\sigma^\mu\chi^\dagger)^*
\end{aligned} \tag{2.143}$$

Note that when taking the complex conjugate of a spinor bilinear, one reverses the order. The spinors here are assumed to be classical fields; for quantum fields the complex conjugation operation in these equations would be replaced by Hermitian conjugation in the Hilbert space operator sense.

Other hepful identities are:

$$(\chi^\dagger\overline{\sigma}^\nu\sigma^\mu\xi^\dagger)^* = \xi\sigma^\mu\overline{\sigma}^\nu\chi \;=\; \chi\sigma^\nu\overline{\sigma}^\mu\xi \;=\; (\xi^\dagger\overline{\sigma}^\mu\sigma^\nu\chi^\dagger)^*, \tag{2.144}$$

$$\sigma^\mu_{\alpha\dot{\alpha}}\,\overline{\sigma}^{\dot{\beta}\beta}_\mu \;=\; 2\delta^\beta_\alpha\delta^{\dot{\beta}}_{\dot{\alpha}}, \tag{2.145}$$

$$\sigma^\mu_{\alpha\dot{\alpha}}\,\sigma_{\mu\beta\dot{\beta}} \;=\; 2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}, \tag{2.146}$$

$$\overline{\sigma}^{\mu\dot{\alpha}\alpha}\,\overline{\sigma}^{\dot{\beta}\beta}_\mu \;=\; 2\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}, \tag{2.147}$$

$$\left[\sigma^\mu\overline{\sigma}^\nu + \sigma^\nu\overline{\sigma}^\mu\right]_\alpha{}^\beta \;=\; 2g^{\mu\nu}\delta^\beta_\alpha, \tag{2.148}$$

$$\left[\overline{\sigma}^\mu\sigma^\nu + \overline{\sigma}^\nu\sigma^\mu\right]^{\dot{\beta}}{}_{\dot{\alpha}} \;=\; 2g^{\mu\nu}\delta^{\dot{\beta}}_{\dot{\alpha}}, \tag{2.149}$$

$$\overline{\sigma}^\mu\sigma^\nu\overline{\sigma}^\rho \;=\; g^{\mu\nu}\overline{\sigma}^\rho + g^{\nu\rho}\overline{\sigma}^\mu - g^{\mu\rho}\overline{\sigma}^\nu - i\epsilon^{\mu\nu\rho\kappa}\overline{\sigma}_\kappa, \tag{2.150}$$

$$\sigma^\mu\overline{\sigma}^\nu\sigma^\rho \;=\; g^{\mu\nu}\sigma^\rho + g^{\nu\rho}\sigma^\mu - g^{\mu\rho}\sigma^\nu + i\epsilon^{\mu\nu\rho\kappa}\sigma_\kappa, \tag{2.151}$$

where $\epsilon^{\mu\nu\rho\kappa}$ is the totally antisymmetric tensor with $\epsilon^{0123} = +1$.
The so called *Fierz identity*, which we will need later, is

$$\chi_\alpha\,(\xi\eta) = -\xi_\alpha\,(\eta\chi) - \eta_\alpha\,(\chi\xi), \tag{2.152}$$

#### 2.3.5.3 Lagrangian for Weyl fermions

With these conventions, the Dirac Lagrangian can now be rewritten:

$$\mathcal{L}_{\text{Dirac}} \;=\; i\xi^\dagger\overline{\sigma}^\mu\partial_\mu\xi + i\chi^\dagger\overline{\sigma}^\mu\partial_\mu\chi - M(\xi\chi + \xi^\dagger\chi^\dagger) \tag{2.153}$$

where we have dropped a total derivative piece $-i\partial_\mu(\chi^\dagger\overline{\sigma}^\mu\chi)$, which does not affect the action.
A four-component Majorana spinor can be obtained from the Dirac spinor of eq. (2.133) by imposing the constraint $\chi = \xi$, so that

$$\Psi_{\text{M}} = \begin{pmatrix}\xi_\alpha\\ \xi^{\dagger\dot{\alpha}}\end{pmatrix}, \qquad\qquad \overline{\Psi}_{\text{M}} = \begin{pmatrix}\xi^\alpha & \xi^\dagger_{\dot{\alpha}}\end{pmatrix}. \tag{2.154}$$

The four-component spinor form of the Lagrangian for a Majorana fermion with mass $M$,

$$\mathcal{L}_{\text{Majorana}} \;=\; \frac{i}{2}\overline{\Psi}_{\text{M}}\gamma^\mu\partial_\mu\Psi_{\text{M}} - \frac{1}{2}M\overline{\Psi}_{\text{M}}\Psi_{\text{M}} \tag{2.155}$$

can therefore be rewritten as

$$\mathcal{L}_{\text{Majorana}} \;=\; i\xi^\dagger\overline{\sigma}^\mu\partial_\mu\xi - \frac{1}{2}M(\xi\xi + \xi^\dagger\xi^\dagger) \tag{2.156}$$

in the more economical two-component Weyl spinor representation. Note that even though $\xi_\alpha$ is anti-commuting, $\xi\xi$ and its complex conjugate $\xi^\dagger\xi^\dagger$ do not vanish, because of the suppressed $\epsilon$ symbol, see eq. (2.140). Explicitly, $\xi\xi = \epsilon^{\alpha\beta}\xi_\beta\xi_\alpha = \xi_2\xi_1 - \xi_1\xi_2 = 2\xi_2\xi_1$.

Any theory involving spin-1/2 fermions can always be written in terms of a collection of left-handed Weyl spinors $\psi_i$ with

$$\mathcal{L} \;=\; i\psi^{\dagger i}\overline{\sigma}^\mu\partial_\mu\psi_i - M^{ij}(\psi^\dagger_i\psi^\dagger_j - \psi_i\psi_j) \tag{2.157}$$

For $i = j$ one has a Majorana mass term, and $i \neq j$ gives Dirac mass term.

Given any expression involving bilinears of four-component spinors

$$\Psi_i = \begin{pmatrix} \xi_i \\ \chi_i^\dagger \end{pmatrix}, \tag{2.158}$$

labelled by a flavor or gauge-representation index $i$, one can translate into two-component Weyl spinor language (or vice versa) using the dictionary:

$$\overline{\Psi}_i P_L \Psi_j = \chi_i \xi_j, \qquad \overline{\Psi}_i P_R \Psi_j = \xi_i^\dagger \chi_j^\dagger, \tag{2.159}$$

$$\overline{\Psi}_i \gamma^\mu P_L \Psi_j = \xi_i^\dagger \overline{\sigma}^\mu \xi_j, \qquad \overline{\Psi}_i \gamma^\mu P_R \Psi_j = \chi_i \sigma^\mu \chi_j^\dagger \tag{2.160}$$

## 2.4 The Standard Model of Particle Physics

### 2.4.1 Gauge Symmetries

The so called standard model of particle physics (SM) is a gauge theory.

The gauge symmetry of the SM is

$$\mathcal{G} = SU(3)_C \times SU(2)_L \times U(1)_Y \tag{2.161}$$

with

- $C$: Colour
- $L$: Left
- $Y$: Hypercharge

### 2.4.2 Particle Content

Before symmetry breaking, the particle content of the SM is

| | | |
|---|---|---|
| Vector Bosons | $B$ | $(\mathbf{1},\mathbf{1})_0$ |
| | $W$ | $(\mathbf{1},\mathbf{3})_0$ |
| | $g$ | $(\mathbf{8},\mathbf{1})_0$ |
| Fermions | $e_R$ | $(\mathbf{1},\mathbf{1})_1$ |
| (3 Generations) | $l$ | $(\mathbf{1},\mathbf{2})_{-1/2}$ |
| | $u_R$ | $(\bar{\mathbf{3}},\mathbf{1})_{-2/3}$ |
| | $d_R$ | $(\bar{\mathbf{3}},\mathbf{1})_{1/3}$ |
| | $q$ | $(\mathbf{3},\mathbf{2})_{1/6}$ |
| Scalar | $H$ | $(\mathbf{1},\mathbf{2})_{1/2}$ |

The last column shows the quantum numbers with respect to $\mathcal{G}$. These quantum numbers are not as random as it might look. Special conditions must be fulfilled to avoid anomalies, e.g.

- Gauge anomalies

$$\sum_f Y(f)^3 \equiv 0 \tag{2.162}$$

- Gauge × gravity anomalies

$$\sum_f Y(f) \equiv 0 \tag{2.163}$$

- Witten anomaly: even number of $SU(2)$ doublets

**Check**:

$$\sum_f Y(f) = \underbrace{3}_{\text{generations}} \times \left( Y(e) + \underbrace{2}_{\text{isospin}} \times Y(l) + \underbrace{3}_{\text{color}} \times Y(u_R) + 3 \times Y(d_R) + 2 \times 3 \times Y(q) \right) \tag{2.164}$$

$$= 3 \times \left( 1 + 2\left(-\frac{1}{2}\right) + 3\left(-\frac{2}{3}\right) + 3\left(\frac{1}{3}\right) + 6\left(\frac{1}{6}\right)\right) \tag{2.165}$$

$$= 3 \times (1 - 1 - 2 + 1 + 1) \tag{2.166}$$

$$= 0 \tag{2.167}$$

$$\sum_f Y(f)^3 = 3 \times \left( 1 + 2\left(-\frac{1}{8}\right) + 3\left(-\frac{8}{27}\right) + 3\left(\frac{1}{27}\right) + 6\left(\frac{1}{216}\right)\right) \tag{2.168}$$

$$= 3 \times \left( 1 - \frac{1}{4} - \frac{8}{9} + \frac{1}{9} + \frac{1}{36} \right) \tag{2.169}$$

$$= 0 \tag{2.170}$$

⇒ One needs to be careful when adding new fermions in order not to introduce anomalies

### 2.4.3 Gauge part of the Lagrangian

The gauge part of the Lagrangian before symmetry breaking reads

$$L = D_\mu H D^\mu H^* + i \sum_f f^\dagger \sigma^\mu D_\mu f + \sum_V V_{\mu\nu} V^{\mu\nu} \tag{2.171}$$

with $f = \{l, e_R, q, d_R, u_R\}$ and $V = \{B, W^a, g^a\}$. Let's be more explicit at some examples. Note, we consider only one generation of fermions because gauge couplings are always flavour diagonal.

- Right leptons

$$e_R^\dagger \sigma^\mu D_\mu e_R = e_R^\dagger \sigma^\mu (\partial_\mu + i g_1 B_\mu) e_R \tag{2.172}$$

- Left leptons carry one isospin index, i.e. $l_i$ with $i = 1, 2$

$$l^\dagger \sigma^\mu D_\mu l = l_i^\dagger \sigma^\mu \left( \partial_\mu \delta_{ij} - i\frac{1}{2} g_1 B_\mu \delta_{ij} + i g_2 \frac{\sigma^a_{ij}}{2} W^a_\mu \right) l_j \tag{2.173}$$

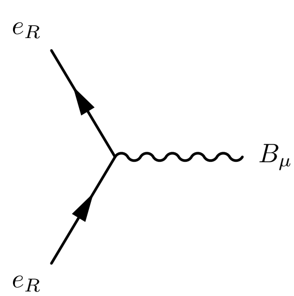
- Right up-quarks carry one colour index, i.e. $u_{R,\alpha}$ with $\alpha = 1, 2, 3$

$$u_R^\dagger \sigma^\mu D_\mu u_R = u_{R,\alpha}^\dagger \sigma^\mu \left( \partial_\mu \delta_{\alpha\beta} - i\frac{2}{3} g_1 B_\mu \delta_{\alpha\beta} + i g_3 \frac{\lambda^a_{\alpha\beta}}{2} G^a \right) u_{R,\beta} \tag{2.174}$$

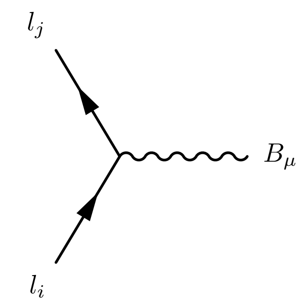
- Left quarks carry one colour and one isospin index, i.e. $q_{i,\alpha}$ with $\alpha = 1, 2, 3,\ i = 1, 2$

$$q^\dagger \sigma^\mu D_\mu q = q^\dagger_{i,\alpha} \sigma^\mu \left( \partial_\mu \delta_{\alpha\beta} \delta_{ij} - i\frac{1}{3} g_1 B_\mu \delta_{\alpha\beta} \delta_{ij} + i g_2 \delta_{\alpha\beta} \frac{\sigma^a_{ij}}{2} W^a + i g_3 \delta_{ij} \frac{\lambda^a_{\alpha\beta}}{2} G^a \right) q_{i.\beta} \qquad (2.175)$$

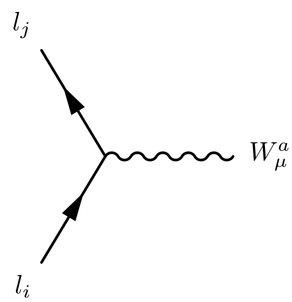
From these expressions the vertices are derived:



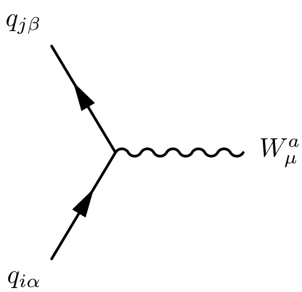

$\sigma_\mu g_1$



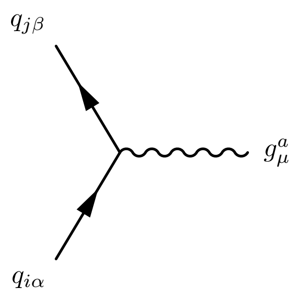

$-\sigma_\mu \delta_{ij} \frac{g_1}{2}$




$\sigma_\mu \sigma^a_{ij} \frac{g_2}{2}$




$\sigma_\mu \delta_{\alpha\beta} \sigma^a_{ij} \frac{g_2}{2}$




$\sigma_\mu \delta_{ij} \lambda^a_{ij} \frac{g_3}{2}$

### 2.4.4 Electroweak symmetry breaking

#### 2.4.4.1 The Higgs potential

The Higgs potential in the SM is given by

$$V(H) = \frac{1}{2}\lambda |H|^4 + \mu^2 |H|^2 \qquad (2.176)$$

Note, different conventions for the normalisation of the quartic coupling exist in literature. $\mu^2 < 0$ causes a spontaneous breaking of the electroweak symmetry (EWSB). The Higgs field is written as

$$\begin{pmatrix} H^+ \\ H^0 \end{pmatrix} \rightarrow \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}} \left( h + i G^0 + v \right) \end{pmatrix} \qquad (2.177)$$

The Higgs potential becomes

$$V = \frac{1}{8}\lambda((G^0)^2 + (h+v)^2 + 2G^+G^-)^2 + \frac{1}{2}\mu^2((G^0)^2 + (h+v)^2 + 2G^+G^-) \tag{2.178}$$

We can calculate the Higgs coupling and masses form this potential

**a) Tadpole conditions**: The condition for being at the minimum of the potential is

$$\frac{\partial V(h = G = G^+ = 0)}{\partial v} \equiv 0 = \frac{\partial}{\partial v}\left(\frac{1}{8}\lambda v^4 + \frac{1}{2}\mu^2 v^2\right) \tag{2.179}$$

$$= \frac{1}{2}\lambda v^3 + \mu^2 v \tag{2.180}$$

$$\to \mu^2 = -\frac{1}{2}v^2\lambda \tag{2.181}$$

Thus, one can eliminate $\mu^2$ from all following expressions.

**b) CP-even mass**: the Higgs mass is given by

$$m_h^2 = \frac{\partial^2 V}{\partial h^2}|_{h=G^0=G^+=0} \tag{2.182}$$

$$= \frac{3}{2}\lambda v^2 + \mu^2 \tag{2.183}$$

$$= \frac{3}{2}\lambda v^2 - \frac{1}{2}\lambda v^2 \tag{2.184}$$

$$= \lambda v^2 \tag{2.185}$$

**c) Goldstone masses**: the mass of $G^0$ becomes

$$m_{G^0}^2 = \frac{\partial^2 V}{\partial {G^0}^2}|_{h=G^0=G^+=0} \tag{2.186}$$

$$= \mu^2 + \frac{1}{2}\lambda v^2 \tag{2.187}$$

$$= 0 \tag{2.188}$$

Since we are working here in Landau gauge, the Goldstone mass vanishes as expected. Similarly, one can show $m_{G^+}^2 = 0$

**d) Cubic Higgs coupling**: the cubic Higgs self-interaction is

$$c_{hhh} = \frac{\partial^3 L}{\partial h^3}|_{h=G^0=G^+=0} \tag{2.189}$$

$$= -3v\lambda \tag{2.190}$$

$$= -3\frac{m_h^2}{v} \tag{2.191}$$

**e) Quartic Higgs coupling**: the quartic Higgs self-interaction is

$$c_{hhhh} = \frac{\partial^4 L}{\partial h^4}|_{h=G^0=G^+=0} \tag{2.192}$$

$$= -3\lambda \tag{2.193}$$

The entire Higgs sector of the SM can be parametrised after EWSB by just two parameters: $\lambda$ (or $m_h$) and $v$.

#### 2.4.4.2 Electroweak gauge bosons

The gauge interactions of the Higgs field become after EWSB:

$$D_\mu H D^\mu H^* = \left(\partial_\mu \delta_{ik} + i\left(\frac{1}{2}g_1 B_\mu \delta_{ik} + g_2 \frac{\sigma^a_{ik}}{2} W^a\right) H_i\right)\left(\partial_\mu \delta_{jk} - i\left(\frac{1}{2}g_1 B_\mu \delta_{jk} + g_2 \frac{\sigma^a_{jk}}{2} W^a\right) H^*_j\right) \tag{2.194}$$

$$\begin{aligned}=&\frac{1}{2}\partial_\mu h \partial^\mu h + \frac{1}{2}\partial_\mu G^0 \partial^\mu G^0 + \partial_\mu G^+ \partial^\mu G^- \\ &+ \frac{1}{4}\left((h+v)^2 + (G^0)^2\right)\left(g_1^2 B^2 - 2 g_1 g_2 B W^3 + g_2^2 (W_1^2 + W_2^2 + W_3^2)\right) \\ &+ \dots\end{aligned} \tag{2.195}$$

On can see in the second line that not only mass terms for the vector bosons are generated, but also a mixing between $B$ and $W^3$ occurs. The neutral mass matrix $M_V$ reads

$$M_V^2 = (B\ W_3)\begin{pmatrix} \frac{1}{4}v^2 g_1^2 & -\frac{1}{4}g_1 g_2 v^2 \\ -\frac{1}{4}g_1 g_2 v^2 & \frac{1}{4}g_2^2 v^2 \end{pmatrix}\begin{pmatrix} B \\ W_3 \end{pmatrix} \tag{2.196}$$

One finds that

$$\det M_V^2 = 0 \tag{2.197}$$

i.e. one eigenvalue is zero. The mixed particles, which appear after diagonalisation, are called photon ($\gamma$) and Z-Boson ($Z$). Their masses are the eigenvalues which are given by

$$m_\gamma = 0 \tag{2.198}$$

$$m_Z^2 = \frac{1}{4}(g_1^2 + g_2^2)v^2 \tag{2.199}$$

The rotation matrix which diagonalises $M_V^2$ is

$$\begin{pmatrix} \gamma \\ Z \end{pmatrix} = \begin{pmatrix} \cos\Theta_W & \sin\Theta_W \\ -\sin\Theta_W & \cos\Theta_W \end{pmatrix}\begin{pmatrix} B \\ W^3 \end{pmatrix} \tag{2.200}$$

with the Weinberg angle $\Theta_W$. This defines the electric charge, the coupling strength of the photon, as:

$$e = g_1 \cos\Theta_W = g_2 \sin\Theta_W \tag{2.201}$$

One remaining massless gauge boson corresponds to one unbroken symmetry. Therefore, the remaining symmetry of the SM is

$$\mathcal{G} \to SU(3)_C \times U(1)_{em} \tag{2.202}$$

Since $W_1$ and $W_2$ are not electromagnet eigenstates, they are combined to new eigenstate of $U(1)_{em}$

$$W^\pm = \frac{1}{\sqrt{2}}(W_1 \pm i W_2) \tag{2.203}$$

The mass of $W^\pm$ is given by

$$M_W^2 = \frac{1}{4} g^2 v^2 \tag{2.204}$$

The massless states $G^0$ and $G^\pm$ are the Goldstone bosons of $Z$ and $W^\pm$ and form their longitudinal components.

Let's count the real components of the particles

| Before EWSB | | | After EWSB | | |
|---|---|---|---|---|---|
| massless vectors: | $B$, $W^a$ | 4 | massless vectors: | $\gamma$ | 1 |
| massive vectors: | - | 0 | massive vectors: | $Z$, $W^\pm$ | 3 |
| complex scalars: | $H^0$, $H^\pm$ | 4 | complex scalars: | $G^\pm$ | 2 |
| real scalars: | - | 0 | real scalars: | $h$, $G^0$ | 2 |

The count of physical degrees of freedom gives

| Before EWSB | | | After EWSB | | |
|---|---|---|---|---|---|
| massless vectors: | $B$, $W^a$ | 8 | massless vectors: | $\gamma$ | 2 |
| massive vectors: | - | 0 | massive vectors: | $Z$, $W^\pm$ | 9 |
| complex scalars: | $H^0$, $H^\pm$ | 4 | complex scalars: | - | 0 |
| real scalars: | - | 0 | real scalars: | $h$ | 1 |

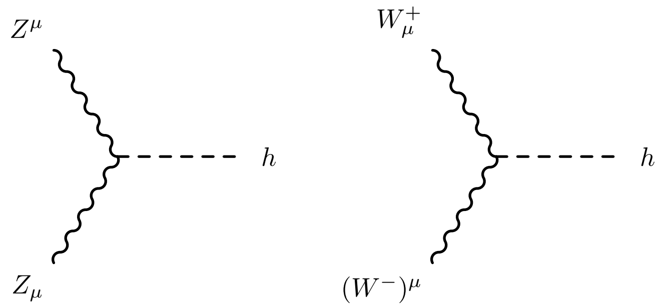
**Higgs interactions with vector bosons** The kinetic term for the mass eigenstates $h$, the SM Higgs boson, becomes after applying all rotations:

$$\begin{aligned}
\mathcal{L} =& \left(\partial_\mu \delta_{ij} + i\frac{1}{2} g_1 \delta_{ij} B_\mu + i\frac{1}{2} g_2 \sigma^a_{ij} W^a_\mu\right) H_i \left(\partial^\mu \delta_{ij} - i\frac{1}{2} g_1 \delta_{ij} B^\mu - i\frac{1}{2} g_2 \sigma^a_{ji} (W^a)^\mu\right) H^*_j \\
=& \dots \\
=& \frac{1}{4}(h+v)^2 \Big[ 2 g_2^2 W^+_\mu (W^-)^\mu + \gamma_\mu Z^\mu \left( (g_1^2 - g_2^2) \sin 2\Theta_W + 2 g_1 g_2 \cos 2\Theta_W \right) + \\
& \qquad \gamma_\mu \gamma^\mu (g_1 \cos\Theta_W - g_2 \sin\Theta_W)^2 + Z_\mu Z^\mu (g_1 \sin\Theta_W + g_2 \cos\Theta)^2 \Big] \\
& + (\partial_\mu + i\gamma_\mu (g_1 \cos\Theta - g_2 \sin\Theta) + i Z_\mu (g_1 \sin\Theta_W + g_2 \cos\Theta))\, h \\
& \qquad (\partial^\mu - i\gamma^\mu (g_1 \cos\Theta - g_2 \sin\Theta) - i Z^\mu (g_1 \sin\Theta_W + g_2 \cos\Theta))\, h \\
& + \mathcal{L}(G^0, G^+, h)
\end{aligned} \tag{2.205}$$

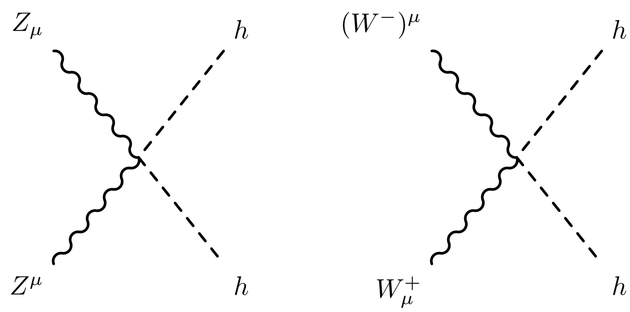
$$= \frac{1}{4} \frac{e^2}{\sin^2\Theta_W} (h+v)^2 \left( 2 W^+_\mu (W^-)^\mu + \frac{1}{\cos^2\Theta_W} Z_\mu Z^\mu \right) + \frac{1}{2} \partial_\mu h \partial^\mu h + \mathcal{L}(G^0, G^+, h) \tag{2.206}$$

Thus, the couplings between the Higgs to the photon drops out after performing all replacements correctly[1]. There is also no $h-h-Z$ interaction (which is forbidden by CP), but only $h-G^0-Z$. On the other side, one finds interactions between one Higgs and two $Z$- or $W$ bosons. The vertices for the Higgs to the gauge bosons are given by

[1] The general rule is: 'At tree-level, the photon couples only to charged particles and the Higgs only to massive ones'




$2iv\frac{e^2}{\sin^2 2\Theta_W}$ $\quad$ $\frac{1}{2}iv\frac{e^2}{\sin^2\Theta_W}$




$2i\frac{e^2}{\sin^2 2\Theta_W}$ $\quad$ $\frac{1}{2}i\frac{e^2}{\sin^2\Theta_W}$

The interaction between one scalar and two-vector bosons is always propotional to a VEV and can exist only after gauge symmetry breaking.

### 2.4.5 Fermion masses and Yukawa sector

It is not possible in the SM to write down mass terms for fermions because of the quantum numbers for left and right fields.

⇒ Fermion masses are spontaneously generated after EWSB via interactions with the Higgs field

The interactions between the Higgs and the SM fermions are called 'Yukawa' interactions.

$$\mathcal{L}_Y = Y_u q u_R H + Y_d q d_R H^* + Y_e l e_R H^* \quad + \text{h.c.} \tag{2.207}$$

In the general case, $Y_f$ are (complex) $3\times 3$ matrices. Thus, in the most general form the Lagrangian reads with all indices written explicitly

$$Y_u q u_R H \equiv \delta_{\alpha\beta} Y_{u,ab} q_{ai\alpha} u_{R,b\beta} \epsilon_{ij} H_j \tag{2.208}$$

with colour indices $\alpha, \beta$, isospin indices $i, j$, and generation indices $a, b$. If we neglect flavour mixing for the moment, one can write

$$\mathcal{L}_{Y_u} = Y_u q_{i\alpha} u_{R,\beta} \epsilon_{ij} H_j \tag{2.209}$$

$$= Y_u (u_{L,\alpha} H_0 - d_{L,\alpha} H^+) u_{R,\beta} \delta_{\alpha\beta} \tag{2.210}$$

what becomes after EWSB

$$\mathcal{L}_{Y_u} = \frac{1}{\sqrt{2}}(v+h)Y_u u_{L\alpha} u_{R\alpha} + \ldots \tag{2.211}$$

i.e. the fermion mass is given by

$$m_u = \frac{1}{\sqrt{2}} v Y_u \tag{2.212}$$

And therefore,

$$\mathcal{L}_{Y_u} = \frac{m_u}{v} h u_{L\alpha} u_{R\alpha} + \ldots \tag{2.213}$$

i.e. the Higgs coupling to SM fermions is proportional to their mass.

If we include flavour mixing, the mass terms for the quarks after EWSB read

$$\mathcal{L}_q = (d_L s_L b_L) \begin{pmatrix} vY_{d,11} & vY_{d,12} & vY_{d,13} \\ vY_{d,21} & vY_{d,22} & vY_{d,33} \\ vY_{d,31} & vY_{d,32} & vY_{d,33} \end{pmatrix} \begin{pmatrix} d_R \\ u_R \\ b_R \end{pmatrix} + (u_L c_L t_L) \begin{pmatrix} vY_{u,11} & vY_{u,12} & vY_{u,13} \\ vY_{u,21} & vY_{u,22} & vY_{u,33} \\ vY_{u,31} & vY_{u,32} & vY_{u,33} \end{pmatrix} \begin{pmatrix} u_R \\ c_R \\ t_R \end{pmatrix} \tag{2.214}$$

where we suppressed colour indices.
The six quark masses are the eigenvalues of the matrices $vY_d$ and $vY_u$. These matrices are diagonalised by four unitary matrices:

$$u_R \to U_R = U_u^* u_R \tag{2.215}$$

$$d_R \to D_R = U_d^* d_R \tag{2.216}$$

$$u_L \to U_L = V_u u_L \tag{2.217}$$

$$d_L \to D_L = V_d u_L \tag{2.218}$$

Only one combination of these matrices is physically relevant and defines the CKM (Cabibbo-Kobayashi-Maskawa) matrix

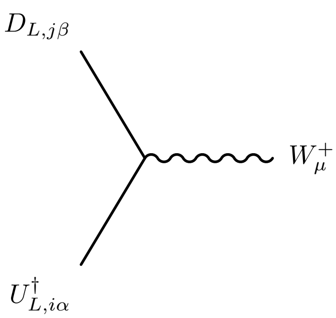
$$V_{\text{CKM}} = V_u^\dagger V_d \tag{2.219}$$

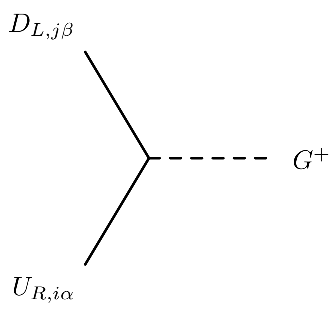
The entire flavour structure of the SM quark sector is encoded in the CKM matrix which can be parametrised by three angles $\Theta_{12}$, $\Theta_{23}$, $\Theta_{13}$ and one phase $\delta$

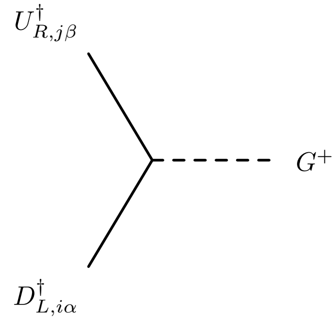
$$V_{\text{CKM}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.220}$$

$\delta$ is the only source of CP violation in the SM and highly restricted by experiments

The CKM matrix shows up explicitly in vertices involving the $W$-boson or the Goldstone $G$. Using two-component fields, one has



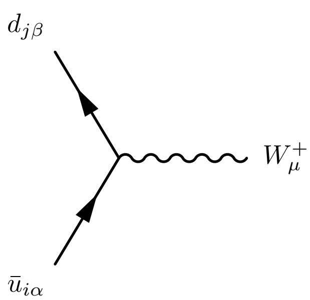



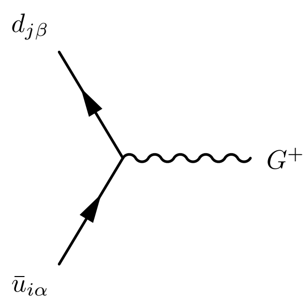

$-\frac{i}{\sqrt{2}}g_2\sigma_\mu V^{*ij}_{CKM}\delta_{\alpha\beta}$ $\qquad$ $i\sqrt{2}\frac{m_{u_i}}{v}V^{*ij}_{CKM}\delta_{\alpha\beta}$ $\qquad$ $i\sqrt{2}\frac{m_{d_j}}{v}V^{*ij}_{CKM}\delta_{\alpha\beta}$

The Dirac Spinors can be built from $D_i, U_i$ as follows

$$d = \begin{pmatrix} D_L \\ D_R^\dagger \end{pmatrix} \qquad u = \begin{pmatrix} U_L \\ U_L^\dagger \end{pmatrix} \tag{2.221}$$

Note, before EWSB is is not possible to write Dirac fermions consisting of left and right degrees of freedom because of different quantum numbers with respect to $U(1)_Y \times SU(2)_L$.




$-\frac{i}{\sqrt{2}}g_2\gamma_\mu V^{*ij}_{CKM}\delta_{\alpha\beta}P_L + 0P_R$ $\qquad$ $i\frac{\sqrt{2}}{v}V^{*ij}_{CKM}\delta_{\alpha\beta}(m_{u_i}P_L + m_{d_j}P_R)$

# Chapter 3

# Supersymmetric Formalities

## 3.1 Basics

### 3.1.1 SUSY transformations

A supersymmetry transformation turns a bosonic state into a fermionic state, and vice versa.

$$Q|\text{Boson}\rangle = |\text{Fermion}\rangle, \qquad Q|\text{Fermion}\rangle = |\text{Boson}\rangle. \tag{3.1}$$

The properties of the operator $Q$ are:

- $Q$ is an anti-commuting spinor
- $Q^\dagger$ is also a symmetry generator
- $Q$, $Q^\dagger$ carry spin $1/2 \to$ SUSY is a space–time symmetry.
- $Q$ and $Q^\dagger$ satisfy the following algebra (schematically):

$$\{Q, Q^\dagger\} = P^\mu, \tag{3.2}$$
$$\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0, \tag{3.3}$$
$$[P^\mu, Q] = [P^\mu, Q^\dagger] = 0, \tag{3.4}$$

  where $P^\mu$ is the four-momentum generator of spacetime translations. Note, we skipped here the spinor indices on $Q$, $Q^\dagger$. (The accurate expressions could be given once we have developed the necessary formalism.)

- $Q$ and $Q^\dagger$ commute with $P^2$
- $Q$ and $Q^\dagger$ commute with all generators of gauge transformations

A non-trivial connection between internal and external symmetries was forbidden by the no-go theorem of *Coleman-Mandula*. However, this doesn't apply to spinor operators.

We consider only the case of a single set of generators $Q$, $Q^\dagger$, what is also called $N = 1$ supersymmetry. $N = 2$ or $N = 4$ theories are mathematically interesting, but phenomenologically not relevant in four space–time dimensions. One would need extra dimensions to get chiral fermions or parity violation.

### 3.1.2 Representations

A supersymmetric theory must consist of states which are irreducible representations of the SUSY algebra. These states are called ”supermultiplets”. The properties of supermultiplets are:

- Each supermultiplet consists of both fermionic and bosonic states. Those are called ”superpartners”
- If $|\Omega\rangle$ and $|\Omega'\rangle$ are members of the same supermultiplet, then $|\Omega'\rangle$ is proportional to some combination of $Q$ and $Q^\dagger$ operators acting on $|\Omega\rangle$ (up to space–time translation or rotation)
- particles within the same supermultiplet must have equal eigenvalues of $P^2$, i.e. equal masses
- particles within the same supermultiplet must sit in the same representation of the gauge groups
- Each supermultiplet contains an equal number of fermionic and bosonic degrees of freedom

$$n_B = n_F \tag{3.5}$$

We are mainly interested in the following two kinds of supermultiplets:

**a) Chiral supermultiplet**: the simplest possibility for a supermultiplet consistent with eq. (3.5) has a single Weyl fermion (with two spin helicity states, so $n_F = 2$) and two real scalars (each with $n_B = 1$). It is convenient to arrange the real scalars as one complex field.

**b) Vector supermultiplet**: the simplest possibibility of a supermultiplet containing gauge fields contains a spin-1 vector boson. We are only interested in renormalizable gauge theories, i.e. the vector boson must be massless (before spontaneous symmetry breaking) and has therefore two degrees of freedom: $n_B = 2$. Its superpartner is therefore a massless spin-1/2 Weyl fermion, again with two helicity states, so $n_F = 2$.

If we include gravity, then the spin-2 graviton (with 2 helicity states, so $n_B = 2$) has a spin-3/2 superpartner called the gravitino. The gravitino would be massless if supersymmetry were unbroken, and so it has $n_F = 2$ helicity states.

One can check that other possible combinations of particles which satisfy $n_B = n_F$ are always reducible. *For example*: If a gauge symmetry could be broken without SUSY breaking then a massless vector supermultiplet would “eat” a chiral supermultiplet. The degrees of freedom of the massive vector supermultiplet are:

$$\begin{aligned} \text{massive vector :} &\quad n_B = 3 \\ \text{massive Dirac fermion :} &\quad n_F = 4 \\ \text{a real scalar :} &\quad n_B = 1 \end{aligned}$$

## 3.2 SUSY Lagrangian

*Based on Steve Martin's primer, sec. 3*

### 3.2.1 A free chiral supermultiplet

We have already seen that the easiest supersymmetric object is a chiral supermultiplet with a single left-handed two-component Weyl fermion $\psi$ and a complex scalar $\phi$. We forget for the moment about all possible interaction or mass terms. Under this assumption, the action of a single supermultiplet can be written in terms of its component fields as:

$$S = \int d^4x \, (\mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{fermion}}) \,, \tag{3.6}$$

with

$$\mathcal{L}_{\text{scalar}} = \partial^\mu \phi^* \partial_\mu \phi \tag{3.7}$$

$$\mathcal{L}_{\text{fermion}} = i\psi^\dagger \overline{\sigma}^\mu \partial_\mu \psi. \tag{3.8}$$

where $\phi$ and $\psi$ are superpartners. This is called the massless, non-interacting *Wess-Zumino model*.

#### 3.2.1.1 SUSY invariance

A SUSY transformation should turn the scalar boson field $\phi$ into something involving the fermion field $\psi_\alpha$. The simplest possibility is

$$\delta\phi = \epsilon\psi, \qquad \delta\phi^* = \epsilon^\dagger \psi^\dagger, \tag{3.9}$$

where $\epsilon^\alpha$ parameterizes the supersymmetry transformation. $\epsilon^\alpha$ is an infinitesimal, anti-commuting, two-component Weyl fermion which we assume for now to be constant, i.e.

$$\partial_\mu \epsilon^\alpha = 0 \tag{3.10}$$

The mass dimension is

$$[\epsilon] = [\phi] - [\psi] = 1 - \frac{3}{2} = -\frac{1}{2} \tag{3.11}$$

Applying the transformation, we find that the scalar part of the Lagrangian transforms as

$$\delta\mathcal{L}_{\text{scalar}} = \epsilon \partial^\mu \psi \, \partial_\mu \phi^* + \epsilon^\dagger \partial^\mu \psi^\dagger \, \partial_\mu \phi. \tag{3.12}$$

This must be canceled by $\delta\mathcal{L}_{\text{fermion}}$ (up to a total derivative). We can guess now how the transformation of the fermion must look like. There is only one chance (up to overall constants) that a cancellation can happen, namely

$$\delta\psi_\alpha = -i(\sigma^\mu \epsilon^\dagger)_\alpha \, \partial_\mu \phi, \qquad \delta\psi^\dagger_{\dot\alpha} = i(\epsilon\sigma^\mu)_{\dot\alpha} \, \partial_\mu \phi^*. \tag{3.13}$$

With this guess, one immediately obtains

$$\begin{aligned}
\delta\mathcal{L}_{\text{fermion}} &= i(\delta\psi^\dagger)\overline{\sigma}^\mu \partial_\mu \psi + i\psi^\dagger \overline{\sigma}^\mu \partial_\mu (\delta\psi) && (3.14)\\
&= i(i\epsilon\sigma^\mu \partial_\mu \phi^*)\overline{\sigma}^\nu \partial_\nu \psi + i\psi^\dagger \overline{\sigma}^\nu \partial_\nu (-i\sigma^\mu \epsilon^\dagger \partial_\mu \phi) && (3.15)\\
&= -\epsilon\sigma^\mu \overline{\sigma}^\nu \partial_\nu \psi \, \partial_\mu \phi^* + \psi^\dagger \overline{\sigma}^\nu \sigma^\mu \epsilon^\dagger \, \partial_\mu \partial_\nu \phi && (3.16)
\end{aligned}$$

This can be simplified by employing the Pauli matrix identities

$$\left[\sigma^\mu \overline{\sigma}^\nu + \sigma^\nu \overline{\sigma}^\mu\right]_\alpha{}^\beta = 2g^{\mu\nu} \delta_\alpha^\beta \tag{3.17}$$

$$\left[\overline{\sigma}^\mu \sigma^\nu + \overline{\sigma}^\nu \sigma^\mu\right]^{\dot\beta}{}_{\dot\alpha} = 2g^{\mu\nu} \delta_{\dot\alpha}^{\dot\beta} \tag{3.18}$$

as follows:

$$\psi^\dagger\overline{\sigma}^\nu\sigma^\mu\epsilon^\dagger\,\partial_\mu\partial_\nu\phi = \frac{1}{2}\psi^\dagger\overline{\sigma}^\nu\sigma^\mu\epsilon^\dagger\,\partial_\mu\partial_\nu\phi + \frac{1}{2}\psi^\dagger\overline{\sigma}^\nu\sigma^\mu\epsilon^\dagger\,\partial_\nu\partial_\mu\phi \tag{3.19}$$

$$= \frac{1}{2}\psi^\dagger\overline{\sigma}^\nu\sigma^\mu\epsilon^\dagger\,\partial_\mu\partial_\nu\phi + \frac{1}{2}\psi^\dagger\overline{\sigma}^\mu\sigma^\nu\epsilon^\dagger\,\partial_\mu\partial_\nu\phi \tag{3.20}$$

$$= \frac{1}{2}\psi^\dagger[\overline{\sigma}^\nu\sigma^\mu + \overline{\sigma}^\mu\sigma^\nu]\epsilon^\dagger\,\partial_\mu\partial_\nu\phi \tag{3.21}$$

$$= \psi^\dagger\epsilon^\dagger\partial^\mu\partial_\mu\phi \tag{3.22}$$

$$= \partial_\mu(\psi^\dagger\epsilon^\dagger\partial^\mu\phi) - (\partial_\mu\psi^\dagger)(\partial^\mu\phi) \tag{3.23}$$

$$\epsilon\sigma^\mu\overline{\sigma}^\nu\partial_\nu\psi\,\partial_\mu\phi^* = \epsilon(2g^{\mu\nu} - \sigma^\nu\overline{\sigma}^\mu)\partial_\nu\psi\partial_\mu\phi^* \tag{3.24}$$

$$= 2\epsilon\partial^\mu\psi\partial_\mu\phi^* - \epsilon\sigma^\nu\overline{\sigma}^\mu\partial_\nu\psi\partial_\mu\phi^* \tag{3.25}$$

$$= 2\epsilon\partial^\mu\psi\partial_\mu\phi^* - \partial_\nu(\epsilon\sigma^\nu\overline{\sigma}^\mu\psi\partial_\mu\phi^*) + \epsilon\sigma^\nu\overline{\sigma}^\mu\psi\partial_\nu\partial_\mu\phi^*) \tag{3.26}$$

$$= 2\epsilon\partial^\mu\psi\partial_\mu\phi^* - \partial_\nu(\epsilon\sigma^\nu\overline{\sigma}^\mu\psi\partial_\mu\phi^*) + \epsilon\psi\partial^\mu\partial_\mu\phi^* \tag{3.27}$$

$$= 2\epsilon\partial^\mu\psi\partial_\mu\phi^* - \partial_\nu(\epsilon\sigma^\nu\overline{\sigma}^\mu\psi\partial_\mu\phi^*) + \partial^\mu(\epsilon\psi\partial_\mu\phi^*) - \epsilon(\partial^\mu\psi)\partial_\mu\phi^*) \tag{3.28}$$

$$= \epsilon\partial^\mu\psi\partial_\mu\phi^* - \partial_\mu(\epsilon\sigma^\mu\overline{\sigma}^\nu\psi\partial_\nu\phi^* - \epsilon\psi\partial^\mu\phi^*) \tag{3.29}$$

and we get

$$\delta\mathcal{L}_{\text{fermion}} = -\epsilon\partial^\mu\psi\,\partial_\mu\phi^* - \epsilon^\dagger\partial^\mu\psi^\dagger\,\partial_\mu\phi - \partial_\mu\left(\epsilon\sigma^\mu\overline{\sigma}^\nu\psi\,\partial_\nu\phi^* - \epsilon\psi\,\partial^\mu\phi^* - \epsilon^\dagger\psi^\dagger\,\partial^\mu\phi\right). \tag{3.30}$$

The first two terms here just cancel against $\delta\mathcal{L}_{\text{scalar}}$, while the remaining contribution is a total derivative. So we arrive at

$$\delta S = \int d^4x\;\;(\delta\mathcal{L}_{\text{scalar}} + \delta\mathcal{L}_{\text{fermion}}) = 0, \tag{3.31}$$

justifying our guess of the numerical multiplicative factor made in eq. (3.13).

#### 3.2.1.2 Closure of the SUSY algebra

We have shown so far that the Wess–Zumino Lagrangian is invariant under a SUSY transformation. However, me must also show that the SUSY algebra closes: the commutator of two SUSY transformations is another symmetry of the theory.

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\phi = (\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})\phi \tag{3.32}$$

$$= \delta_{\epsilon_2}(\delta_{\epsilon_1}\phi) - \delta_{\epsilon_1}(\delta_{\epsilon_2}\phi) \tag{3.33}$$

$$= \delta_{\epsilon_2}(\epsilon_1\psi) - \delta_{\epsilon_1}(\epsilon_\psi\phi) \tag{3.34}$$

$$= i(-\epsilon_1\sigma^\mu\epsilon_2^\dagger + \epsilon_2\sigma^\mu\epsilon_1^\dagger)\,\partial_\mu\phi. \tag{3.35}$$

Here, we used $\delta\phi = \epsilon\psi$ and $\delta\psi_\alpha = -i(\sigma^\mu\epsilon^\dagger)_\alpha\,\partial_\mu\phi$.
We have found that the commutator of two supersymmetry transformations gives us back the derivative of the original field. In the Heisenberg picture of quantum mechanics $i\partial_\mu$ corresponds to the generator of spacetime translations $P_\mu$. Thus, this result agrees with our expectations from the SUSY algebra.

We must repeat the exercise for the fermionic case.

$$[\delta_{\epsilon_1},\delta_{\epsilon_2}]\psi_\alpha = (\delta_{\epsilon_2}\delta_{\epsilon_1}-\delta_{\epsilon_1}\delta_{\epsilon_2})\psi_\alpha \tag{3.36}$$

$$= \delta_{\epsilon_2}(-i(\sigma^\mu\epsilon_1^\dagger)_\alpha\,\partial_\mu\phi) - \delta_{\epsilon_1}(-i(\sigma^\mu\epsilon_2^\dagger)_\alpha\,\partial_\mu\phi)) \tag{3.37}$$

$$= -i\underbrace{(\sigma^\mu\epsilon_1^\dagger)_\alpha}_{\chi_\alpha}\;\underbrace{\epsilon_2}_{\xi}\;\underbrace{\partial_\mu\psi}_{\eta} + i(\sigma^\mu\epsilon_2^\dagger)_\alpha\,\epsilon_1\partial_\mu\psi \quad \text{using} \quad \chi_\alpha\,(\xi\eta) = -\xi_\alpha\,(\eta\chi) - \eta_\alpha\,(\chi\xi) \tag{3.38}$$

$$= i\left[\epsilon_{2\alpha}((\partial_\mu\psi)(\sigma^\mu\epsilon_1^\dagger)) + (\partial_\mu\psi)_\alpha((\sigma^\mu\epsilon_1^\dagger)\epsilon_2)) - (\epsilon_1\leftrightarrow\epsilon_2)\right] \tag{3.39}$$

Using the identity,

$$(\chi^\dagger\overline{\sigma}^\mu\xi)^* \;=\; \xi^\dagger\overline{\sigma}^\mu\chi \;=\; -\chi\sigma^\mu\xi^\dagger \;=\; -(\xi\sigma^\mu\chi^\dagger)^* \tag{3.40}$$

this becomes

$$[\delta_{\epsilon_1},\delta_{\epsilon_2}]\psi_\alpha = i(-\epsilon_1\sigma^\mu\epsilon_2^\dagger + \epsilon_2\sigma^\mu\epsilon_1^\dagger)\,\partial_\mu\psi_\alpha + i\epsilon_{1\alpha}\,\epsilon_2^\dagger\overline{\sigma}^\mu\partial_\mu\psi - i\epsilon_{2\alpha}\,\epsilon_1^\dagger\overline{\sigma}^\mu\partial_\mu\psi \tag{3.41}$$

Thus, if we apply the Dirac equation

$$\overline{\sigma}^\mu\partial_\mu\psi = 0 \tag{3.42}$$

we find

$$[\delta_{\epsilon_1},\delta_{\epsilon_2}]\psi_\alpha = i(-\epsilon_1\sigma^\mu\epsilon_2^\dagger + \epsilon_2\sigma^\mu\epsilon_1^\dagger)\,\partial_\mu\psi_\alpha \tag{3.43}$$

which is very similar to the scalar case.

We found so far that the SUSY algebra closes **only on-shell**. In order to consider the off-shell case, we play a trick and introduce another ingredient, so called *auxiliary fields* $F$. $F$ are complex scalar fields which don't propagate. Their Lagrangian is just

$$\mathcal{L}_{\text{auxiliary}} = F^*F\,. \tag{3.44}$$

Note, the mass dimension of $F$ is 2. One can easily check that the equation of motion from $\mathcal{L}_{\text{auxiliary}}$ is

$$F = F^* = 0 \tag{3.45}$$

We impose the following property of $F$ under a SUSY transformation:

$$\delta F = \quad -i\epsilon^\dagger\overline{\sigma}^\mu\partial_\mu\psi \tag{3.46}$$

$$\delta F^* = \quad i\partial_\mu\psi^\dagger\overline{\sigma}^\mu\epsilon. \tag{3.47}$$

Now the auxiliary part of the Lagrangian density transforms as

$$\delta\mathcal{L}_{\text{auxiliary}} = -i\epsilon^\dagger\overline{\sigma}^\mu\partial_\mu\psi\,F^* + i\partial_\mu\psi^\dagger\overline{\sigma}^\mu\epsilon\,F, \tag{3.48}$$

which vanishes on-shell, but not for arbitrary off-shell field configurations. We also modify the transformation properties of our fermions:

$$\delta\psi_\alpha = \quad -i(\sigma^\mu\epsilon^\dagger)_\alpha\,\partial_\mu\phi + \epsilon_\alpha F, \tag{3.49}$$

$$\delta\psi^\dagger_{\dot\alpha} = \quad i(\epsilon\sigma^\mu)_{\dot\alpha}\,\partial_\mu\phi^* + \epsilon^\dagger_{\dot\alpha}F^*, \tag{3.50}$$

One can check that the additional contribution to $\delta\mathcal{L}_{\text{fermion}}$ cancels the ones from $\delta\mathcal{L}_{\text{auxiliary}}$, up to a total derivative term. Thus

$$\delta\mathcal{L} = \quad \delta\mathcal{L}_{\text{scalar}} + \delta\mathcal{L}_{\text{fermion}} + \delta\mathcal{L}_{\text{auxiliary}} = 0 \tag{3.51}$$

If we now repeat the calculation from before, one finds for all fields $X = \phi, \phi^*, \psi, \psi^\dagger, F, F^*$

$$(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})X \quad = \quad i(-\epsilon_1\sigma^\mu\epsilon_2^\dagger + \epsilon_2\sigma^\mu\epsilon_1^\dagger)\,\partial_\mu X \tag{3.52}$$

also without applying the equations of motion. So, we found that the SUSY algebra closes all off-shell once we include the auxiliary fields.

**What is the interpreation of all that?** On-shell, the complex scalar field $\phi$ has two real propagating degrees of freedom, matching the two spin polarization states of $\psi$. Off-shell, however, the Weyl fermion $\psi$ is a complex two-component object, so it has four real degrees of freedom. (Going on-shell eliminates half of the propagating degrees of freedom for $\psi$, because the Lagrangian is linear in time derivatives, so that the canonical momenta can be re-expressed in terms of the configuration variables without time derivatives and are not independent phase space coordinates.) To make the numbers of bosonic and fermionic degrees of freedom match off-shell as well as on-shell, we had to introduce two more real scalar degrees of freedom in the complex field $F$, which are eliminated when one goes on-shell. This counting is by

| | $\phi$ | $\psi$ | $F$ |
|---|---|---|---|
| on-shell ($n_B = n_F = 2$) | 2 | 2 | 0 |
| off-shell ($n_B = n_F = 4$) | 2 | 4 | 2 |

We can summarize the main outcame as follows:

A **chiral superfield** consists of

- A complex scalar $\phi$
- A Weyl fermion $\psi$
- An auxiliary field $F$

and the free Lagrangian is given by

$$\mathcal{L}_{\text{free chiral}} = \partial^\mu\phi^*\partial_\mu\phi + i\psi^\dagger\overline{\sigma}^\mu\partial_\mu\psi + F^*F \tag{3.53}$$

### 3.2.2 Interactions of chiral supermultiplets

We go now one step further and consider several chiral supermultiplets which can interact among each other. We won't introduce gauge interactions, yet.

We start with the Lagrangina density of several free chiral supermultiplets labelled by an index $i$. We can easily generalise the result for one fields by writing

$$\mathcal{L}_{\text{free}} \quad = \quad \partial^\mu\phi^{*i}\partial_\mu\phi_i + i\psi^{\dagger i}\overline{\sigma}^\mu\partial_\mu\psi_i + F^{*i}F_i, \tag{3.54}$$

where we sum over repeated indices $i$. This Lagrangian is invariant under the individual supersymmetry transformation

$$\delta\phi_i = \epsilon\psi_i, \qquad \delta\phi^{*i} = \epsilon^\dagger\psi^{\dagger i}, \tag{3.55}$$

$$\delta(\psi_i)_\alpha = -i(\sigma^\mu\epsilon^\dagger)_\alpha\,\partial_\mu\phi_i + \epsilon_\alpha F_i, \qquad \delta(\psi^{\dagger i})_{\dot\alpha} = i(\epsilon\sigma^\mu)_{\dot\alpha}\,\partial_\mu\phi^{*i} + \epsilon^\dagger_{\dot\alpha}F^{*i}, \tag{3.56}$$

$$\delta F_i = -i\epsilon^\dagger\overline{\sigma}^\mu\partial_\mu\psi_i, \qquad \delta F^{*i} = i\partial_\mu\psi^{\dagger i}\overline{\sigma}^\mu\epsilon\,. \tag{3.57}$$

We want to find the most general set of renormalizable interactions that respects SUSY invariance. We start by writing down:

$$\mathcal{L}_{\text{int}} = \left(-\frac{1}{2}W^{ij}\psi_i\psi_j + W^iF_i + x^{ij}F_iF_j\right) + \text{c.c.} - U, \tag{3.58}$$

where the different coefficients are polynomials in the scalar fields $\phi_i, \phi^{*i}$ of the schematic form:

- $W^{ij} \sim \phi_a$
- $W^i \sim \phi_a\phi_b$
- $x^{ij} \sim \text{const}$
- $U \sim \phi_a\phi_b\phi_c\phi_d$

We must now require that $\mathcal{L}_{\text{int}}$ is invariant under the supersymmetry transformations, since $\mathcal{L}_{\text{free}}$ was already invariant by itself. The very schematic transformation properties of the different terms are

- $\delta(W^{ij}\psi_i\psi_j) \sim (\epsilon\psi)\psi^2 + \phi(\epsilon(\partial_\mu\phi + F))\psi$
- $\delta(W^iF_i) \sim (\epsilon\psi)\phi F + \phi^2(\epsilon\partial_\mu\psi)$
- $\delta U \sim (\epsilon\psi)\phi^3$
- $\delta x^{ij}F_iF_j \sim (\epsilon\partial\psi)F$

There is no possibility that to cancel the terms arising from $U$ and $x^{ij}$ against something else. So, we are left with

$$\mathcal{L}_{\text{int}} = \left(-\frac{1}{2}W^{ij}\psi_i\psi_j + W^iF_i\right) + \text{c.c.} \tag{3.59}$$

At this point, we are not assuming that $W^{ij}$ and $W^i$ are related, but we will see that thye are. From

$$\xi\chi \equiv \xi^\alpha\chi_\alpha = \xi^\alpha\epsilon_{\alpha\beta}\chi^\beta = -\chi^\beta\epsilon_{\alpha\beta}\xi^\alpha = \chi^\beta\epsilon_{\beta\alpha}\xi^\alpha = \chi^\beta\xi_\beta \equiv \chi\xi \tag{3.60}$$

we see that $W^{ij}$ is symmetric under $i \leftrightarrow j$.

We want to find the most general form which $W^{ij}$ and $W^i$ can have which is in agreement with the SUSY transformations. For this purpose we can check different pieces which must cancel separately.

**a)** We start with the part that contains four spinors.

$$\delta\mathcal{L}_{\text{int}}|_{4-\text{spinor}} = \left[-\frac{1}{2}\frac{\delta W^{ij}}{\delta\phi_k}(\epsilon\psi_k)(\psi_i\psi_j) - \frac{1}{2}\frac{\delta W^{ij}}{\delta\phi^{*k}}(\epsilon^\dagger\psi^{\dagger k})(\psi_i\psi_j)\right] + \text{c.c.} \tag{3.61}$$

The term proportional to $(\epsilon\psi_k)(\psi_i\psi_j)$ cannot cancel against any other term. However, the Fierz identity

$$\chi_\alpha\,(\xi\eta) = -\xi_\alpha\,(\eta\chi) - \eta_\alpha\,(\chi\xi) \tag{3.62}$$

implies

$$(\epsilon\psi_i)(\psi_j\psi_k) + (\epsilon\psi_j)(\psi_k\psi_i) + (\epsilon\psi_k)(\psi_i\psi_j) = 0, \tag{3.63}$$

Thus, in order to get $\delta\mathcal{L}_{\text{int}} = 0$, the term $\delta W^{ij}/\delta\phi_k$ must be totally symmetric under interchange of $i, j, k$. Consequently, $W$ can only involve $\phi$ but not $\phi^*$, **i.e. $W^{ij}$ is a holomorphic function of the complex fields $\phi$.**

Combining what we have learned so far, we can write

$$W^{ij} = M^{ij} + y^{ijk}\phi_k \tag{3.64}$$

Because of this form, we can write $W^{ij}$ as

$$W^{ij} = \frac{\delta^2}{\delta\phi_i\delta\phi_j}W \tag{3.65}$$

where we have introduced a useful object

$$W = \frac{1}{2}M^{ij}\phi_i\phi_j + \frac{1}{6}y^{ijk}\phi_i\phi_j\phi_k, \tag{3.66}$$

called the *superpotential*.

**b)** We turn to the parts of $\delta\mathcal{L}_{\text{int}}$ that contain a spacetime derivative:

$$\delta\mathcal{L}_{\text{int}}|_\partial \;=\; \left(iW^{ij}\partial_\mu\phi_j\,\psi_i\sigma^\mu\epsilon^\dagger + iW^i\,\partial_\mu\psi_i\sigma^\mu\epsilon^\dagger\right) + \text{c.c.} \tag{3.67}$$

Here we have used the again the identity

$$(\chi^\dagger\overline{\sigma}^\mu\xi)^* \;=\; \xi^\dagger\overline{\sigma}^\mu\chi \;=\; -\chi\sigma^\mu\xi^\dagger \;=\; -(\xi\sigma^\mu\chi^\dagger)^* \tag{3.68}$$

on the second term, which came from $(\delta F_i)W^i$. Now we can use eq. (3.65) to observe that

$$W^{ij}\partial_\mu\phi_j = \partial_\mu\left(\frac{\delta W}{\delta\phi_i}\right). \tag{3.69}$$

Therefore, eq. (3.67) will be a total derivative if

$$W^i = \frac{\delta W}{\delta\phi_i} = M^{ij}\phi_j + \frac{1}{2}y^{ijk}\phi_j\phi_k\;, \tag{3.70}$$

which explains why we chose its name as we did.

**c)** The remaining terms in $\delta\mathcal{L}_{\text{int}}$ are all linear in $F_i$ or $F^{*i}$. We can use the results for $W^i$ and $W^{ij}$ to check that these cancel as well:

$$\delta\mathcal{L}_{\text{int}}|_F = -\frac{1}{2}\left(M^{ij} + y^{ijk}\phi_k\right)((\epsilon F_i)\psi_j + \psi_i(\epsilon F_j)) + \left(M^{ij}(\epsilon\psi_j) + \frac{1}{2}y^{ijk}((\epsilon\psi_j)\phi_k + \phi_j(\epsilon\psi_k)\right) F_i \tag{3.71}$$

$$=0 \tag{3.72}$$

We have found that the most general non-gauge interactions for chiral supermultiplets are determined by a single holomorphic function of the complex scalar fields, the **superpotential** $W$. The general form of the superpotential in terms of scalar fields is

$$W(\phi) = L^i\phi_i + \frac{1}{2}M^{ij}\phi_i\phi_j + \frac{1}{6}y^{ijk}\phi_i\phi_j\phi_k \tag{3.73}$$

With

- $M^{ij}$ is a symmetric mass matrix for the fermion fields
- $y^{ijk}$ is a (Yukawa) coupling of a scalar $\phi_k$ and two fermions $\psi_i\psi_j$ that must be totally symmetric under interchange of $i, j, k$
- $L^i$ a linear (tadpole) term which is only possible for pure gauge singlets

The auxiliary fields $F_i$ and $F^{*i}$ can be eliminated using their classical equations of motion.

$$\mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}} = F_iF^{*i} + W^iF_i + W_i^*F^{*i} + \dots \tag{3.74}$$

where the dots represent all terms independent of $F$, $F^*$. The equations of motion are

$$F_i = -W_i^*, \qquad F^{*i} = -W^i\,. \tag{3.75}$$

Thus the auxiliary fields can be expressed in terms of the scalar fields. Therefore, the Lagrangian can be written as

$$\mathcal{L} = \partial^\mu\phi^{*i}\partial_\mu\phi_i + i\psi^{\dagger i}\bar{\sigma}^\mu\partial_\mu\psi_i - \frac{1}{2}\left(W^{ij}\psi_i\psi_j + W_{ij}^*\psi^{\dagger i}\psi^{\dagger j}\right) - W^iW_i^*. \tag{3.76}$$

The scalar potential of the theory *without gauge interactions and unbroken supersymmetry* is completely fixed by the superpotential:

$$V(\phi,\phi^*) = W^kW_k^* = F^{*k}F_k \tag{3.77}$$

$$= M_{ik}^*M^{kj}\phi^{*i}\phi_j + \frac{1}{2}M^{in}y_{jkn}^*\phi_i\phi^{*j}\phi^{*k} + \frac{1}{2}M_{in}^*y^{jkn}\phi^{*i}\phi_j\phi_k + \frac{1}{4}y^{ijn}y_{kln}^*\phi_i\phi_j\phi^{*k}\phi^{*l} \tag{3.78}$$

This part is also called the $F$-term potential which has the following properties:

- This $F$-term potential is automatically bounded from below and even non-negative
- The scalar masses are given by $M_{ik}^*M^{kj}$

- Th cubic and quartic scalar interactions are not free parameters but prop. to the Yukawa-like interactions

We have finally found the most general form of the full interacting Lagrangian stemming from chiral superfields:

$$\begin{aligned}\mathcal{L}_{\text{chiral}} &= \partial^\mu\phi^{*i}\partial_\mu\phi_i - V(\phi,\phi^*) + i\psi^{\dagger i}\overline{\sigma}^\mu\partial_\mu\psi_i - \frac{1}{2}M^{ij}\psi_i\psi_j - \frac{1}{2}M^*_{ij}\psi^{\dagger i}\psi^{\dagger j} \\ &\quad -\frac{1}{2}y^{ijk}\phi_i\psi_j\psi_k - \frac{1}{2}y^*_{ijk}\phi^{*i}\psi^{\dagger j}\psi^{\dagger k}. \end{aligned} \tag{3.79}$$

### 3.2.3 Lagrangians for vector supermultiplets

We want to include now gauge interactions. As we already mentioned, the gauge fields $A^a_\mu$ are part of vector supermultiplets. The other (propagating!) degrees of freedom are those of a a two-component Weyl fermion $\lambda^a$ which we will call 'gaugino'. The index $a$ here runs over the adjoint representation of the gauge group. The gauge transformations of the vector supermultiplet fields are

$$A^a_\mu \rightarrow A^a_\mu - \partial_\mu\Lambda^a + gf^{abc}A^b_\mu\Lambda^c, \tag{3.80}$$
$$\lambda^a \rightarrow \lambda^a + gf^{abc}\lambda^b\Lambda^c, \tag{3.81}$$

Before we start to check the SUSY properties, we count this time first the degrees of freedom in the on- and off-shell case:

- The on-shell degrees of freedom for $A^a_\mu$ and $\lambda^a_\alpha$ amount to two bosonic and two fermionic helicity states (for each $a$), as required by supersymmetry.
- Off-shell $\lambda^a_\alpha$ consists of two complex, or four real, fermionic degrees of freedom, while $A^a_\mu$ only has three real bosonic degrees of freedom one degree of freedom is removed by the inhomogeneous gauge transformation eq. (3.80).

We will see that we need one real bosonic auxiliary field $D^a$ to balance the degrees of freedom (and to close the SUSY algebra off-shell). The counting of degrees of freedom is summarized as

| | $A_\mu$ | $\lambda$ | $D$ |
|---|---|---|---|
| on-shell ($n_B = n_F = 2$) | 2 | 2 | 0 |
| off-shell ($n_B = n_F = 4$) | 3 | 4 | 1 |

The properties of the $D$ field are:

- $D$ transforms in the adjoint representation of the gauge group
- $(D^a)^* = D^a$ holds
- $D$ fields have mass dimension of 2 as $F$ fields
- $D$ fields don't propagate, i.e.their Lagrangian is

$$\mathcal{L}_{\text{auxiliary}} = \frac{1}{2}D^aD^a \tag{3.82}$$

Therefore, the Lagrangian density for the components of a vector supermultiplet are

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + i\lambda^{\dagger a}\overline{\sigma}^{\mu}D_{\mu}\lambda^a + \frac{1}{2}D^aD^a, \tag{3.83}$$

where

$$D_{\mu}\lambda^a = \partial_{\mu}\lambda^a - gf^{abc}A^b_{\mu}\lambda^c \tag{3.84}$$

is the covariant derivative of the gaugino field.

Of course, $\mathcal{L}_{\text{gauge}}$ must form a supersymmetric theory. That means that is must be invariant under SUSY transformations and that the SUSY algebra closes. One can guess how SUSY transformations might look like which fulfill these properties. They must have the following properties:

- they should be linear in the infinitesimal parameters $\epsilon, \epsilon^{\dagger}$ which have mass dimension $\frac{1}{2}$
- $\delta A^a_{\mu}$ is real
- $\delta D^a$ should be real and proportional to the field equations for the gaugino (in analogy with the role of the auxiliary field $F$)

Up to multiplicative constants, this results in

$$\delta A^a_{\mu} = -\frac{1}{\sqrt{2}}\left(\epsilon^{\dagger}\overline{\sigma}_{\mu}\lambda^a + \lambda^{\dagger a}\overline{\sigma}_{\mu}\epsilon\right), \tag{3.85}$$

$$\delta\lambda^a_{\alpha} = -\frac{i}{2\sqrt{2}}(\sigma^{\mu}\overline{\sigma}^{\nu}\epsilon)_{\alpha}\,F^a_{\mu\nu} + \frac{1}{\sqrt{2}}\epsilon_{\alpha}\,D^a, \tag{3.86}$$

$$\delta D^a = \frac{i}{\sqrt{2}}\left(-\epsilon^{\dagger}\overline{\sigma}^{\mu}D_{\mu}\lambda^a + D_{\mu}\lambda^{\dagger a}\overline{\sigma}^{\mu}\epsilon\right). \tag{3.87}$$

The factors of $\sqrt{2}$ are chosen so that the action obtained by integrating $\mathcal{L}_{\text{gauge}}$ is indeed invariant. After some (tedious) work, which we skip here, one finds that

- $\delta\mathcal{L}_{\text{gauge}} = 0$
- $(\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2})X = i(-\epsilon_1\sigma^{\mu}\epsilon_2^{\dagger} + \epsilon_2\sigma^{\mu}\epsilon_1^{\dagger})D_{\mu}X$ for $X = \{F^a_{\mu\nu}, \lambda^a, \lambda^{\dagger a}, D^a\}$

If we had not included the auxiliary field $D^a$, then the supersymmetry algebra would hold only after using the equations of motion for $\lambda^a$ and $\lambda^{\dagger a}$. The auxiliary fields satisfies a trivial equation of motion $D^a = 0$, but this is modified if one couples the gauge supermultiplets to chiral supermultiplets, as we now do.

### 3.2.4 Supersymmetric gauge interactions

The final step to obtain the full Lagrangian for a supersymmetric theory is to add gauge interactions between vector and chiral supermultiplets. As we already mentioned, supersymmetric and gauge transformations commute, i.e. the scalar, fermion, and auxiliary component of a chiral superfield is in the same representation of the gauge group, so

$$X_i \quad\rightarrow\quad X_i + ig\Lambda^a(T^aX)_i \tag{3.88}$$

for $X_i = \phi_i, \psi_i, F_i$. Exactly as for non-supersymmetric models, we obtain a supersymmetric gauge theory by replacing ordinary derivatives $\partial_\mu$, by covariant derivatives:

$$D_\mu \phi_i = \partial_\mu \phi_i + ig A^a_\mu (T^a \phi)_i \tag{3.89}$$
$$D_\mu \phi^{*i} = \partial_\mu \phi^{*i} - ig A^a_\mu (\phi^* T^a)^i \tag{3.90}$$
$$D_\mu \psi_i = \partial_\mu \psi_i + ig A^a_\mu (T^a \psi)_i. \tag{3.91}$$

In that way, we couple the vector bosons to the matter fields. Note, we have not yet checked that this replacement is in agreement with SUSY invariance! Moreover, the difference compared to non-supersymmetric models is that the vector superfields includes also gauginos and auxiliary fields. Thus, for full generality, we need to check if those can also couple to the components of the chiral superfield. If we restrict ourselves to renormalizable couplings, there are only three possibilities which one can write donwn

**a)** $(\phi^* T^a \psi)\lambda^a$

**b)** $\lambda^{\dagger a}(\psi^\dagger T^a \phi)$

**c)** $(\phi^* T^a \phi) D^a$

We must now check if these terms can – or even must – be included to obtain a supersymmetric theory. And if, what their overall coefficients are. To that end, we need to change our SUSY transformations as follows:

- Normal derivatives must be replaced by covariant derivatives
- $\delta F_i$ must include a new term involving gauginos

The full SUSY transformations for matter fields become:

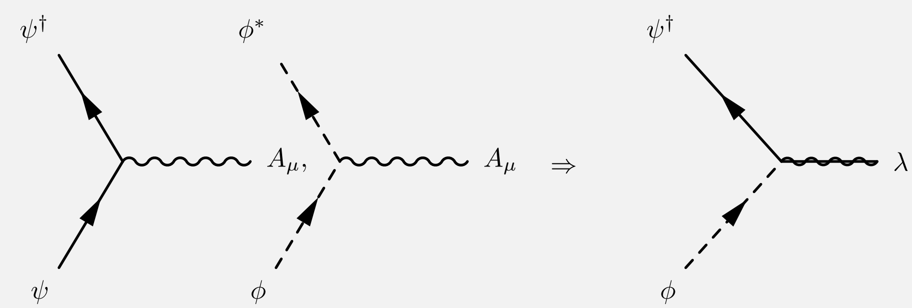
$$\delta \phi_i = \epsilon \psi_i \tag{3.92}$$
$$\delta \psi_{i\alpha} = -i(\sigma^\mu \epsilon^\dagger)_\alpha \, D_\mu \phi_i + \epsilon_\alpha F_i \tag{3.93}$$
$$\delta F_i = -i\epsilon^\dagger \overline{\sigma}^\mu D_\mu \psi_i \; + \; \sqrt{2} g (T^a \phi)_i \, \epsilon^\dagger \lambda^{\dagger a}. \tag{3.94}$$

which result in a supersymmetric theory if the additional terms in the Lagrangian are

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{\text{chiral}} + \mathcal{L}_{\text{gauge}} \\ &\quad -\sqrt{2} g (\phi^* T^a \psi)\lambda^a - \sqrt{2} g \lambda^{\dagger a}(\psi^\dagger T^a \phi) + g(\phi^* T^a \phi) D^a. \end{aligned} \tag{3.95}$$

There is actually a 'naive' explanation for the first two terms in the second line: one takes the usual interaction between a vector boson and two fermions and replaces two particles by their superpartners.

In a supersymmetric gauge theory, the supersymmetrized version of a coupling of a gauge boson to a pair of scalars or fermions becomes the interaction of a gaugino to a fermion/scalar which are superpartners:




$$\mathcal{L}_\lambda = -\sqrt{2}g(\phi^* T^a \psi)\lambda^a - \sqrt{2}g\lambda^{\dagger a}(\psi^\dagger T^a \phi) \tag{3.96}$$

With the last term in eq. (3.96), the Lagrangian for the $D$ fields becomes

$$\mathcal{L}_D = \frac{1}{2}D^a D^a + g(\phi^* T^a \phi)D^a \tag{3.97}$$

which results in the equation of motion

$$D^a = -g(\phi^* T^a \phi). \tag{3.98}$$

Thus, like the auxiliary fields $F_i$ and $F^{*i}$, the $D^a$ can be expressed by a pair of the scalar fields. Consequently, $D^a D^a$ corresponds to a $\phi^4$ term which is part of the scalar potential.

The full scalar potential of the theory is a sum of $D$- and $F$-term contributions

$$V(\phi, \phi^*) = F^{*i}F_i + \frac{1}{2}\sum_a D^a D^a = W_i^* W^i + \frac{1}{2}\sum_a g_a^2 (\phi^* T^a \phi)^2. \tag{3.99}$$

Here, we have explicitly written $\sum_a$ which is the sum over all gauge groups of the theory. In contrast to non-supersymmetric models, the scalar potential has no free parameters (quartic couplings) but is completely fixed by gauge and Yukawa interactions.

### 3.2.5 Superfields and superspace

All the results which we have derived so far could also be obtained using so called 'superfield methods'. This approach is mathematically more elegant but also more involved. Therefore, we give here only the basic idea.
The so called superspace extends the four space–time coordinates by four additional coordinates Points in superspace are labeled by coordinates:

$$x^\mu,\ \theta^\alpha,\ \theta^\dagger_{\dot\alpha}. \tag{3.100}$$

Here $\theta^\alpha$ and $\theta^\dagger_{\dot\alpha}$ are constant complex anti-commuting two-component spinors (Grassmann coordinates). Considering a single Grassmann variable $\eta$ with

$$\eta^2 = 0 \tag{3.101}$$

one can express any function $f(\eta)$ as

$$f(\eta) = f_0 + \eta f_1 \tag{3.102}$$

Integration and derivation with respect to Grassmann variables are defined as:

$$\frac{df}{d\eta} = f_1 \tag{3.103}$$

$$\left.\begin{array}{l}\int d\eta = 0\\ \int d\eta\eta = 1\end{array}\right\} \int d\eta f = f_1 \tag{3.104}$$

One can write a superfields as function of Grassmann coordinates:

$$\hat\Phi = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y), \tag{3.105}$$

The superpotential can be written in terms of superfields

$$W(\hat\Phi) = L_i\hat\Phi^i + \frac{1}{2}M_{ij}\hat\Phi^i\hat\Phi^j + \frac{1}{6}y_{ijk}\hat\Phi^i\hat\Phi^j\hat\Phi^k \tag{3.106}$$

from which the Lagrangian can be calculted as

$$\mathcal{L} = \int d^2\theta\theta(W(\hat\Phi) + \text{c.c.}) \tag{3.107}$$

We find that a product of three superfields becomes

$$\begin{aligned}\hat\Phi_i\Phi_j\Phi_k = \; & \phi_i\phi_j\phi_k + \sqrt{2}\theta(\psi_i\phi_j\phi_k + \psi_j\phi_i\phi_k + \psi_k\phi_i\phi_j)\\ & + \theta\theta(\phi_i\phi_jF_k + \phi_i\phi_kF_j + \phi_j\phi_kF_i - \psi_i\psi_j\phi_k - \psi_i\psi_k\phi_j - \psi_j\psi_k\phi_i)\end{aligned} \tag{3.108}$$

Thus,

$$\mathcal{L} = \int d^2\theta\theta\hat\Phi_i\Phi_j\Phi_k \tag{3.109}$$

$$= (\phi_i\phi_jF_k + \phi_i\phi_kF_j + \phi_j\phi_kF_i - \psi_i\psi_j\phi_k - \psi_i\psi_k\phi_j - \psi_j\psi_k\phi_i) \tag{3.110}$$

Where we recovered the Yukawa-like interactions ($\psi\psi\phi$) and $F$-terms ($F\phi\phi$).

In order to define a supersymmetric theory, often the superpotential in terms of superfields is given:

$$W(\hat{\Phi}) = L_i \hat{\Phi}^i + \frac{1}{2} M_{ij} \hat{\Phi}^i \hat{\Phi}^j + \frac{1}{6} y_{ijk} \hat{\Phi}^i \hat{\Phi}^j \hat{\Phi}^k \tag{3.111}$$

The obtained Lagrangian from

$$\mathcal{L} = \int d^2\theta\theta (W(\hat{\Phi}) + \text{c.c.}) \tag{3.112}$$

is identical to the one which one gets from

$$W(\phi) = L^i \phi_i + \frac{1}{2} M^{ij} \phi_i \phi_j + \frac{1}{6} y^{ijk} \phi_i \phi_j \phi_k \tag{3.113}$$

and

$$\mathcal{L} = \left( -\frac{1}{2} \frac{\delta^2 W}{\delta\phi_i \delta\phi_j} \psi_i \psi_j + \frac{\delta W}{\delta\phi_i} F_i \right) + \text{c.c.} \tag{3.114}$$

## 3.3 SUSY breaking

*Based on Steve Martin's primer, sec. 7*
We have so far assumed that SUSY is an unbroken symmmetry. In this case all components of the supermultiplets have the same masses. However,this would rule out the theory immediately because it predicts for instance a fundamental scalar with the same mass and charge as the electron:

$$m_{\tilde{e}} = m_e \tag{3.115}$$

Such a particle, called selectron, would have been discovered long ago. The current limits for the selectron mass are actually about 100 GeV. Therefore, one needs to introduce a mass splitting between the superpartners. In other words, SUSY must be broken. We want to discuss two different approaches for SUSY breaking:

**a)** spontaneous SUSY breaking

**b)** Hidden sector SUSY breaking

We will see that the first attempt to break SUSY similar to gauge theory spontaneously is phenomenological not possible. Nevertheless, we are going to discuss this case because it gives important insights.

### 3.3.1 Spontaneous SUSY breaking

#### 3.3.1.1 General considerations

We start with a discussion of spontaneous SUSY breaking. By definition, this means that the vacuum state $|0\rangle$ is not invariant under supersymmetry transformations, so $Q_\alpha|0\rangle \neq 0$ and $Q^\dagger_{\dot{\alpha}}|0\rangle \neq 0$. Now, in global supersymmetry, the Hamiltonian operator $H$ is related to the supersymmetry generators through the algebra

$$\{Q_\alpha, Q^\dagger_{\dot{\alpha}}\} = -2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \tag{3.116}$$

Note, here we have added the explicit form of the spinor indices. For $H = P^0$ we get

$$H = P^0 = \frac{1}{4}(Q_1 Q_1^\dagger + Q_1^\dagger Q_1 + Q_2 Q_2^\dagger + Q_2^\dagger Q_2). \tag{3.117}$$

We can distinguish two cases:

**a)** $H|0\rangle = 0$: SUSY is unbroken and the vacuum has zero energy

**b)** $H|0\rangle \neq 0$: SUSY is broken and the vacuum energy is

$$\langle 0|H|0\rangle = \frac{1}{4}\Big(\|Q_1^\dagger|0\rangle\|^2 + \|Q_1|0\rangle\|^2 + \|Q_2^\dagger|0\rangle\|^2 + \|Q_2|0\rangle\|^2\Big) > 0 \tag{3.118}$$

This is positive for a positive norm in the Hilbert space. From $\langle 0|H|0\rangle = \langle 0|V|0\rangle$ we get the condition

$$\langle 0|V_F|0\rangle + \langle 0|V_D|0\rangle > 0 \tag{3.119}$$

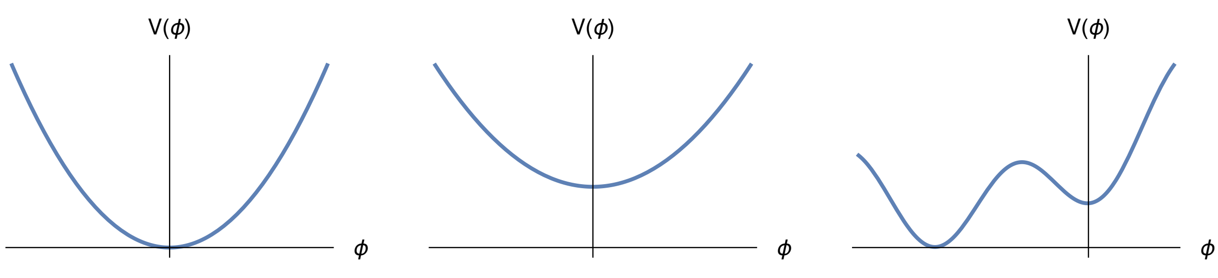
Where $V_F$ and $V_D$ are the $F$- and $D$-term potentials.

SUSY is spontaneously broken if and only if it is not possible to satisfy

$$F_i = 0 \qquad \vee \qquad D^a = 0 \tag{3.120}$$

for any field configuration.

If any state exists in which all $F_i$ and $D^a$ vanish, then it will have zero energy, implying that supersymmetry is not spontaneously broken in the true ground state. Aanother possibility is that the vacuum state in which we live is not the true ground state (which may preserve supersymmetry), but is instead a higher energy metastable supersymmetry-breaking state with lifetime at least of order the present age of the universe. Finite temperature effects can indeed cause the early universe to prefer the metastable supersymmetry-breaking local minimum of the potential over the supersymmetry-breaking global minimum. The potential for the three possibilities looks schematically like




If SUSY is broken spontaneously, a massless Nambu-Goldstone particle must be present. Since SUSY generators are spinors, this particle is a massless, neutral Weul fermion called the Goldstino. If we would consider *local* supersymmetry in which SUSY is combined with gravity, the Goldstino would get eaten up by the Gravitino. The Gravitino is a spin 3/2 particle and the superpartner of the Graviton. This is called *Super-Higgs mechanism*.

#### 3.3.1.2 Sum rule

We want to show an important relation between the masses of fermions, scalars and vector-bosons after spontaneous SUSY breaking. For this purpose, we consider the general mass matrices for these fields.

**a) Scalars** We are interested in the masses of the scalar after symmetry breaking. Those masses are the eigenvalues of the matrix $\mathbf{m}_{\mathrm{S}}^2$ which can be used to express the Lagrangian as

$$V = \frac{1}{2}\begin{pmatrix}\phi^{*j} & \phi_j\end{pmatrix}\mathbf{m}_{\mathrm{S}}^2\begin{pmatrix}\phi_i \\ \phi^{*i}\end{pmatrix}. \qquad (3.121)$$

The general scalar potential reads

$$V = W_k^* W^k + \frac{1}{2} g_a^2 (\phi_k^* T^a \phi_l)(\phi_m^* T^a \phi_n) \qquad (3.122)$$

From that, we get for $\mathbf{m}_{\mathrm{S}}^2$

$$\mathbf{m}_{\mathrm{S}}^2 = \begin{pmatrix} W_{jk}^* W^{ik} + g_a^2 (T^a\phi)_j(\phi^* T^a)^i - g_a T_j^{ai} D^a & W_{ijk}^* W^k + g_a^2 (T^a\phi)_i (T^a\phi)_j \\ W^{ijk} W_k^* + g_a^2 (\phi^* T^a)^i (\phi^* T^a)^j & W_{ik}^* W^{jk} + g_a^2 (T^a\phi)_i (\phi^* T^a)^j - g_a T_i^{aj} D^a \end{pmatrix} \qquad (3.123)$$

Here, we used $W^{ijk} = \delta^3 W / \delta\phi_i \delta\phi_j \delta\phi_k$, and we kept the scalar fields which are replaced by their VEVs. The sum of the two eigenvalues is just the trace of this matrix. This trace is calculated to

$$\mathrm{Tr}(\mathbf{m}_{\mathrm{S}}^2) = 2 W_{ik}^* W^{ik} + 2 g_a^2 C_a(i) \phi^{*i}\phi_i - 2 g_a \mathrm{Tr}(T^a) D^a, \qquad (3.124)$$

with the Casimir invariants $C_a(i)\delta_{ij} = (T^a T^a)_j^i$.

**b) Fermions**: bilinear fermion terms can appear after symmetry breaking in the terms coming from the superpotential as well as in gaugino-fermion-scalar interactions.

$$\mathcal{L} = -\sqrt{2} g_a \phi_i T^a \lambda^a \psi_i - W^{ik}\psi_i\psi_j + \text{c.c.} \qquad (3.125)$$

Therefore, the mass matrix $m_{\mathrm{F}}$ defined as

$$V = -\begin{pmatrix}\lambda_j^a & \psi_j\end{pmatrix}\mathbf{m}_{\mathrm{F}}\begin{pmatrix}\lambda_i^a \\ \psi_i\end{pmatrix}. \qquad (3.126)$$

is

$$\mathbf{m}_{\mathrm{F}} = \begin{pmatrix} 0 & \sqrt{2} g_a (T^a\phi)_i \\ \sqrt{2} g_a (T^a\phi)_j & W^{ij}\end{pmatrix} \qquad (3.127)$$

Thus, the mass matrix squared becomes

$$\mathbf{m}_{\mathrm{F}}^\dagger \mathbf{m}_{\mathrm{F}} = \begin{pmatrix} 2 g_a g_b (\phi^* T^a T^b \phi) & \sqrt{2} g_b (T^b\phi)_k W^{ik} \\ \sqrt{2} g_a (\phi^* T^a)^k W_{jk}^* & W_{jk}^* W^{ik} + 2 g_c^2 (T^c\phi)_j (\phi^* T^c)^i \end{pmatrix}, \qquad (3.128)$$

so the sum of the two-component fermion squared masses is

$$\mathrm{Tr}(\mathbf{m}_{\mathrm{F}}^\dagger \mathbf{m}_{\mathrm{F}}) = W_{ik}^* W^{ik} + 4 g_a^2 C_a(i) \phi^{*i}\phi_i. \qquad (3.129)$$

**c) Vectors**: mass terms from vector always come from the kinetic terms of scalars. The general form of the mass matrix is

$$\mathbf{m}_{\mathrm{V}}^2 \;=\; g_a^2(\phi^*\{T^a, T^b\}\phi), \tag{3.130}$$

so

$$\mathrm{Tr}(\mathbf{m}_{\mathrm{V}}^2) \;=\; 2g_a^2 C_a(i)\phi^{*i}\phi_i. \tag{3.131}$$

It follows that the *supertrace* of the tree-level squared-mass eigenvalues, defined in general by a weighted sum over all particles with spin $j$:

$$\mathrm{STr}(m^2) \;\equiv\; \sum_j (-1)^{2j}(2j+1)\mathrm{Tr}(m_j^2), \tag{3.132}$$

satisfies the sum rule

$$\mathrm{STr}(m^2) \;=\; \mathrm{Tr}(\mathbf{m}_{\mathrm{S}}^{\mathbf{2}}) - 2\mathrm{Tr}(\mathbf{m}_{\mathrm{F}}^\dagger \mathbf{m}_{\mathrm{F}}) + 3\mathrm{Tr}(\mathbf{m}_{\mathrm{V}}^{\mathbf{2}}) \;=\; -2g_a \mathrm{Tr}(T^a)D^a \;=\; 0. \tag{3.133}$$

The last equality assumes that the traces of the $U(1)$ charges over the chiral superfields are 0. This holds for any non-anomalous gauge symmetry.

The sum rules are a handy tool to check ...

- ...if SUSY is broken spontaneously
- ...the calculated masses for a supersymmetric model which should fulfill this rule when taking the limit of unbroken supersymmetry

#### 3.3.1.3 Example: $F$-term supersymmetry breaking

We want to discuss an explicit example of $F$-term SUSY breaking. These models are also called *O'Raifeartaigh models*. The basic idea is to find a set of chiral supermultiplets $\Phi_i \supset (\phi_i, \psi_i, F_i)$ and a superpotential $W$ in such a way that the equations $F_i = -\delta W^*/\delta\phi^{*i} = 0$ have no simultaneous solution. The simplest example for this has three chiral supermultiplets $\Phi_{1,2,3}$ and the superpotential

$$W_{\mathrm{O'R}} = -k\Phi_1 + m\Phi_2\Phi_3 + \frac{y}{2}\Phi_1\Phi_3^2. \tag{3.134}$$

Note, that the linear term $k$ is crucial. Otherwise, $\phi_i = 0$ will also correspond to a supersymetric conserving vacuum. Without loss of generality, we can choose $k$, $m$, and $y$ to be real and positive: The scalar potential following from $W_{\mathrm{O'R}}$ is

$$V_{\mathrm{tree-level}} = |F_1|^2 + |F_2|^2 + |F_3|^2 \tag{3.135}$$

with

$$F_1 = k - \frac{y}{2}\phi_3^{*2} \tag{3.136}$$

$$F_2 = -m\phi_3^* \tag{3.137}$$

$$F_3 = -m\phi_2^* - y\phi_1^*\phi_3^*. \tag{3.138}$$

Obviously, it is ont possible to get $F_1 = 0$ and $F_2 = 0$ at the same time, ie SUSY must be broken.
We assume from now on $m^2 > yk$, then the absolute minimum of the classical potential is

$$V(\phi_1, \phi_2 = 0, \phi_3 = 0) = k^2 \tag{3.139}$$

$\phi_1$, which doesn't lift the vacuum, is called a 'flat direction'.
We can now check the masses for the different fields. For this purpose, we parametrize the complex scalars as real fields

$$\phi_1 = \frac{1}{\sqrt{2}}(v_1 + \varphi_1 + i\sigma_1) \tag{3.140}$$

$$\phi_2 = \frac{1}{\sqrt{2}}(\varphi_2 + i\sigma_2) \tag{3.141}$$

$$\phi_3 = \frac{1}{\sqrt{2}}(\varphi_3 + i\sigma_3) \tag{3.142}$$

Note the VEV for $p_1$ which is responsible for SUSY breaking. The mass matrix for the scalars in the basis $(\varphi_1, \sigma_1, \varphi_2, \sigma_2, \varphi_3, \sigma_3)$ is

$$m_S^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m^2 & 0 & \frac{mv_1y}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & m^2 & 0 & \frac{mv_1y}{\sqrt{2}} \\ 0 & 0 & \frac{mv_1y}{\sqrt{2}} & 0 & m^2 + \frac{v_1^2y^2}{2} + ky & 0 \\ 0 & 0 & 0 & \frac{mv_1y}{\sqrt{2}} & 0 & m^2 + \frac{1}{2}y\left(v_1^2y - 2k\right) \end{pmatrix} \tag{3.143}$$

and the eigenvalues of this matrix are

$$m_{\varphi,1}^2 = 0 \tag{3.144}$$

$$m_{\varphi,2}^2 = 0 \tag{3.145}$$

$$m_{\varphi,3}^2 = \frac{1}{4}\left(y\left(-\sqrt{\left(v_1^2y - 2k\right)^2 + 8m^2v_1^2} - 2k + v_1^2y\right) + 4m^2\right) \tag{3.146}$$

$$m_{\varphi,4}^2 = \frac{1}{4}\left(y\left(\sqrt{\left(v_1^2y - 2k\right)^2 + 8m^2v_1^2} - 2k + v_1^2y\right) + 4m^2\right) \tag{3.147}$$

$$m_{\varphi,5}^2 = \frac{1}{4}\left(y\left(-\sqrt{\left(2k + v_1^2y\right)^2 + 8m^2v_1^2} + 2k + v_1^2y\right) + 4m^2\right) \tag{3.148}$$

$$m_{\varphi,6}^2 = \frac{1}{4}\left(y\left(\sqrt{\left(2k + v_1^2y\right)^2 + 8m^2v_1^2} + 2k + v_1^2y\right) + 4m^2\right) \tag{3.149}$$

$$\tag{3.150}$$

and we find

$$\mathrm{Tr}(m_S^2) = 4m^2 + v_1^2y^2 \tag{3.151}$$

We turn now to the fermion sector. The Yukawa-like potential of this model is

$$\mathcal{L}_{\mathrm{FFS}} = m\psi_2\psi_3 + \frac{1}{2}\phi_1^*\psi_3^2y + y\phi_3^*\psi_1\psi_3 + \mathrm{h.c.} \tag{3.152}$$

The fermionic mass matrix in the basis $(\psi_1, \psi_2, \psi_3)$ is

$$m_F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & m \\ 0 & m & \frac{1}{\sqrt{2}} v_1 y \end{pmatrix} \tag{3.153}$$

from what the eigenvalues of $m_F m_F^\dagger$ are calculated to

$$m_{\psi,1}^2 = 0 \tag{3.154}$$

$$m_{\psi,2}^2 = -\frac{1}{4}\sqrt{2m^2 v_1^2 y^2 + v_1^4 y^4} + m^2 + \frac{1}{4} v_1^2 y^2 \tag{3.155}$$

$$m_{\psi,3}^2 = \frac{1}{4}\sqrt{2m^2 v_1^2 y^2 + v_1^4 y^4} + m^2 + \frac{1}{4} v_1^2 y^2 \tag{3.156}$$

and we have

$$\mathrm{Tr}(m_F^2) = 2m^2 + \frac{1}{2} v_1^2 y^2 \tag{3.157}$$

Thus, we can now verify that

$$\mathrm{STr}(m^2) = \mathrm{Tr}(m_S^2) - 2\mathrm{Tr}(m_F^2) = 0 \tag{3.158}$$

holds.

#### 3.3.1.4 Example: $D$-term supersymmetry breaking

It is in principle also possible to break SUSY via $D$-terms. This option is known as *Fayet-Iliopoulos mechanism*. If the gauge symmetry includes a $U(1)$ factor, then one can write down a term linear in the auxiliary field of the corresponding gauge supermultiplet,

$$\mathcal{L}_{\mathrm{FI}} = -\kappa D, \tag{3.159}$$

where $\kappa$ is a constant with dimensions of [mass]$^2$. This term is gauge-invariant and supersymmetric by itself. The relevant part of the scalar potential become

$$V = \kappa D - \frac{1}{2} D^2 - gD \sum_i q_i |\phi_i|^2. \tag{3.160}$$

Here the $q_i$ are the charges of the scalar fields $\phi_i$ under the $U(1)$ gauge group in question. The presence of the Fayet-Iliopoulos term modifies the equation of motion the $D$-field to

$$D = \kappa - g \sum_i q_i |\phi_i|^2. \tag{3.161}$$

Now suppose that the scalar fields $\phi_i$ that are charged under the $U(1)$ all have non-zero superpotential masses $m_i$. Then the potential will have the form

$$V = \sum_i |m_i|^2 |\phi_i|^2 + \frac{1}{2}(\kappa - g \sum_i q_i |\phi_i|^2)^2. \tag{3.162}$$

Since this cannot vanish, supersymmetry must be broken. This is also obvious from the masses:

$$m_{S,i}^2 = |m_i|^2 - g q_i \kappa \tag{3.163}$$

$$m_{F,i}^2 = \frac{1}{2}|m_i|^2 \tag{3.164}$$

The supertrace becomes

$$\mathrm{STr}(m^2) = -g\kappa \mathrm{Tr}(q_i) \tag{3.165}$$

which vanishes if the $U(1)$ is anomaly free.

However, one needs to state that there are some problems building a realistic model with $D$-term breaking

- In SUSY versions of the SM, it is not possible to use $U(1)_Y$ because many fields don't get superpotential mass terms. (We will see this explicitly in the next chapter)
- If another (additional) $U(1)$ is used, this group must not couple to SM particles. However, this makes it difficult to generate appropriate masses for all superpartners of SM fields.

#### 3.3.1.5 The problem of spontaneous SUSY breaking

The sum rules obtainted so far are relations between all masses in the theory. However, we could assume that some particles don't mix with the rest. In that case, individual sum rules are found for the sub-sets of fields that mix. A well motivated choice is to assume that the mixing of the (s)electron with other fields is negligble small or zero. This would predict the following relation between the two selectrons and the electron:

$$m_{\tilde{e}_1}^2 + m_{\tilde{e}_2}^2 = 2m_e^2, \tag{3.166}$$

mass. Even small deviations from lepton flavour violation won't change the conclusion that the sum rules rule out phenomenologically acceptable SUSY masses. Therefore, we need to search for possibilities how to circumvent the sum rules.

### 3.3.2 Soft supersymmetry breaking interactions

We have discussed so far the spontaneus SUSY breaking as origin of a mass splitting between scalars and fermions of the same multiplet. For pratical purposes, one can also choose another approach and ask the questions: which terms can I add to my Lagrangian in order to keep the most important SUSY properties? The guiding principle is that all terms which we add only introduce a **soft breaking** of SUSY. 'Soft' means, that no quadratic divergences appear. This means that only dimensionful parameters can be added [1]. The most important soft supersymmetry-breaking terms in the Lagrangian of a general theory are

$$\mathcal{L}_{\text{soft}} = -\left(\frac{1}{2}M_a\,\lambda^a\lambda^a + \frac{1}{6}t^{ijk}\phi_i\phi_j\phi_k + \frac{1}{2}b^{ij}\phi_i\phi_j + l^i\phi_i\right) + \text{c.c.} - (m^2)^i_j\phi^{j*}\phi_i, \tag{3.167}$$

These terms are

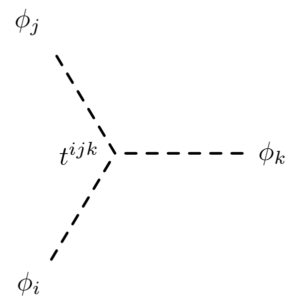
- gaugino masses $M_a$ for each vector superfield. Gaugino masses $M_a$ are always allowed by gauge symmetry.

[1] We will discuss this in more detail at the example of the MSSM

- scalar squared-mass terms $(m^2)_i^j$ for each chiral superfield. The $(m^2)_j^i$ terms are allowed for $i, j$ such that $\phi_i$, $\phi^{j*}$ transform in complex conjugate representations of each other under all gauge symmetries; in particular this is true of course when $i = j$, so every scalar is eligible to get a mass in this way if supersymmetry is broken.

- holomorphic soft-terms $l_i$, $b^{ij}$, $t^{ijk}$ of one to three scalars. The $t^{ijk}$, $b^{ij}$, and $l^i$ terms have the same form as the $y^{ijk}$, $M^{ij}$, and $L^i$ terms in the superpotential, so they will each be allowed by gauge invariance if and only if a corresponding superpotential term is allowed.

The $t$ terms are special in that sense that they modify scalar interactions, while all other terms enter the masses/mass matrices:




In addition, two other possibilities exists which, however, are not studied as intensively as these standard terms:

**a) Non-holomorphic soft term**:

$$\mathcal{L}_{\rm non-holomorphic} = c_i^{jk}\phi^{*i}\phi_j\phi_k + \tilde{\mu}^{ij}\psi_i\psi_j + \text{c.c.} \tag{3.168}$$

These terms can only be added if no singlet is involved because they are not soft otherwise.

**b) Dirac Gaugino mass terms**:

$$\mathcal{L} \,=\, -M^a_{\rm Dirac}\lambda^a\psi_a + \text{c.c.} \tag{3.169}$$

These terms need a chiral superfield in the adjoint representation

### 3.3.3 Hidden Sector SUSY breaking

One the one side, spontaneus SUSY breaking is not in ageement with observation, one the other side, putting soft-masses by hand is often unsatisfying.

If we consider the MSSM, there are 107 parameters appeing in the most general soft-breaking Lagrangian. Therefore, is very attractive to relate them to a SUSY breaking mechanism which reduces the number of free parameters and which explains why CP phases and/or dangerous flavour changing parameters are small.

A possible solution to both problems is that soft terms arise indirectly or radiatively, rather than from tree-level renormalizable couplings to the supersymmetry-breaking order parameters:

Supersymmetry breaking evidently occurs in a "hidden sector" of particles that have no (or only very small) direct couplings to the "visible sector" chiral supermultiplets of the MSSM. However, the two sectors do share some interactions that are responsible for mediating supersymmetry breaking from the hidden sector to the visible sector, resulting in the MSSM soft terms.

| hidden sector: SUSY-Breaking | interactions $\longleftrightarrow$ | visible sector: MSSM |
|---|---|---|

In this scenario, the tree-level squared mass sum rules need not hold, even approximately, for the physical masses of the visible sector fields, so that a phenomenologically viable superpartner mass spectrum is, in principle, achievable.
There are two main mechanims how SUSY breaking is mediated:

**a) Gravitational interactions**:
These scenarios are associated with the new physics, including gravity, that enters near the Planck scale. If supersymmetry is broken in the hidden sector by a VEV $\langle F\rangle$, then the soft terms in the visible sector should be roughly

$$m_{\text{soft}} \sim \langle F\rangle/M_{\text{P}}, \tag{3.170}$$

by dimensional analysis. The reason is

- we know that $m_{\text{soft}}$ must vanish in the limit $\langle F\rangle \to 0$ where supersymmetry is unbroken
- in the limit $M_{\text{P}} \to \infty$ (corresponding to $G_{\text{Newton}} \to 0$) in which gravity becomes irrelevant, $m_{\text{soft}}$ must also become zero

For $m_{\text{soft}}$ of order a few hundred GeV, one would therefore expect that the scale associated with the origin of supersymmetry breaking in the hidden sector should be roughly $\sqrt{\langle F\rangle} \sim 10^{10}$ or $10^{11}$ GeV.

A bit more explicitly, the soft Lagrangian can be expected to have the form

$$\mathcal{L}_{\text{soft}} = \left(-\frac{\langle F\rangle}{2M_{\text{P}}} f_a \lambda^a \lambda^a - \frac{\langle F\rangle}{M_{\text{P}}} n_i^j \phi_j W^i_{\text{MSSM}} + \cdots + \text{c.c.}\right) - \frac{\langle F\rangle^2}{M_{\text{P}}^2}(k^i_j + n^i_p \overline{n}^p_j)\phi^{*j}\phi_i, \tag{3.171}$$

with dimensionless parameters $f_a$, $n$, $\overline{n}$, $k$ which are given by the setup in the hidden sector. Under the radical assumption that the hidden sector is very simple and couple the same to all scalars and gauginos, all soft-terms are fixed by just four parameters

$$M_{1/2} = f\frac{\langle F\rangle}{M_{\text{P}}}, \qquad m_0^2 = (k+n^2)\frac{|\langle F\rangle|^2}{M_{\text{P}}^2}, \qquad A_0 = (\alpha+3n)\frac{\langle F\rangle}{M_{\text{P}}}, \qquad B_0 = (\beta+2n)\frac{\langle F\rangle}{M_{\text{P}}}. \tag{3.172}$$

Here, $M_{1/2}$ and $m_0^2$ are universal soft-masses for all scalars and gauginos:

$$(m^2)^i_j \phi^i \phi^{j*} = \delta_{ij} m_0^2 |\phi_i|^2 \tag{3.173}$$

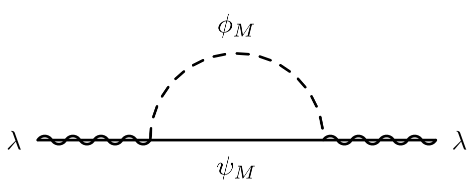
$$M_a \lambda^a \lambda^a = M_{1/2} \lambda^a \lambda^a \tag{3.174}$$

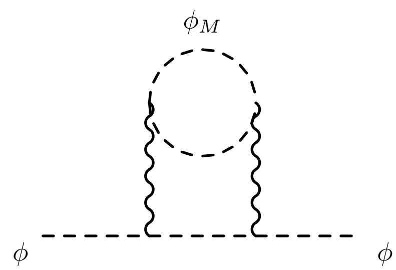
$A_0$ and $B_0$ define the proportionality between soft-parameters and superpotential parameters:

$$t_{ijk} = A_0 y_{ijk} \qquad b_{ij} = B_0 M_{ij} \tag{3.175}$$

This ansatz is called *minimal supergravity* (mSugra).

**b) Gauge interactions**:
In that case, the flavor-blind mediating interactions for supersymmetry breaking are the ordinary electroweak and QCD gauge interactions. In this *gauge-mediated supersymmetry breaking* (GMSB) scenario, the MSSM soft terms come from loop diagrams involving some *messenger* particles. The messengers are new chiral supermultiplets that couple to a supersymmetry-breaking VEV $\langle F \rangle$, and also have $SU(3)_C \times SU(2)_L \times U(1)_Y$ interactions, which provide the necessary connection to the MSSM. Examples for the diagrams which generate SUSY breaking masses are




Then, using dimensional analysis, one estimates for the MSSM soft terms

$$m_{\text{soft}} \sim \frac{\alpha_a}{4\pi} \frac{\langle F \rangle}{M_{\text{mess}}} \tag{3.176}$$

where the $\alpha_a/4\pi$ is a loop factor for Feynman diagrams involving gauge interactions, and $M_{\text{mess}}$ is a characteristic scale of the masses of the messenger fields. So if $M_{\text{mess}}$ and $\sqrt{\langle F \rangle}$ are roughly comparable, then the scale of supersymmetry breaking can be as low as about $\sqrt{\langle F \rangle} \sim 10^4$ GeV (much lower than in the gravity-mediated case!) to give $m_{\text{soft}}$ of the right order of magnitude.

## 3.4 Summary: How to construct the SUSY Lagrangian

In a supersymmetric theory, the interactions of all particles are fixed by three ingredients (up to gauge fixing)

**a)** the gauge transformation properties:

$$\begin{aligned}\mathcal{L}_{\text{free+gauge}} =& D^\mu \phi^{*i} D_\mu \phi_i + i\psi^{\dagger i}\overline{\sigma}^\mu D_\mu \psi_i - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + i\lambda^{\dagger a}\overline{\sigma}^\mu D_\mu \lambda^a \\ & - (\sqrt{2}g(\phi^*_i T^a \psi_i)\lambda^a + \text{c.c.}) - \frac{1}{2}\sum_a g_a^2(\phi_i^* T^a \phi_i)^2 \end{aligned} \tag{3.177}$$

**b)** the superpotential $W$

$$W(\phi) = L^i\phi_i + \frac{1}{2}M^{ij}\phi_i\phi_j + \frac{1}{6}y^{ijk}\phi_i\phi_j\phi_k \tag{3.178}$$

which results in

$$\mathcal{L}_{\text{int}} = -\frac{1}{2}\left(\frac{\delta^2 W}{\delta\phi_i\delta\phi_j}\psi_i\psi_j + \text{c.c.}\right) - \left|\frac{\delta W}{\delta\phi_i}\right|^2 \tag{3.179}$$

**c)** Soft-susy breaking terms

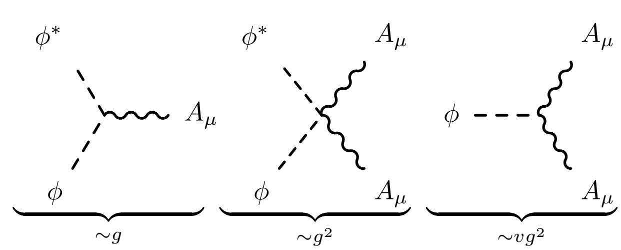
$$\begin{aligned}\mathcal{L}_{\text{soft}} =& -\left(\frac{1}{2}M_a\,\lambda^a\lambda^a + \frac{1}{6}t^{ijk}\phi_i\phi_j\phi_k + \frac{1}{2}b^{ij}\phi_i\phi_j + l^i\phi_i + \text{c.c.}\right) - (m^2)^i_j\phi^{j*}\phi_i \\ & + (\text{non}-\text{standard soft}-\text{terms})\end{aligned} \tag{3.180}$$

The scalar potential, which is for instance responsible for gauge symmetry breaking, is the sum of

$$\begin{aligned} V =& V_F + V_D + V_{\text{soft}} \\ =& \left|\frac{\delta W}{\delta\phi_i}\right|^2 + \frac{1}{2}\sum_a g_a^2(\phi_i^* T^a\phi_i)^2 + (m^2)^i_j\phi^{j*}\phi_i + \left(\frac{1}{6}t^{ijk}\phi_i\phi_j\phi_k + \frac{1}{2}b^{ij}\phi_i\phi_j + l^i\phi_i + \text{c.c.}\right)\end{aligned} \tag{3.181}$$

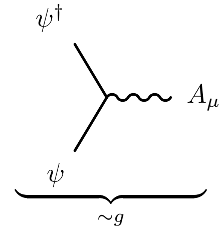
From the different parts of the Lagrangian different kinds of interactions arise

**a)** **Kinetic Terms for scalars**
$\mathcal{L} = D_\mu\phi D^\mu\phi^*$



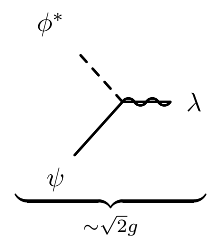

(3.182)

**b) Kinetic Terms for Fermions**
$\mathcal{L} = \psi^\dagger D^\mu \psi*$



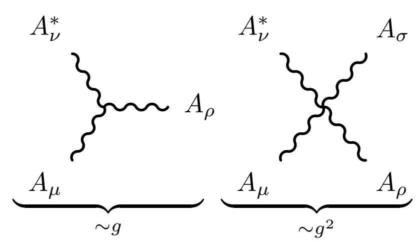

(3.183)

**c) Gaugino-Matter interactions**
$\mathcal{L} = \sqrt{2} g \phi^* T^a \lambda \psi$



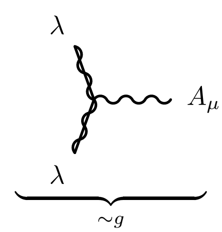

(3.184)

**d) Vector Self-interactions**
$\mathcal{L} = F^{\mu\nu} F_{\mu\nu}$



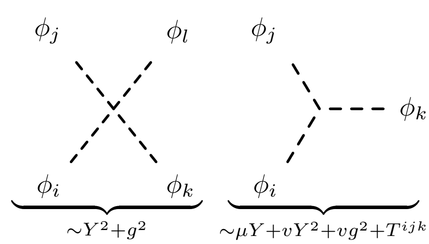

(3.185)

**e) Vector Gaugino-Interactions**
$\mathcal{L} = i\lambda^{\dagger a} \overline{\sigma}^\mu D_\mu \lambda^a$




(3.186)

**f) Scalar interactions**
$\mathcal{L} = -|F|^2 - D^2 - T^{ijk} \phi_i \phi_j \phi_k$




(3.187)

# Chapter 4

# The Minimal Supersymmetric Standard Model

It's time now to write down the minimal supersymmetric extensions of the SM, the MSSM (minimal supersymmetric standard model).

## 4.1 Particle Content and Superpotential

The building blocks of the MSSM are

**a) Vector Superfields**: since the gauge sector of the SM consists of three gauge groups, we need the three corresponding vector superfields. The naming conventions for the superfield names as well as their component fields are as follows:

| Name | SF | spin $\frac{1}{2}$ | spin 1 | $SU(3)_C, SU(2)_L, U(1)_Y$ |
|---|---|---|---|---|
| gluino, gluon | $\hat{g}_\alpha$ | $\tilde{g}_\alpha$ | $g_\alpha$ | $(\mathbf{8}, \mathbf{1}, 0)$ |
| winos, W bosons | $\hat{W}_i$ | $\tilde{W}_i$ | $W_i$ | $(\mathbf{1}, \mathbf{3}, 0)$ |
| bino, B boson | $\hat{B}$ | $\tilde{B}^0$ | $B^0$ | $(\mathbf{1}, \mathbf{1}, 0)$ |

**b) Chiral Superfields**: each SM fermion needs a scalar superpartner. Therefore, five chiral superfield are needed to arrange the matter sector. Also the Higgs doublets needs to be arranged in one superfield. However, it turns out that a second Higgs superfield is needed for two reason:

(a) One needs two fermions to cancel all gauge anomalies

(b) Since the superpotential is a holomorphic function, it is not possible to write down Yukawa terms for up- and down-quarks with a single Higgs Doublet as in the SM

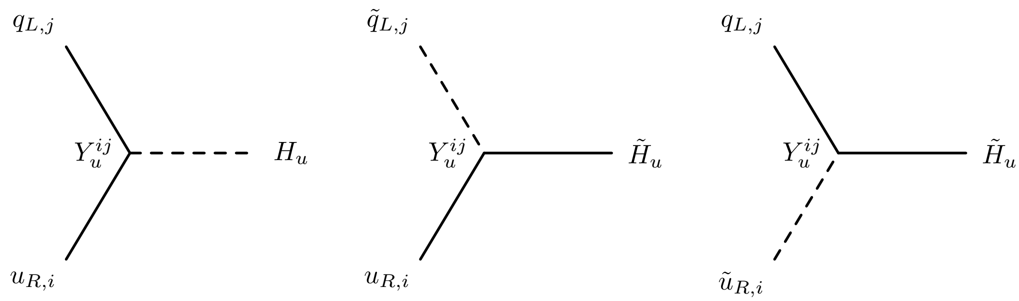
Therefore, the full list of chiral supermultiplets in the SM is

| Name | SF | spin 0 | spin $\frac{1}{2}$ | $SU(3)_C \times SU(2)_L \times U(1)_Y$ |
|---|---|---|---|---|
| squarks, quarks | $\hat{Q}$ | $\tilde{Q} = (\tilde{u}_L\, \tilde{d}_L)^T$ | $Q = (u_L\, d_L)^T$ | $(\mathbf{3}, \mathbf{2}, \frac{1}{6})$ |
| (3 generations) | $\hat{u}$ | $\tilde{u}^*_R$ | $u^*_R$ | $(\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})$ |
| | $\hat{d}$ | $\tilde{d}^*_R$ | $d^*_R$ | $(\bar{\mathbf{3}}, \mathbf{1}, \frac{1}{3})$ |
| sleptons, leptons | $\hat{L}$ | $\tilde{L} = (\tilde{\nu}\, \tilde{e}_L)^T$ | $L = (\nu\, e_L)^T$ | $(\mathbf{1}, \mathbf{2}, -\frac{1}{2})$ |
| (3 generations) | $\hat{e}$ | $\tilde{e}^*_R$ | $e^*_R$ | $(\mathbf{1}, \mathbf{1}, 1)$ |
| Higgs, Higgsinos | $\hat{H}_u$ | $H_u = (H^+_u\, H^0_u)^T$ | $\tilde{H}_u = (\tilde{H}^+_u\, \tilde{H}^0_u)^T$ | $(\mathbf{1}, \mathbf{2}, \frac{1}{2})$ |
| | $\hat{H}_d$ | $H_d = (H^0_d\, H^-_d)^T$ | $\tilde{H}_d = (\tilde{H}^0_d\, \tilde{H}^-_d)^T$ | $(\mathbf{1}, \mathbf{2}, -\frac{1}{2})$ |

**c) Superpotential**: the MSSM superpotential in terms of superfields is

$$W = Y_e^{ab}\, \hat{L}^j_a\, \hat{e}_b\, \hat{H}^i_d\, \epsilon_{ij} + Y_d^{ab}\, \hat{Q}^{j\alpha}_a\, \hat{d}_{\alpha b}\, \hat{H}^i_d\, \epsilon_{ij}\delta_{\alpha\beta} + Y_u^{ab}\hat{Q}^{i\alpha}_a\, \hat{u}_{\alpha b}\, \hat{H}^j_u\, \epsilon_{ij}\delta_{\alpha\beta} + \mu\, \hat{H}^i_u\, \hat{H}^j_d \epsilon_{ij}\ . \tag{4.1}$$

- Here, $Y_{d,u,e}$ are Yukawa couplings, which are in general complex $3 \times 3$ matrices, known from the SM. However, not only the coupling strength of the Higgs to two fermions is given by these couplings, but also the Higgsino interactions with a (s)fermion pair.




- $\mu$ is a supersymmetric mass term for the Higgs superfields:

$$m_{\tilde{H}_i} = \mu \tag{4.2}$$
$$m^2_{H_i}\mu^2 \tag{4.3}$$

Often, the simplified assumption is made that only third generation Yukawa couplings are non-negligible. This corresponds to

$$Y_u \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Y_t \end{pmatrix}, \quad Y_d \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Y_b \end{pmatrix}, \quad Y_e \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Y_\tau \end{pmatrix} \tag{4.4}$$

The superpotential becomes in this limit

$$\begin{aligned} W_{\mathrm{MSSM}} \approx\ & Y_t(t_L t_R H^0_u - b_L t_R H^+_u) + Y_b(t_L b_R H^-_d - b_L b_R H^0_d) + Y_\tau(\nu_\tau \tau_R H^-_d - \tau_L \tau_R H^0_d) \\ & + \mu(H^+_u H^-_d - H^0_u H^0_d). \end{aligned} \tag{4.5}$$

where we have expanded the isospin indices. We are using this simplified version of the superpotential to list the other interactions stemming from the superpotential:

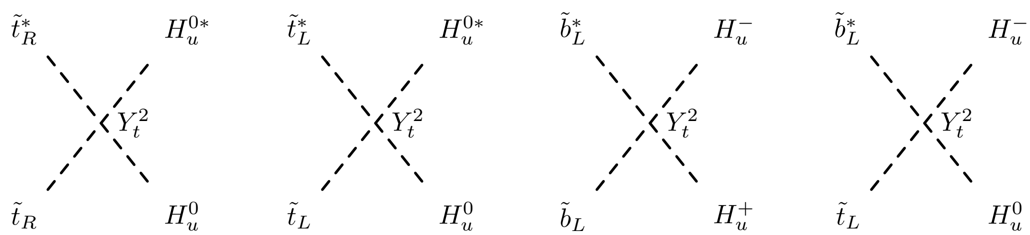
(a) *Yukawa interactions*

$$
\begin{aligned}
\mathcal{L}_{\text{Yuk}} = &-\frac{1}{2}Y_t[(t_L t_R H_u^0 - b_L t_R H_u^+) + (t_L \tilde{t}_R \tilde{H}_u^0 - b_L \tilde{t}_R \tilde{H}_u^+) + (\tilde{t}_L t_R \tilde{H}_u^0 - \tilde{b}_L t_R \tilde{H}_u^+) + \text{c.c.}] \\
&-\frac{1}{2}Y_b[(t_L b_R H_d^- - b_L b_R H_d^0) + (\tilde{t}_L b_R \tilde{H}_d^- - \tilde{b}_L b_R \tilde{H}_d^0) + (t_L \tilde{b}_R \tilde{H}_d^- - b_L \tilde{b}_R \tilde{H}_d^0) + \text{c.c.}]+ \\
&-\frac{1}{2}Y_\tau[(\nu_\tau \tau_R H_d^- - \tau_L \tau_R H_d^0) + (\tilde{\nu}_\tau \tau_R \tilde{H}_d^- - \tilde{\tau}_L \tau_R \tilde{H}_d^0) + (\nu_\tau \tilde{\tau}_R \tilde{H}_d^- - \tau_L \tilde{\tau}_R \tilde{H}_d^0) + \text{c.c.}]
\end{aligned} \tag{4.6}
$$

(b) *Quartic scalar interactions*: the $F$-terms involving four scalars are

$$
\begin{aligned}
\mathcal{L}_{\text{quartic}} = &Y_t^2 \left( (|\tilde{t}_L|^2 + |\tilde{t}_R|^2)|H_u^0|^2 + |\tilde{b}_L|^2 |H_u^+|^2 + (\tilde{t}_L \tilde{b}_L^* H_u^0 H_u^- + \text{c.c.} \right) \\
&+ Y_b^2 \left( (|\tilde{b}_L|^2 + |\tilde{b}_R|^2)|H_d^0|^2 + |\tilde{t}_L|^2 |H_d^-|^2 + (\tilde{b}_L \tilde{t}_L^* H_d^0 H_d^+ + \text{c.c.} \right) \\
&+ Y_\tau^2 \left( (|\tilde{\tau}_L|^2 + |\tilde{\tau}_R|^2)|H_d^0|^2 + |\tilde{\nu}_{\tau,L}|^2 |H_d^-|^2 + (\tilde{\nu}_{\tau,L} \tilde{\tau}_L^* H_d^0 H_d^+ + \text{c.c.} \right) \\
&- Y_t Y_b \left( \tilde{t}_R H_u^0 \tilde{b}_R^* H_d^+ + \tilde{t}_R H_u^+ \tilde{b}_R^* H_d^{0*} + \text{c.c.} \right) \\
&+ Y_b Y_\tau \left( \tilde{\tau}_L \tilde{\tau}_R \tilde{b}_L^* \tilde{b}_R^* - \tilde{\nu}_\tau \tilde{\tau}_R \tilde{t}_L^* \tilde{b}_R + \text{c.c.} \right)
\end{aligned} \tag{4.7}
$$

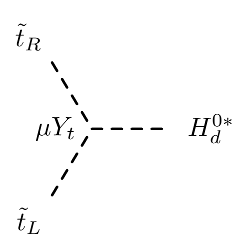
From these many terms, one can see how economic the superfield notation is. The Feynman diagrams corresponding to the couplings prop. to $Y_t^2$ are




The relation between the quartic couplings of Higgs to stops and the Yukawa coupling Higgs to tops will be crucial when we discuss the SUSY solution to the hierarchy problem.

(c) *Cubic scalar interactions*: also trilinear, scalar couplings arise due to the presence of the $\mu$-term:

$$
\begin{aligned}
\mathcal{L}_{\text{cubic}} = \quad & Y_t \mu^* (t_L t_R H_d^{0*} + b_L t_R H_d^{-*}) + Y_b \mu^* (b_L b_R H_u^{0*} + t_L b_R H_u^{+*}) + \\
& Y_\tau \mu^* (\tau_L \tau_R H_u^{0*} + \nu_L \tau_R H_u^{+*}) + \text{c.c.}
\end{aligned} \tag{4.8}
$$



**d) Soft-Breaking terms**: the general soft-breaking interactions in the MSSM are

$$\begin{aligned}\mathcal{L}_{\text{soft}}^{\text{MSSM}} &= -\frac{1}{2}\left(M_3\tilde{g}\tilde{g} + M_2\tilde{W}\tilde{W} + M_1\tilde{B}\tilde{B} + \text{c.c.}\right) \\ &\quad -\tilde{Q}^\dagger\, m_{\tilde{Q}}^2\,\tilde{Q} - \tilde{L}^\dagger\, m_{\tilde{L}}^2\,\tilde{L} - \tilde{u}_R^T\, m_{\tilde{u}}^2\,\tilde{u}_R^* - \tilde{d}_R^T\, m_{\tilde{d}}^2\,\tilde{d}_R^* - \tilde{e}_R^T\, m_{\tilde{e}}^2\,\tilde{e}_R^* - m_{H_u}^2 H_u^* H_u - m_{H_d}^2 H_d^* H_d \\ &\quad -\left(T_u\,\tilde{Q}\,\tilde{u}_R^*\,H_u + T_d\,\tilde{Q}\,\tilde{d}_R^*\,H_d + T_e\,\tilde{L}\,\tilde{e}_R^*\,H_d + \text{c.c.}\right) - (B_\mu H_u H_d + \text{c.c.})\,. \end{aligned} \tag{4.9}$$

Here, we have

- $M_3$, $M_2$, and $M_1$ are the gluino, wino, and bino mass terms. These are complex parameters with mass dimension of 1.
- Mass squared terms for all chiral superfields. While $m_{\tilde{f}}^2$ ($f = \{Q, L, u, d, e\}$) are Hermitian $3 \times 3$ matrices, $m_{H_d}^2$ and $m_{H_u}^2$ are real parameters.
- For each term in the superpotentiala purely scalar interaction appears. These are trilinear scalar couplings $T_i$ ($i = \{d, u, e\}$) which are complex $3 \times 3$ matrices as well as Higgs mixing term $B_\mu$ which is complex as well.

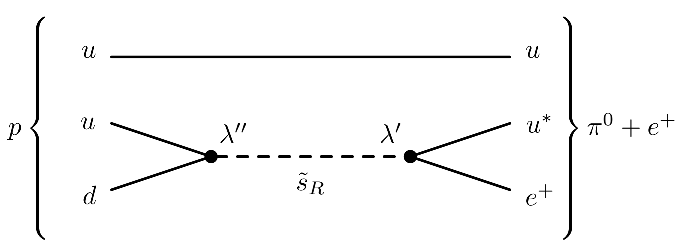
### 4.1.1 $R$-parity

The superpotential which we have written down does not include all interactions which are allowed by gauge invariance. Other possible terms would be

$$W_{\Delta\text{L}=1} = \frac{1}{2}\lambda^{ijk}\,\hat{L}_i\,\hat{L}_j\,\hat{e}_k + \lambda'^{ijk}\,\hat{L}_i\,\hat{Q}_j\,\hat{d}_k + \epsilon^i\,\hat{L}_i\,\hat{H}_u \tag{4.10}$$

$$W_{\Delta\text{B}=1} = \frac{1}{2}\lambda''^{ijk}\,\hat{u}_i\,\hat{d}_j\,\hat{d}_k \tag{4.11}$$

The terms in $W_{\Delta\text{L}=1}$ violate total lepton number by 1 unit and those in $W_{\Delta\text{B}=1}$ violate baryon number by 1 unit. Note, $\epsilon_i$ in this context is not the anti-symmetric tensor but a common nomenclature for a superpotential term.
The presence of such terms is highly constrained by proton decay. For instance, the combination $\lambda' \cdot \lambda''$ can trigger proton decay via diagrams like




We can estimate the life-time of the proton just from a dimensional analysis as:

$$\Gamma_{p\to e^+\pi^0} \sim m_{\text{proton}}^5 \sum_{i=2,3} |\lambda'^{11i}\lambda''^{11i}|^2/m_{\tilde{d}_i}^4, \tag{4.12}$$

$i = 1$ vanishes because of the anti-symmetry of $\lambda''$.
The lower limit on the proton life-time is $10^{32}$ years what corresponds to $10^{-64}\text{GeV}^{-1}$. Thus, for SUSY masses in the TeV range, we obtain a limit of

$$|\lambda'\lambda''| < 10^{-26} \tag{4.13}$$

Such tight constraints usually point towards a symmetry which completely forbids the underlying process. Therefore, a new symmetry called $R$-parity is introduced.

$R$-parity is a $Z_2$ symmetry which is defined as

$$P_R = (-1)^{3(B-L)+2s} \tag{4.14}$$

where $s$ is the spin of the particle, while $B$ and $L$ are its baryon respectively lepton number.

With this definition, it turns out that all SM particles are even under $R$-parity, while their superpartners are odd. Therefore, 'supersymmetric particles' or 'sparticles' are a synonym for particles with $P_R = -1$. This has tremendous consequences:

- If $R$-parity is conserved, there can't be any mixing between supersymmetric particles and SM ones
- All interaction vertices must involve an even number of supersymmetric particles
- The lightest supersymmetric particle (LSP) can't decay, i.e. it might be a candidate for dark matter
- Always an even number of supersymmetric particles is produced at colliders

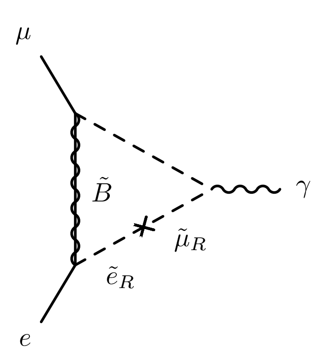
Note, sometimes also 'matter parity' is used which is a $Z_2$ for the entire superfield. The matter parity of a superfield is identical to the $R$-parity of its scalar component. Therefore, both symmetries allow equivalent terms in the superpotential independently of the choice if $W$ is expressed in terms of superfields or scalars.

If either $B$ or $L$ violating operators are present, proton decay is not possible. Therefore, one could also relax the condition of $R$-parity violation and study models with $\lambda$, $\lambda'$ and $\epsilon$, or with $\lambda''$ alone. Such $R$-parity violating scenarios can be motivated by other symmetries and have interesting phenomenological consequences like neutrino masses via a neutrino–neutralino mixing.

### 4.1.2 Constraints on Soft-terms

We have written down the most general form of soft-breaking terms. However, there exist stringent constraints on their shape, because large off-diagonal entries would trigger flavour violating processes while sizeable complex phases have an impact on CP observables like dipole moments. Let's discuss two examples:

**a) Flavour violation:** Suppose that $m^2_{\bar{e}}$ is not diagonal in the basis $(\tilde{e}_R, \tilde{\mu}_R, \tilde{\tau}_R)$. In that case, flavor mixing in the slepton occurs, so the individual lepton numbers will not be conserved, even for processes that only involve the sleptons as virtual particles. A particularly strong limit on this possibility comes from the experimental bound on the process $\mu \to e\gamma$, which could arise from the one-loop diagram as



Here, "×" on the slepton line represents the insertion of the off-diagonal soft-term $m^2_{\tilde{\mu}^*_R \tilde{e}_R}$. The result of calculating this diagram gives roughly

$$\mathrm{Br}(\mu \to e\gamma) \quad \simeq \quad 10^{-6} \left( \frac{|m^2_{\tilde{\mu}^*_R \tilde{e}_R}|}{m^2_{\tilde{\ell}_R}} \right)^2 \left( \frac{100\,\mathrm{GeV}}{m_{\tilde{\ell}_R}} \right)^4 \tag{4.15}$$

where we assumed that ...

- the diagonal entries are degenerated $m^2_{\tilde{e}_R \tilde{e}_R} = m^2_{\tilde{\mu}_R \tilde{\mu}_R} = m^2_{\tilde{l}_R}$
- $m^2_{\tilde{l}_R} \gg m^2_{\tilde{\mu}^*_R \tilde{e}_R}$ to get mass eigenstates with mass $m_{\tilde{l}_R}$.
- The bino contribution dominates over similar wino/Higgsino diagrams
- The bino mass is similar to the slepton mass

This result has to be compared to the present experimental upper limit

$$\mathrm{Br}(\mu \to e\gamma)_{\mathrm{exp}} < 5.7 \times 10^{-13} \tag{4.16}$$

So, we find for 1 TeV slepton masses that

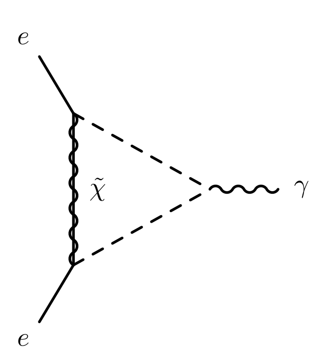
$$\left( \frac{|m^2_{\tilde{\mu}^*_R \tilde{e}_R}|}{m^2_{\tilde{\ell}_R}} \right) < 0.1 \tag{4.17}$$

must hold.
Similar constraints come from the experimental limits of

- $\mu \to 3e$
- $B \to X_s \gamma$
- $B \to \mu^+ \mu^-$
- $\Delta M_B$
- ...

Thus, the soft-matrices can't be random but must follow some hierarchy.

**b) CP violation**:
Additional sources of CP violation are constrained by the measured dipole moments of the SM particles. For instance, the dipole moment of the electron receives new SUSY contributions from the imaginary part of diagrams like



In the case, that the neutralino is a Higgsino-Wino mixture, one gets

$$d_e \simeq \frac{1}{2} m_e g_2^2 |M_2 \mu| \tan\beta \sin\Phi_{\rm CP} \times K(m_{\tilde{e}_L}^2, \mu^2, M^2) \tag{4.18}$$

with a kinematic function $K$. Here, $\sin\Phi_{\rm CP}$ is the misalignment of the phases of $M_2$ and $\mu$ and must be below 0.01 for SUSY masses of about 1 TeV.

The form of the soft-breaking parameters in the MSSM is highly constrained by:

- Experimental limits on flavour violating processes. Those restrict the size of off-diagonal elements in the soft-matrices.
- Experimental limits on electric dipole moments. Those restrict the size of complex phases in the soft-terms.

One could avoid these constraints if specific assumptions are made about the form of the soft-breaking terms *at the SUSY scale*. A very simple ansatz is:

- Consider an idealised limit in which the squark and slepton squared-mass matrices are flavor-blind, each proportional to the $3 \times 3$ identity matrix in family space:

  $$m_{\tilde{Q}}^2 = m_Q^2 \mathbf{1}, \quad m_{\tilde{u}}^2 = m_{\bar{u}}^2 \mathbf{1}, \quad m_{\tilde{d}}^2 = m_{\bar{d}}^2 \mathbf{1}, \quad m_{\tilde{L}}^2 = m_L^2 \mathbf{1}, \quad m_{\tilde{e}}^2 = m_{\bar{e}}^2 \mathbf{1} \tag{4.19}$$

  Then all squark and slepton mixing angles are rendered trivial, because squarks and sleptons with the same electroweak quantum numbers will be degenerate in mass and can be rotated into each other at will.
  Supersymmetric contributions to flavor-changing neutral current processes will therefore be very small in such an idealised limit, up to mixing induced by $T_u$, $T_d$, $T_e$.

- Making the further assumption that the (scalar)$^3$ couplings are each proportional to the corresponding Yukawa coupling matrix,

  $$T_u = A_u Y_u, \qquad T_d = A_d Y_d, \qquad T_e = A_e Y_e, \tag{4.20}$$

  will ensure that only the squarks and sleptons of the third family can have large (scalar)$^3$ couplings. Now, the entire flavour violation originates from the SM CKM matrix. This scenario is called *minimal flavour violation*. .

- Finally, one can avoid disastrously large CP-violating effects by assuming that the soft parameters do not introduce new complex phases. This is automatic for $m_{H_u}^2$ and $m_{H_d}^2$, and for sfermion soft masses if eq. (4.19) is assumed. One can also fix $\mu$ in the superpotential and $B_\mu$ in soft potential to be real, by appropriate phase rotations of fermion and scalar components of the $H_u$ and $H_d$ supermultiplets. The remaining phases which need to be small are

  $$\mathrm{Im}(M_1), \mathrm{Im}(M_2), \mathrm{Im}(M_3), \mathrm{Im}(A_u), \mathrm{Im}(A_d), \mathrm{Im}(A_e) \tag{4.21}$$

  If those exactly vanish, then the only CP-violating phase in the theory will be the usual CKM phase found in the ordinary Yukawa couplings.

The MSSM with these flavor- and CP-preserving relations imposed has far fewer parameters than the most general case. The new parameters beside the SM ones, are

- 3 independent real gaugino masses
- 5 real squark and slepton squared mass parameters
- 3 real scalar cubic coupling parameters
- 4 Higgs mass parameters (2 can be eliminated by the minimum conditions as we will see)

Including $\tan\beta = \frac{v_u}{v_d}$, there are 14 free parameters.
Other common choices which are often called *phenomenological MSSM (pMSSM)* are for instance

- pMSSM-10:

$$\begin{aligned} m_{\tilde{L}} = m_{\tilde{e}} &= m_L \\ m_{\tilde{u},11} = m_{\tilde{u},22} = m_{\tilde{Q},11} = m_{\tilde{Q},22} = m_{\tilde{d},11} = m_{\tilde{d},22} &= m_{\tilde{q}} \\ m_{\tilde{u},33} = m_{\tilde{Q},33} = m_{\tilde{d},33} &= m_{\tilde{q}_3} \\ A_t = A_b = A_\tau &= A \\ M_1, M_2, M_3 & \\ M_A, \mu, \tan\beta & \end{aligned}$$

- pMSSM-19

$$\begin{aligned} m_{\tilde{L},11} = m_{\tilde{L},22} &= m_L \\ m_{\tilde{e},11} = m_{\tilde{e},22} &= m_E \\ m_{\tilde{e},33}, m_{\tilde{L},33} & \\ m_{\tilde{u},11} = m_{\tilde{u},22} &= m_u \\ m_{\tilde{Q},11} = m_{\tilde{Q},22} &= m_q \\ m_{\tilde{d},11} = m_{\tilde{d},22} &= m_d \\ m_{\tilde{u},33}, m_{\tilde{Q},33}, m_{\tilde{d},33} & \\ A_t, A_b, A_\tau & \\ M_1, M_2, M_3 & \\ M_A, \mu, \tan\beta & \end{aligned}$$

## 4.2 Gauge Coupling Unification and SUSY Breaking mechanism

### 4.2.1 RGE Running

Lagrangian parameters are scaled dependent, i.e. they change with the energy at which the test the model. This energy dependence is described by the renormalisation group equations (RGEs). The RGEs are calculated in a chosen renormalisation scheme. For non-supersymmetric models, it is convenient to choose the so called '$\overline{\text{MS}}$ scheme' (dimensional regularisation). In this scheme , the number of space-time dimensions is continued to $d = 4 - 2\epsilon$. The $\beta$-functions calculated in $\overline{\text{MS}}$, which describe the energy dependence of the parameters $\Theta$, are defined as

$$\beta_i = \mu \frac{d\Theta_i}{d\mu} = \frac{d\Theta_i}{d\ln\mu} = \frac{d}{dt}\Theta_i \tag{4.22}$$

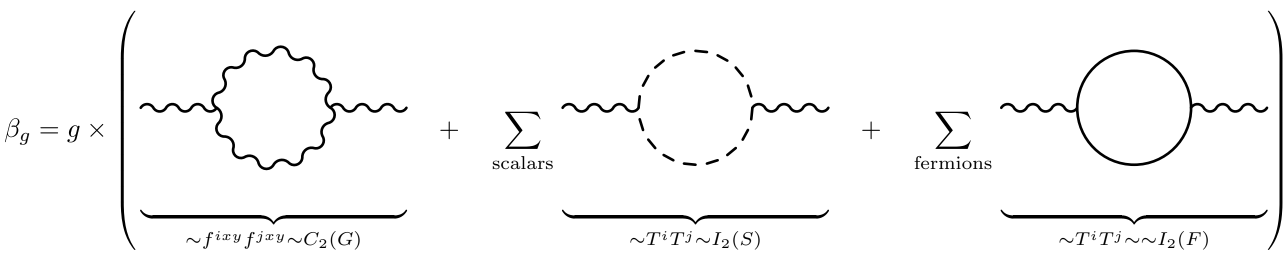
Here, $\mu$ is an arbitrary mass scale. $\beta_i$ can be expanded in a perturbative series:

$$\beta_i = \sum_n \frac{1}{(16\pi^2)^n} \beta_i^{(n)} \tag{4.23}$$

$\beta_i^{(1)}$ is the one-loop contributions which we will consider here. The $\beta$ function for gauge couplings is related to the anomalous dimension of the corresponding vector field via

$$\beta_g = g\gamma \tag{4.24}$$

The one-loop RGEs for gauge couplings can be calculated from diagrams as




More precisely, we need to calculate the divergent part of each contribution to the wave function renormalisation constant. When doing that, one can derive a general expression for the $\beta$ function of the gauge coupling in an arbitrary model.

The one-loop $\beta$-function of a gauge coupling in a general quantum field theory can be calculated from

$$\beta_{g_a}^{(1)} = \frac{d}{dt} g_a = \frac{g_a^3}{16\pi^2} \left( \frac{2}{3} I_2(F) + \frac{1}{6} I_2(S) - \frac{11}{3} C_2(G) \right) \tag{4.25}$$

Here, $C_2(G)$ is the quadratic Casimir index of the group, and $I_2(F)$, $I_2(S)$ are the Dynkin indices summed over all fermions respectively scalars. $S$ counts the *real* scalar degrees of freedom.

We get for the Standard model

$$\beta_{g_1}^{(1)} = g_1^3 \left[ \underbrace{\frac{2}{3} \times 3 \times 3 \times \left( 2 \times \frac{1}{36} + \frac{1}{9} + \frac{4}{9} \right)}_{\text{quarks}} + \underbrace{\frac{2}{3} \times 3 \times \left( 2 \times \frac{1}{4} + 1 \right)}_{\text{leptons}} + \underbrace{\frac{1}{6} \times 2 \times 2 \times \left( \frac{1}{4} \right)}_{\text{Higgs}} \right] \frac{5}{3} = \frac{41}{10} g_1^3 \tag{4.26}$$

$$\beta_{g_2}^{(1)} = g_2^3 \left[ \underbrace{-\frac{11}{3} \times (2)}_{\text{W bosons}} + \underbrace{\frac{2}{3} \times 3 \times 3 \times \left( \frac{1}{2} \right)}_{\text{left quarks}} + \underbrace{\frac{2}{3} \times 3 \times \left( \frac{1}{2} \right)}_{\text{left leptons}} + \underbrace{\frac{1}{6} \times 2 \times \left( \frac{1}{2} \right)}_{\text{Higgs}} \right] = -\frac{19}{6} g_2^3 \tag{4.27}$$

$$\beta_{g_3}^{(1)} = g_3^3 \left[ \underbrace{-\frac{11}{3} \times (3)}_{\text{gluon}} + \underbrace{\frac{2}{3} \times 3 \times \left( 1 + \frac{1}{2} + \frac{1}{2} \right)}_{\text{quarks}} \right] = -7 g_3^3 \tag{4.28}$$

Note the additional factor 2 for Higgs particles because the expressions are given in terms of real scalars. Moreover, we have added a factor $\sqrt{5/3}$ which is the 'GUT normalisation' from the embedding

$$U(1) \times SU(2) \times SU(3) \subset SU(5) \tag{4.29}$$

If we turn to a supersymmetric theory, it is actually not possible to perform the calculation in $\overline{\text{MS}}$ scheme: it introduces a mismatch between the numbers of gauge boson degrees of freedom and the gaugino degrees of freedom off-shell. Therefore, one uses the slightly different '$\overline{\text{DR}}$ scheme' (dimensional reduction). In this scheme, all momentum integrals are still performed in $d = 4 - 2\epsilon$ dimensions, but the vector index $\mu$ on the gauge boson fields $A^a_\mu$ now runs over all 4 dimensions to maintain the match with the gaugino degrees of freedom. It turns out, that one loop $\beta$-functions are always the same in these two schemes. Therefore, one can immediately derive the expression for the $\beta$-function in a supersymmetric theory from the result above.

- The gaugino contributions can be written as

$$\frac{2}{3} I_2(F_\lambda) \to \frac{2}{3} C_2(G) \tag{4.30}$$

- The scalar and fermionic contributions are related

$$I_2(S) = 2 \times I_2(F) \tag{4.31}$$

We have used here that one gaugino appears in the adjoint representation and that appear always together.

The one-loop $\beta$ function for a gauge coupling in a supersymmetric theory is given by

$$\beta^{(1)}_{g_a} = \frac{g_a^3}{16\pi^2} \left( I_2(\Phi) - 3 C_2(G) \right), \tag{4.32}$$

Here, $I_2(\Phi)$ is the Dynkin summed over all superfields

We get now for the MSSM

$$\beta^{(1)}_{g_1} = g_1^3 \left[ \underbrace{3 \times 3 \times \left( 2 \times \frac{1}{36} + \frac{1}{9} + \frac{4}{9} \right)}_{\text{quarks}} + \underbrace{3 \times \left( 2 \times \frac{1}{4} + 1 \right)}_{\text{leptons}} + \underbrace{2 \times 2 \times \left( \frac{1}{4} \right)}_{\text{Higgs}} \right] \frac{5}{3} = \frac{33}{10} g_1^3 \tag{4.33}$$

$$\beta^{(1)}_{g_2} = g_2^3 \left[ \underbrace{-3 \times 2}_{\text{Gauge}} + \underbrace{3 \times 3 \times \left( \frac{1}{2} \right)}_{\text{left quarks}} + \underbrace{3 \times \left( \frac{1}{2} \right)}_{\text{left leptons}} + \underbrace{2 \times \left( \frac{1}{2} \right)}_{\text{Higgs}} \right] = g_2^3 \tag{4.34}$$

$$\beta^{(1)}_{g_3} = g_3^3 \left[ \underbrace{-3 \times 3}_{\text{Gauge}} + \underbrace{3 \times \left( 1 + \frac{1}{2} + \frac{1}{2} \right)}_{\text{quarks}} \right] = -3 g_3^3 \tag{4.35}$$

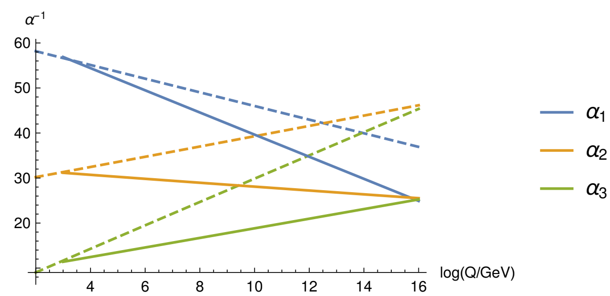
In order to check the running, it is helpful to introduce

$$\alpha_i = \frac{g_i^2}{4\pi} \tag{4.36}$$

and to use

$$\frac{d}{dt}\alpha^{-1} = \frac{d}{dt}\frac{4\pi}{g^2} = \frac{-8\pi}{g^3}\frac{d}{dt}g = \frac{-8\pi}{g^3}g_3\beta = -8\pi\beta \tag{4.37}$$

The difference in the running looks as follows:




Here, we assumed a SUSY scale of 1 TeV and started with following values:

$$g_1(m_Z) = 0.36\sqrt{5/3} \tag{4.38}$$
$$g_2(m_Z) = 0.645 \tag{4.39}$$
$$g_3(m_Z) = 1.18 \tag{4.40}$$

One sees that the gauge couplings unify in the MSSM (within the theoretical uncertainty) at a scale $M = 2 \cdot 10^{16}$ GeV. This scale is often called GUT scale, $M_{\text{GUT}}$ because it might be the scale at which a GUT theory ($SO(10)$, String theory) gets broken to the MSSM.

### 4.2.2 Boundary conditions at the GUT scale

The unification of the gauge couplings at a scale of $M = 2 \cdot 10^{16}$ GeV have motivated many studies how SUSY could be broken at that scale and what the impact on the soft masses at the SUSY scale is. A very popular assumption is that the SUSY breaking parameters of the MSSM are induced at that scale via gravitational interactions. As we have seen, one can built explicit models how this can happen, e.g. minimal supergravity. The outcome is that in the minimal version only all soft-breaking terms in the MSSM are fixed by only three parameters at the GUT scale. These are

**a)** A common mass $m_0$ for scalar fields

**b)** A common mass $M_{1/2}$ for gaugino fields

**c)** A parameter $A_0$ which relates the trilinear soft-terms and the Yukawa interactions.

The boundary conditions at the GUT scale are

$$m^2_{\tilde{Q}} = m^2_{\tilde{d}} = m^2_{\tilde{u}} = m^2_{\tilde{L}} = m^2_{\tilde{e}} \equiv \mathbf{1}m_0^2 \tag{4.41}$$
$$m^2_{H_d} = m^2_{H_u} \equiv m_0^2 \tag{4.42}$$
$$M_1 = M_2 = M_3 \equiv M_{1/2} \tag{4.43}$$
$$T_u \equiv A_0 Y_u \tag{4.44}$$
$$T_d \equiv A_0 Y_d \tag{4.45}$$
$$T_l \equiv A_0 Y_l \tag{4.46}$$

Although $B_\mu$ would also be predicted at the GUT scale in mSugra, it is usually fixed by the condition that tadpole equations (=minimum conditions of the scalar potential) are fulfilled at the SUSY scale. Also $|\mu|^2$ is fixed by this condition. Thus, the only free parameter in this setup are

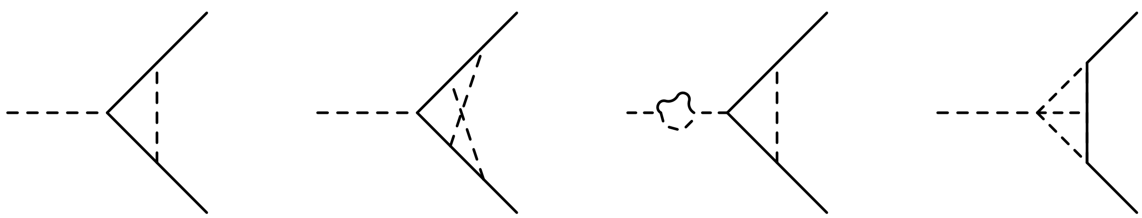
$$m_0, \quad M_{1/2}, \quad A_0, \quad \tan\beta, \quad \text{phase}(\mu) \tag{4.47}$$

This version of the MSSM is called the 'Constrained MSSM' because of obvious reasons.

## 4.3 Renormalisation Group Equations for Superpotential parameters and Soft-SUSY breaking Terms

### 4.3.1 Superpotential Terms

A general and powerful result known as the *supersymmetric non-renormalization theorem* defines the form of the renormalization group equations for supersymmetric theories. This theorem implies that the logarithmically divergent contributions to a particular process can always be written in terms of wave-function renormalizations, without any coupling vertex renormalization. Thus, we don't need to worry about diagrams like



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For the parameters appearing in the superpotential the implication is that the $\beta$-functions can be calculated from

$$\beta_{y^{ijk}} \equiv \frac{d}{dt} y^{ijk} = \gamma^i_n y^{njk} + \gamma^j_n y^{ink} + \gamma^k_n y^{ijn}, \tag{4.48}$$

$$\beta_{M^{ij}} \equiv \frac{d}{dt} M^{ij} = \gamma^i_n M^{nj} + \gamma^j_n M^{in}, \tag{4.49}$$

$$\beta_{L^i} \equiv \frac{d}{dt} L^i = \gamma^i_n L^n, \tag{4.50}$$

where the $\gamma^i_j$ are anomalous dimension matrices associated with the superfields, which generally have to be calculated in a perturbative loop expansion. At the 1-loop level, one finds

$$\gamma^i_j = \frac{1}{16\pi^2}\left[\frac{1}{2} y^{imn} y^*_{jmn} - 2g_a^2 C_a(i)\delta^i_j\right], \tag{4.51}$$

The first term includes Yukawa contributions. If we consider for example $\hat{H}_u$, we find

$$y^{H_u mn} y^*_{H_u mn} = \sum_{\alpha,\beta} y^{H_u Q_\alpha u_\beta} y^*_{H_u Q_\alpha u_\beta} \delta_{\alpha\beta} \tag{4.52}$$

$$= 3 y^{H_u Q u} y^*_{H_u Q u} \tag{4.53}$$

$$= 3 Y_t Y_t^* \tag{4.54}$$

Thus, a colour factor of three appears. For right stau we have

$$y^{\tau_R mn} y^*_{\tau_R mn} = \sum_{i,j} y^{\tau_R H_d^i L^j} y^*_{\tau_R H_d^i L^j} \epsilon_{ij} \tag{4.55}$$
$$= 2 y^{\tau_R H_d L} y^*_{\tau_R H_d L} \tag{4.56}$$
$$= 2 Y^*_\tau Y_\tau \tag{4.57}$$

Thus, a factor of 2 appears from the sum over internal isospin states. With this procedure, one finds the following anomalous dimensions for all superfields:

$$\gamma_{\hat{H}_u} = \frac{1}{16\pi^2}\left[3|Y_t|^2 - \frac{3}{2}g_2^2 - \frac{3}{10}g_1^2\right] \tag{4.58}$$
$$\gamma_{\hat{H}_d} = \frac{1}{16\pi^2}\left[3|Y_b|^2 + |Y_\tau|^2 - \frac{3}{2}g_2^2 - \frac{3}{10}g_1^2\right] \tag{4.59}$$
$$\gamma_{\hat{Q}_3} = \frac{1}{16\pi^2}\left[|Y_t|^2 + |Y_b|^2 - \frac{8}{3}g_3^2 - \frac{3}{2}g_2^2 - \frac{1}{30}g_1^2\right] \tag{4.60}$$
$$\gamma_{\hat{u}_3} = \frac{1}{16\pi^2}\left[2|Y_t|^2 - \frac{8}{3}g_3^2 - \frac{8}{15}g_1^2\right] \tag{4.61}$$
$$\gamma_{\hat{d}_3} = \frac{1}{16\pi^2}\left[2|Y_b|^2 - \frac{8}{3}g_3^2 - \frac{2}{15}g_1^2\right] \tag{4.62}$$
$$\gamma_{\hat{L}_3} = \frac{1}{16\pi^2}\left[|Y_\tau|^2 - \frac{3}{2}g_2^2 - \frac{3}{10}g_1^2\right] \tag{4.63}$$
$$\gamma_{\hat{e}_3} = \frac{1}{16\pi^2}\left[2|y_\tau|^2 - \frac{6}{5}g_1^2\right] \tag{4.64}$$

For the first two generations of (S)fermions, the expressions are obtained by setting the Yukawa contributions to zero. We used only one index for $\gamma$ because in the MSSM there is no mixing between the wave-functions of different superfields, i.e.

$$\gamma_x \equiv \gamma_x^x \tag{4.65}$$

This would change if we include $R$-parity violating terms

$$W = W_{\text{MSSM}} + \frac{1}{2}\lambda \hat{L}\hat{L}\hat{E} + \lambda' \hat{L}\hat{Q}\hat{d} \tag{4.66}$$

Now, we find

$$\gamma_{\hat{H}_d}^{\hat{L}_i} = y^{H_d mn} y^*_{L_i mn} + \dots \tag{4.67}$$
$$= y^{H_d L_x E_y} y^*_{L_i L_x E_y} + y^{H_d L_x E_y} y^*_{L_x L_i E_y} + 3 y^{H_d Q_x d_y} y^*_{L_i Q_x d_y} + \dots \tag{4.68}$$
$$= \frac{1}{2} Y_{l,xy} \lambda'^*_{ixy} + \frac{1}{2} Y_{l,xy} \lambda'^*_{xiy} + 3 Y_{d,xy} \lambda''^*_{ixy} + \dots \tag{4.69}$$
$$= Y_{l,xy} \lambda'^*_{ixy} + 3 Y_{d,xy} \lambda''^*_{ixy} + \dots \tag{4.70}$$

Coming back to the MSSM. With all anomalous dimensions at hand, we can easily get the RGEs for the Yukawa couplings and the $\mu$-term:

$$\beta_{Y_t}^{(1)} = Y_t\Big[\gamma_{\hat{H}_u} + \gamma_{\hat{Q}_3} + \gamma_{\hat{u}_3}\Big]$$

$$= Y_t\Big[6|Y_t|^2 + |Y_b|^2 - \frac{16}{3}g_3^2 - 3g_2^2 - \frac{13}{15}g_1^2\Big] \quad (4.71)$$

$$\beta_{y_b}^{(1)} = Y_b\Big[6|Y_b|^2 + |Y_t|^2 + |Y_\tau|^2 - \frac{16}{3}g_3^2 - 3g_2^2 - \frac{7}{15}g_1^2\Big] \quad (4.72)$$

$$\beta_{y_\tau}^{(1)} = Y_\tau\Big[4|Y_\tau|^2 + 3|Y_b|^2 - 3g_2^2 - \frac{9}{5}g_1^2\Big] \quad (4.73)$$

$$\beta_{\mu}^{(1)} = \mu\Big[3|Y_t|^2 + 3|Y_b|^2 + |Y_\tau|^2 - 3g_2^2 - \frac{3}{5}g_1^2\Big]. \quad (4.74)$$

The RGEs of superpotential parameters is not affected by soft-SUSY breaking. This means:

**a)** The non-renormalisation theorem still holds after SUSY breaking, i.e. only wave-function renormalisation is needed

**b)** Soft-SUSY breaking don't appear in the RGEs of superpotential terms. However, superpotential terms appear in the RGEs of soft-SUSY breaking parameters.

## 4.3.2 Running Soft-Masses

We want to discuss how the behaviour of the different soft-breaking parameters under RGE evaluation is. Even if we use in the given examples parameter values which are motivated by the CMSSM setup, the qualitative behaviour of the different contributions is independent of that choice.

### 4.3.2.1 Gaugino mass parameters

The generic RGEs for the gaugino masses are given by

$$\beta_{M_a}^{(1)} = \frac{g_a^2}{8\pi^2}\left(I_2(\Phi) - 3C_2(G)\right)M_a \quad (4.75)$$

Thus, the $\beta$ function can directly be read off from the RGEs of the gauge couplings, i.e. on has

$$\beta_{M_1}^{(1)} = \frac{66}{5}g_1^2 M_1 \quad (4.76)$$

$$\beta_{M_2}^{(1)} = 2g_2^2 M_2 \quad (4.77)$$

$$\beta_{M_3}^{(1)} = -6g_3^2 M_3 \quad (4.78)$$

At the one-loop level, it is possible to solve the coupled RGEs of gauge couplings and gauginos analytically. We re-write

$$\frac{d}{dt}g_a^2 = 2g_a\frac{d}{dt}g_a = 2g_a \times \frac{1}{16\pi^2}g_a^3\beta_a = \frac{1}{16\pi^2}2\beta_a(g_a^2)^2 \quad (4.79)$$

$$\frac{d}{dt}M_a = \frac{1}{16\pi^2}2\beta_a g_a^2 M_a \quad (4.80)$$

The general solution of the system

$$\alpha' = 2b\alpha^2 \tag{4.81}$$

$$M' = 2b\alpha M \tag{4.82}$$

is

$$\alpha(t) = -\frac{1}{2bt + c_1} \tag{4.83}$$

$$M(t) = \frac{c_2}{2bt + c_1} \tag{4.84}$$

with

$$\alpha(0) = g(M_{\rm GUT})^2 \tag{4.85}$$

$$M(0) = M_{\rm GUT} \tag{4.86}$$

we finally have

$$g_a(t)^2 = \frac{g_a(M_{\rm GUT})^2}{1 - 2\beta_a g_a(M_{\rm GUT})^2 t} \tag{4.87}$$

$$M_a(t) = \frac{M_a(M_{\rm GUT})}{1 - 2\beta_a g_a(M_{\rm GUT})^2 t} \tag{4.88}$$

Thus

$$\frac{M_a(t)}{g_a(t)^2} = \frac{M_a(M_{\rm GUT})}{g_a(M_{\rm GUT})^2} \tag{4.89}$$

We find that the ration of the gaugino mass parameters at SUSY scale are given by

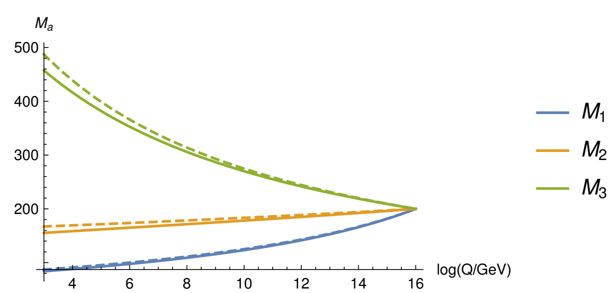
$$\frac{M_{1/2}}{g_{\rm GUT}^2} = \frac{M_1}{g_1^2} = \frac{M_2}{g_2^2} = \frac{M_3}{g_3^2} \tag{4.90}$$

From $g_1 \simeq 0.45$, $g_2 \simeq 0.65$ and $g_3 \simeq 1.1$, we get

$$M_1 : M_2 : M_3 \sim 1 : 2 : 6 \tag{4.91}$$

This is the main reasons why in many cases the lightest neutralino, i.e. the dark matter candidate is assumed to be bino-like. The big hierarchy between the bino and gluino is also very helpful from the experimental point view: the production cross section of the gluino at the LHC is very high, i.e. it's mass limit is already pushed in the TeV range. Nevertheless, light electroweakinos are still possible even in this simple, unified scenario.

At the two-loop level this prediction is (moderately) changed, because the higher order corrections for the gluino are more important than for the bino and wino. We show the running at one-loop (dashed) and two-loop (full lines) in the following plot:



#### 4.3.2.2 Scalar masses

Also for all other soft-masses generic expressions exist up to full two-loop and even partial three-loop level. However, the explicit expressions are not necessary for the following discussion. We only want to understand the general features qualitatively.
The $\beta$-function for scalar soft masses $m^2$ is of the form:

$$\beta_{m^2}^{(1)} = \frac{1}{16\pi^2}\left(\underbrace{Y^2(m^2+A^2)}_{\text{Yukawa}} - \underbrace{g^2|M|^2}_{\text{Gauge}}\right) \tag{4.92}$$

where we have used $T = YA$, ie assumed a proportionality between trilinear soft-terms and Yukawa couplings. We see, that the gauge contributions enter with a different sign than the Yukawa contributions. This observation leads to many important features.

If we assume that all scalar soft-masses unify at the GUT scale, we will find at the SUSY scale:

- Squarks are usually heavier than sleptons because of the additional contributions from the strong gauge coupling
- Left-Sfermions are heavier than right Sfermions because of the contributions from $g_2$
- Third generation particles are usually *lighter* than first and second generation because of bigger Yukawa contributions (what is opposite to the mass hierarchy in the SM masses)
- The stops are lighter than the sbottoms because of the bigger Yukawa contributions
- The lightest Sfermion gauge eigenstate is often $\tilde{t}_R$.
- The up-Higgs runs much faster than the down-Higgs because of the contribution from the top Yukawa coupling.

Even if these hierarchies are motivated by a GUT theory based on unified masses they influence often also the assumptions made about SUSY spectra at the low scale (Bino LSP, light stops) even if no explicit GUT model is assumed. However, this bias can be dangerous because interesting mass hierarchies can be missed.

We compare in the following numerical the running of the following particles to confirm our estimates:

$\tilde{\tau}_L$, $\tilde{\tau}_R$, $\tilde{e}_L$, $\tilde{t}_L$, $\tilde{t}_R$, $\tilde{b}_R$, $\tilde{d}_L$, $\tilde{d}_R$. The explicit RGEs are given by

$$\beta^{(1)}_{m^2_{\tilde{\tau}_L}} = -\left(\frac{6}{5}g_1^2|M_1|^2 + 6g_2^2|M_2|^2\right) + 2(m_L^2 Y_\tau^2 + T_\tau^2) \tag{4.93}$$

$$\beta^{(1)}_{m^2_{\tilde{\tau}_R}} = -\frac{24}{5}g_1^2|M_1|^2 + 4(m_L^2 Y_\tau^2 + T_\tau^2) \tag{4.94}$$

$$\beta^{(1)}_{m^2_{\tilde{e}_L}} = -\left(\frac{6}{5}g_1^2|M_1|^2 + 6g_2^2|M_2|^2\right) \tag{4.95}$$

$$\beta^{(1)}_{m^2_{\tilde{t}_L}} = -\left(\frac{2}{15}g_1^2|M_1|^2 + \frac{32}{3}g_3^2|M_3|^2 + 6g_2^2|M_2|^2\right) + 2(m_D^2 Y_b^2 + T_b^2) + 2(m_U^2 Y_t^2 + T_t^2) \tag{4.96}$$

$$\beta^{(1)}_{m^2_{\tilde{t}_R}} = -\frac{32}{15}g_1^2|M_1|^2 - \frac{32}{3}g_3^2|M_3|^2 + 4(m_U^2 Y_t^2 + T_t^2) \tag{4.97}$$

$$\beta^{(1)}_{m^2_{\tilde{b}_R}} = -\frac{8}{15}g_1^2|M_1|^2 - \frac{32}{3}g_3^2|M_3|^2 + 4(m_D^2 Y_b^2 + T_b^2) \tag{4.98}$$

$$\beta^{(1)}_{m^2_{\tilde{d}_R}} = -\frac{8}{15}g_1^2|M_1|^2 - \frac{32}{3}g_3^2|M_3|^2 \tag{4.99}$$

with

$$m_L^2 = m_{H_d}^2 + m_{\tilde{l}_3}^2 + m_{\tilde{\tau}_R}^2 \tag{4.100}$$

$$m_U^2 = m_{H_u}^2 + m_{\tilde{t}_R}^2 + m_{\tilde{q}_3} \tag{4.101}$$

$$m_D^2 = (m_{H_d}^2 + m_{\tilde{b}_R}^2 + m_{\tilde{q}_3}^2) \tag{4.102}$$

In order to be able to run down the RGEs from the GUT to the SUSY scale, one needs to know the values of the gauge and Yukawa couplings at the GUT scale. Therefore, the common approach is:

**a)** Calculate $g_i$, $Y_i$ at the low scale from experimental data (SM fermion masses, $e$, $\sin\Theta_W$, $\alpha_s$)

**b)** Run $g_i$, $Y_i$ to the GUT scale

**c)** Run all parameters down to the SUSY scale.

This procedure needs to be iterated if in step (1) the SUSY corrections in the matching between experimental data and $g_i$, $Y_i$ are included. However, we will use only a 'tree-level matching' and neglect flavour violation. Thus, we have[1]

$$Y_u = \sqrt{2}\frac{m_u}{v}\frac{\sqrt{1+\tan^2\beta}}{\tan\beta} \tag{4.103}$$

$$Y_d = \sqrt{2}\frac{m_d}{v}\sqrt{1+\tan^2\beta} \tag{4.104}$$

$$Y_l = \sqrt{2}\frac{m_l}{v}\sqrt{1+\tan^2\beta} \tag{4.105}$$

$$g_1 = \frac{e}{\cos\Theta_W} \tag{4.106}$$

$$g_2 = \frac{e}{\sin\Theta_W} \tag{4.107}$$

$$g_3 = \sqrt{4\pi\alpha_s} \tag{4.108}$$

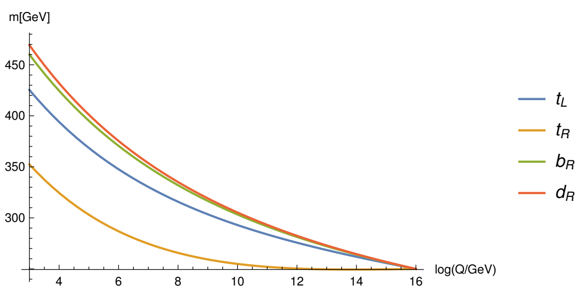
[1] the origin of the factors involving $\tan\beta$ are explained in detail when we discuss the Higgs sector of the MSSM

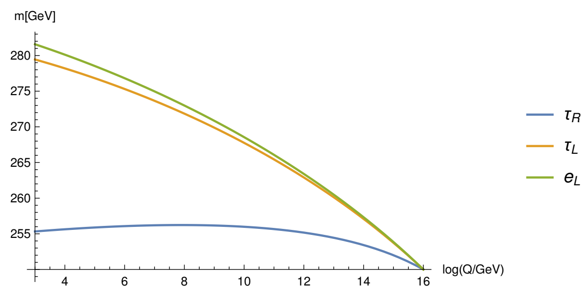
where $m_{u,dl}$ are the masses for up-quarks, down-quarks and leptons.

We use for the following example

$$m_0 = 250\text{ GeV}, \quad M_{1/2} = 200\text{ GeV}, \quad A_0 = 0, \quad \tan\beta = 10 \tag{4.109}$$

The running masses as function of the scale $Q$ evolve like this:



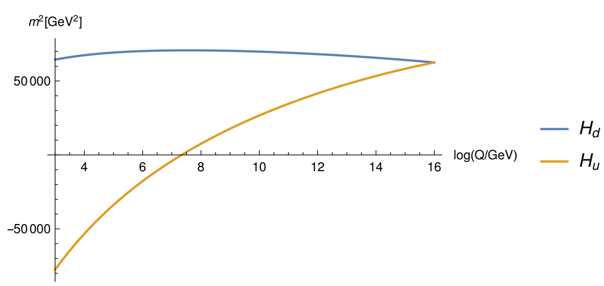




Finally, we can also check the running of the two Higgs soft masses. Their $\beta$ functions are given by

$$\beta^{(1)}_{m^2_{H_d}} = -\frac{6}{5}g_1^2|M_1|^2 - 6g_2^2|M_2|^2 + 6[m_D^2 Y_b^2 + T_b^2] + 2[m_L^2 Y_\tau^2 + T_\tau^2] \tag{4.110}$$

$$\beta^{(1)}_{m^2_{H_u}} = -\frac{6}{5}g_1^2|M_1|^2 - 6g_2^2|M_2|^2 + 6[m_U^2 Y_t^2 + T_t^2] \tag{4.111}$$

and we get:




Note, we have plotted here the mass squared. Thus, we can see that $m^2_{H_u}$ runs negative. This alone is not yet sufficient for EWSB, because

$$m_H^2 = m^2_{H_u} + \mu^2 < 0 \tag{4.112}$$

must hold to break the ew symmetry. However, also this condition is often satisfied in larger parameter regions of the CMSSM.

> In simple GUT scenarios of the MSSM like the CMSSM, one finds that the up-Higgs mass term runs negative and triggers EWSB. This is called 'radiative symmetry breaking' which seems more natural than the situation in the SM, where no good reason exists with $\mu^2_{\text{SM}}$ must be negative. The fast running of $m_{H_u}$ is caused by the top Yukawa coupling. Since radiative symmetry breaking wouldn't occur for lighter top masses, one might interpret this as a non-trivial relation between the large top mass and EWSB.

## 4.4 From gauge to mass eigenstates

We have so far only considered the so called 'gauge eigenstates' before electroweak symmetry breaking. However, these particles mix after EWSB to new 'mass eigenstates'. And these mass eigenstates the particles which we would observe at colliders if SUSY exists. This is analogue to the rotation of $B$ and $W^3$ bosons to $\gamma$ and $Z$-boson.
We are going to assume that the electroweak symmetry gets broken by Higgs VEVs

$$\langle H_d^0 \rangle = \frac{1}{\sqrt{2}} v_d \tag{4.113}$$

$$\langle H_u^0 \rangle = \frac{1}{\sqrt{2}} v_u \tag{4.114}$$

with $v = \sqrt{v_d^2 + v_u^2} \simeq 246$ GeV. And the ratio of $v_d$, $v_u$ defines

$$\tan\beta = \frac{v_u}{v_d} \tag{4.115}$$

Much more details about the Higgs sector will be given in a dedicated section. The mixing takes place between particles of same quantum numbers after symmetry breaking. We start with list of all mixing which we will discuss step by step in the following. In the Sfermion sector, the mixing depends on the assumptions about flavour violation, i.e. if it is important to include flavour violating effects or not:

**a) Sneutrino**

$$\tilde{\nu}_e, \tilde{\nu}_\mu, \tilde{\nu}_\tau \quad \rightarrow \quad \begin{cases} \text{Flavour mixing}: \tilde{\nu}_1, \tilde{\nu}_1, \tilde{\nu}_3 \\ \text{No Flavour mixing}: \tilde{\nu}_e, \tilde{\nu}_\mu, \tilde{\nu}_\tau \end{cases} \tag{4.116}$$

Thus, without flavour mixing, the gauge and mass eigenstates are identical.

**b) Charged Sleptons**

$$\tilde{e}_L, \tilde{\mu}_L, \tilde{\tau}_L, \tilde{e}_R, \tilde{\mu}_R, \tilde{\tau}_R \quad \rightarrow \begin{cases} \text{Flavour mixing}: \quad \tilde{e}_1, \dots, \tilde{e}_6 \\ \text{No Flavour mixing}: \tilde{e}_{1,2}(\simeq \tilde{e}_{L,R}), \tilde{\mu}_{1,2}(\simeq \tilde{\mu}_{L,R}), \tilde{\tau}_{1,2} \end{cases} \tag{4.117}$$

If flavour mixing is neglected also the left-right mixing for the first two generations is often assumed to be negligible, i.e. a mixing exists only for the third generation.

**c) Squarks**

$$\tilde{d}_L, \tilde{s}_L, \tilde{b}_L, \tilde{d}_R, \tilde{d}_R, \tilde{b}_R \quad \rightarrow \quad \begin{cases} \text{Flavour mixing}: \tilde{d}_1, \dots, \tilde{d}_6 \\ \text{No Flavour mixing}: \tilde{d}_{1,2}(\simeq \tilde{d}_{L,R}), \tilde{s}_{1,2}(\simeq \tilde{s}_{L,R}), \tilde{b}_{1,2} \end{cases} \tag{4.118}$$

$$\tilde{u}_L, \tilde{c}_L, \tilde{t}_L, \tilde{u}_R, \tilde{c}_R, \tilde{t}_R \quad \rightarrow \quad \begin{cases} \text{Flavour mixing}: \tilde{u}_1, \dots, \tilde{u}_6 \\ \text{No Flavour mixing}: \tilde{u}_{1,2}(\simeq \tilde{u}_{L,R}), \tilde{c}_{1,2}, \tilde{t}_{1,2} \end{cases} \tag{4.119}$$

**d) Neutralinos**

$$\tilde{B}, \tilde{W}^0, \tilde{H}_d^0, \tilde{H}_u^0 \quad \rightarrow \quad \tilde{\chi}_1^0, \dots, \tilde{\chi}_4^0 \tag{4.120}$$

**e) Charginos**

$$\tilde{H}_d^-, \tilde{W}^- / \tilde{H}_u^+, \tilde{W}^+ \qquad \rightarrow \qquad \tilde{\chi}_1^\pm, \tilde{\chi}_2^\pm \tag{4.121}$$

**f) Gluino** The gluino $\tilde{g}$ doesn't mix.

**g) Higgs**

$$H_d^0, H_u^0 \qquad \rightarrow \qquad h, H, G^0, A^0 \tag{4.122}$$

$$H_d^-, H_u^+ \qquad \rightarrow \qquad G^\pm, H^\pm \tag{4.123}$$

### 4.4.1 Sfermion Sector

#### 4.4.1.1 Sleptons

We start with the mixing in the Sfermion sector, more specifically with sleptons. For simplicity, we start with the assumption that only third generation Yukawas contribute and no flavour mixing is present. In that case, we get slightly different Lagrangians for the staus and the first two generations of sleptons. The important parts to understand the stau sector are

$$\begin{aligned} -\mathcal{L}_{\tilde{\tau}} =& \underbrace{m_{\tilde{\tau}_R}^2 |\tilde{\tau}_R|^2 + m_{\tilde{\tau}_L}^2 |\tilde{\tau}_L|^2 + (T_\tau H_d^0 \tilde{\tau}_R^* \tilde{\tau}_L + \text{c.c.})}_{\text{soft terms}} \\ &+ \underbrace{Y_\tau^2 |H_d|^2 (|\tilde{\tau}_L|^2 + |\tilde{\tau}_R|^2) - \mu Y_\tau (H_u \tilde{\tau}_R^* \tilde{\tau}_L + \text{c.c.})}_{\text{F-terms}} \\ &+ \underbrace{\frac{1}{2} g_1^2 (|H_d^0|^2 - |H_u^0|^2)(\frac{1}{2}|\tilde{\tau}_L|^2 - |\tilde{\tau}_R|^2) + \frac{1}{4} g_2^2 (|H_d^0|^2 - |H_u^0|^2)|\tilde{\tau}_L|^2}_{\text{D-terms}} \end{aligned} \tag{4.124}$$

The first line are the soft-terms, the second lines comes from the $F$-term potential and the third line from the $D$-term potential. After inserting the Higgs VEVs, we get

$$\begin{aligned} -\mathcal{L}_{\tilde{\tau}} =& m_{\tilde{\tau}_R}^2 |\tilde{\tau}_R|^2 + m_{\tilde{\tau}_L}^2 |\tilde{\tau}_L|^2 + (\frac{1}{\sqrt{2}} T_\tau v_d \tilde{\tau}_R^* \tilde{\tau}_L + \text{c.c.}) \\ &+ \frac{1}{2} Y_\tau^2 v_d^2 (|\tilde{\tau}_L|^2 + |\tilde{\tau}_R|^2) - \frac{1}{\sqrt{2}} \mu Y_\tau (v_u \tilde{\tau}_R^* \tilde{\tau}_L + \text{c.c.}) \\ &+ \frac{1}{4} g_1^2 (v_d^2 - v_u^2)(\frac{1}{2}|\tilde{\tau}_L|^2 - |\tilde{\tau}_R|^2) - \frac{1}{4} g_2^2 (v_d^2 + v_u^2)|\tilde{\tau}_L|^2 \end{aligned} \tag{4.125}$$

We see that there are mass contributions from SUSY breaking and EWSB, but also left-right mixing appears. Therefore, one writes the Lagrangian as

$$-\mathcal{L}_{\tilde{\tau}} = (\tilde{\tau}_L^* \tilde{\tau}_R^*) M_\tau^2 \begin{pmatrix} \tilde{\tau}_L \\ \tilde{\tau}_R \end{pmatrix} \tag{4.126}$$

with the mass matrix given by

$$M_{\tilde{\tau}}^2 = \begin{pmatrix} \frac{1}{2} v_d^2 |Y_\tau|^2 + \frac{1}{8}\left(g_1^2 - g_2^2\right)\left(v_d^2 - v_u^2\right) + m_{\tilde{\tau}_L}^2 & \frac{1}{\sqrt{2}}\left(v_d T_\tau^* - v_u \mu Y_\tau\right) \\ \frac{1}{\sqrt{2}}\left(v_d T_\tau^* - v_u \mu Y_\tau\right) & \frac{1}{2} v_d^2 |Y_\tau|^2 - \frac{1}{4} g_1^2 \left(v_d^2 - v_u^2\right) + m_{\tilde{\tau}_R}^2 \end{pmatrix} \tag{4.127}$$

The physical masses $m_{\tilde{\tau}_1}$, $m_{\tilde{\tau}_2}$ are the eigenvalues of this matrix. The relation between mass and gauge eigenstates is given by a rotation matrix

$$\begin{pmatrix} \tilde{\tau}_1 \\ \tilde{\tau}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta_\tau & \sin\theta_\tau \\ -\sin\theta_\tau & \cos\theta_\tau \end{pmatrix} \begin{pmatrix} \tilde{\tau}_L \\ \tilde{\tau}_R \end{pmatrix} \tag{4.128}$$

with

$$m_{\tilde{\tau}_1} < m_{\tilde{\tau}_2} \tag{4.129}$$

From this result, one can simply derive the mass matrices for the first two generations of of sleptons. They are given by

$$M^2_{\tilde{l}} = \begin{pmatrix} +\frac{1}{8}\Big(g_1^2 - g_2^2\Big)\Big(v_d^2 - v_u^2\Big) + m^2_{\tilde{\tau}_L} & 0 \\ 0 & -\frac{1}{4}g_1^2\Big(v_d^2 - v_u^2\Big) + m^2_{\tilde{\tau}_R} \end{pmatrix} \tag{4.130}$$

Thus, in our approximation there is no left-right mixing and the mass eigenstates correspond to gauge eigenstates. The mass eigenstates are therefore

$$\tilde{e}_L,\ \tilde{e}_R,\ \tilde{\mu}_L,\ \tilde{\mu}_R,\ \tilde{\tau}_1,\ \tilde{\tau}_2$$

The effects from first generations Yukawas and from flavour mixing are usually only a small perturbation. If one works in the fully general setup, a $6 \times 6$ matrix needs to be considered:

$$-\mathcal{L}_{\tilde{L}} = (\tilde{e}^*_L \tilde{\mu}^*_L \tilde{\tau}^*_L \tilde{e}^*_R \tilde{\mu}^*_R \tilde{\tau}^*_R) M^2_L \begin{pmatrix} \tilde{e}_L \\ \tilde{\mu}_L \\ \tilde{\tau}_L \\ \tilde{e}_R \\ \tilde{\mu}_R \\ \tilde{\tau}_R \end{pmatrix} \tag{4.131}$$

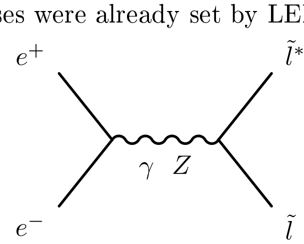
with

$$M^2_L = \begin{pmatrix} \frac{1}{2}v_d^2 Y_e^\dagger Y_e + \frac{1}{8}\Big(-g_2^2 + g_1^2\Big)\mathbf{1}\Big(-v_u^2 + v_d^2\Big) + m_l^2 & \frac{1}{\sqrt{2}}\Big(v_d T_e^\dagger - v_u \mu Y_e^\dagger\Big) \\ \frac{1}{\sqrt{2}}\Big(v_d T_e - v_u Y_e \mu^*\Big) & \frac{1}{2}v_d^2 Y_e Y_e^\dagger + \frac{1}{4}g_1^2\mathbf{1}\Big(-v_d^2 + v_u^2\Big) + m_e^2 \end{pmatrix} \tag{4.132}$$

In this case, a $6 \times 6$ matrix is needed to diagonalise the mass matrix. The eigenstates are called $\tilde{e}_i$ with $i = 1, \dots 6$. The ordering is

$$m_{\tilde{e}_1} < m_{\tilde{e}_2} < \cdots < m_{\tilde{e}_6} \tag{4.133}$$

Strong limits on charged Slepton masses were already set by LEP from processes as



which let to

$$m_{\tilde{l}} \gtrsim 100 \text{ GeV} \tag{4.134}$$

Depending on the neutralino mass and the assumption of the decay mode of the slepton, the LHC could improve this limit by a few hundred GeV.

#### 4.4.1.2 Squarks

The mass matrices for Squarks can be derived in exactly the same way by replacing the corresponding parameters. A bit care is just necessary for the terms proportional to $g_1^2$. The Lagrangian with flavour violation is written as

$$-\mathcal{L}_{\tilde{Q}} = (\tilde{d}_L^* \tilde{s}_L^* \tilde{b}_L^* \tilde{d}_R^* \tilde{s}_R^* \tilde{b}_R^*) M_D^2 \begin{pmatrix} \tilde{d}_L \\ \tilde{s}_L \\ \tilde{b}_L \\ \tilde{d}_R \\ \tilde{s}_R \\ \tilde{b}_R \end{pmatrix} + (\tilde{u}_L^* \tilde{c}_L^* \tilde{t}_L^* \tilde{u}_R^* \tilde{c}_R^* \tilde{t}_R^*) M_U^2 \begin{pmatrix} \tilde{u}_L \\ \tilde{c}_L \\ \tilde{t}_L \\ \tilde{u}_R \\ \tilde{c}_R \\ \tilde{t}_R \end{pmatrix} \tag{4.135}$$

with

$$M_D^2 = \begin{pmatrix} -\frac{1}{24}\Big(3g_2^2 + g_1^2\Big)\mathbf{1}\Big(-v_u^2 + v_d^2\Big) + \frac{1}{2}\Big(2m_q^2 + v_d^2 Y_d^\dagger Y_d\Big) & \frac{1}{\sqrt{2}}\Big(v_d T_d^\dagger - v_u \mu Y_d^\dagger\Big) \\ \frac{1}{\sqrt{2}}\Big(v_d T_d - v_u Y_d \mu^*\Big) & \frac{1}{12} g_1^2 \mathbf{1}\Big(-v_d^2 + v_u^2\Big) + \frac{1}{2}\Big(2m_d^2 + v_d^2 Y_d Y_d^\dagger\Big) \end{pmatrix} \tag{4.136}$$

$$M_U^2 = \begin{pmatrix} -\frac{1}{24}\Big(-3g_2^2 + g_1^2\Big)\mathbf{1}\Big(-v_u^2 + v_d^2\Big) + \frac{1}{2}\Big(2m_q^2 + v_u^2 Y_u^\dagger Y_u\Big) & \frac{1}{\sqrt{2}}\Big(-v_d \mu Y_u^\dagger + v_u T_u^\dagger\Big) \\ \frac{1}{\sqrt{2}}\Big(-v_d Y_u \mu^* + v_u T_u\Big) & \frac{1}{2}\Big(2m_u^2 + v_u^2 Y_u Y_u^\dagger\Big) + \frac{1}{6} g_1^2 \mathbf{1}\Big(-v_u^2 + v_d^2\Big) \end{pmatrix} \tag{4.137}$$

From diagonalising these matrices, one obtains twice six mass eigenstates $\tilde{d}_i$, $\tilde{u}_i$ with $i = 1, \dots 6$. The relation between mass and gauge eigenstates is given by unitary $6 \times 6$ matrices called $Z^D$ and $Z^U$.

The full $6 \times 6$ mass matrices are usually needed when considering flavour observables. In other cases, flavour mixing as well as the mixing of the first two generations can often be neglected. Thus, only the two stops and sbottom mix to two mass eigenstates each

$$-\mathcal{L}_{\tilde{Q}_3} = (\tilde{b}_L^* \tilde{b}_R^*) M_{\tilde{b}}^2 \begin{pmatrix} \tilde{b}_L \\ \tilde{b}_R \end{pmatrix} + (\tilde{t}_L^* \tilde{t}_R^*) M_{\tilde{t}}^2 \begin{pmatrix} \tilde{t}_L \\ \tilde{t}_R \end{pmatrix} \tag{4.138}$$

with

$$M_{\tilde{b}}^2 = \begin{pmatrix} -\frac{1}{24}\Big(3g_2^2 + g_1^2\Big)\Big(-v_u^2 + v_d^2\Big) + \frac{1}{2}\Big(2m_{\tilde{b}_L}^2 + v_d^2 |Y_b|^2\Big) & \frac{1}{\sqrt{2}}\Big(v_d T_b^* - v_u \mu Y_b^*\Big) \\ \frac{1}{\sqrt{2}}\Big(v_d T_b - v_u Y_b \mu^*\Big) & \frac{1}{12} g_1^2\Big(-v_d^2 + v_u^2\Big) + \frac{1}{2}\Big(2m_{\tilde{b}_R}^2 + v_d^2 |Y_b|^2\Big) \end{pmatrix} \tag{4.139}$$

$$M_{\tilde{t}}^2 = \begin{pmatrix} -\frac{1}{24}\Big(-3g_2^2 + g_1^2\Big)\Big(-v_u^2 + v_d^2\Big) + \frac{1}{2}\Big(2m_{\tilde{t}_L}^2 + v_u^2 |Y_t|^2\Big) & \frac{1}{\sqrt{2}}\Big(-v_d \mu Y_t^* + v_u T_t^*\Big) \\ \frac{1}{\sqrt{2}}\Big(-v_d Y_t \mu^* + v_u T_t\Big) & \frac{1}{2}\Big(2m_{\tilde{t}_R}^2 + v_u^2 |Y_t|^2\Big) + \frac{1}{6} g_1^2\Big(-v_u^2 + v_d^2\Big) \end{pmatrix} \tag{4.140}$$

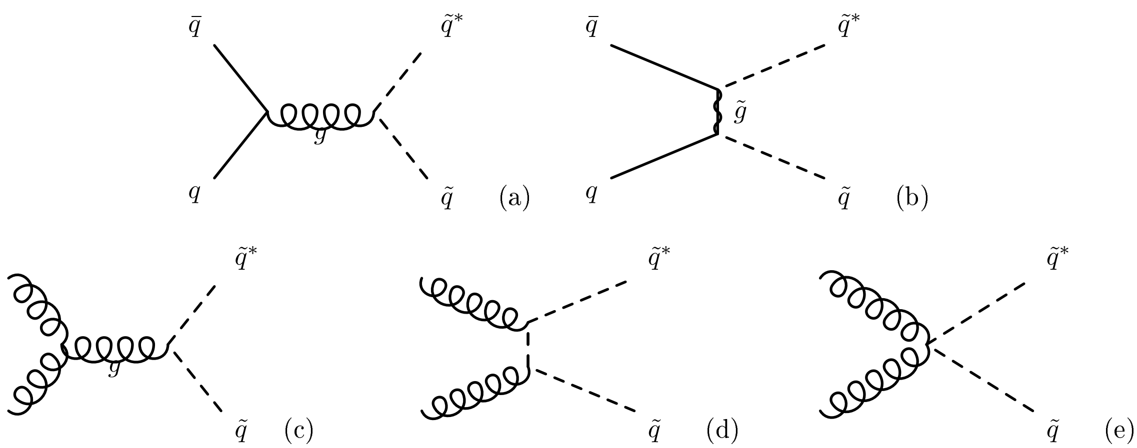
These mass matrices are then diagonalised by two orthogonal matrices which depend only on the angles $\Theta_{\tilde{t}}$ respectively $\Theta_{\tilde{b}}$.

The importance of these mixing angles are for instance visible from the couplings of the (SM-like) Higgs to the stop mass eigenstates which are

$$v_{h\tilde{t}_1\tilde{t}_1} = i(v_u \cos\alpha Y_t^2 + \sqrt{2}\cos\Theta_{\tilde{t}}\sin\Theta_{\tilde{t}}(\mu\sin\alpha Y_t + \cos\alpha T_t)) + \mathcal{O}(e^2) \tag{4.141}$$

We skipped here all terms sub-dominant proportional to $e^2$, and the angle $\alpha$ is the rotation angle in the Higgs sector which we will discuss in detail later. We can see from this expression that the second term becomes only very large if there is a large mixing in the stop sector.

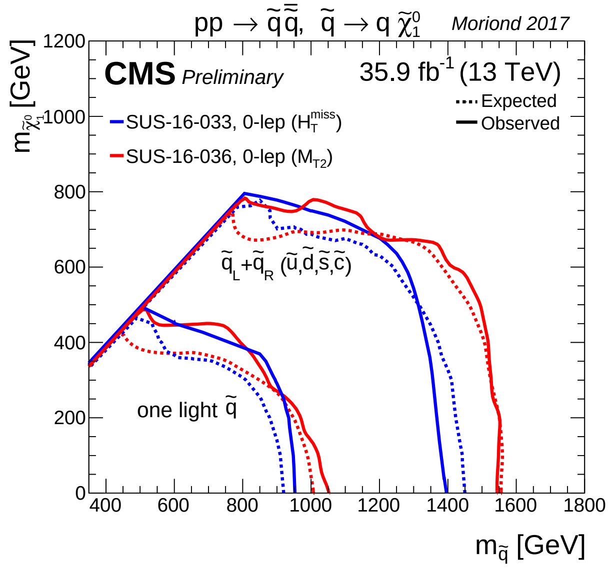
The production rate of squarks is much larger at the LHC than it was at LEP. Therefore, the limits on the squark masses have strongly increased in the last years. The dominant production processes are




Note, that (b) is highly suppressed for third generation squarks. The limit on the squark masses depend not only on the production rate but also on the decays/branching ratios. The strongest limits come from simplified models in which a 100% branching ratio of the squark into a (massless) neutralino LSP and a quark is considered. The current limits on the masses of first/second generation squarks under these assumptions are

$$m_{\tilde{q}} > 1500 \text{ GeV} \tag{4.142}$$

However, they can be also much weaker, e.g. if the neutralino and squark is nearly degenerated:

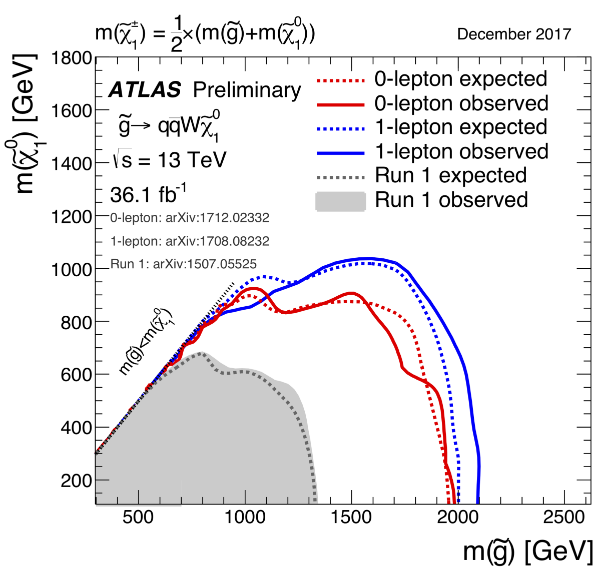



### 4.4.2 The Gluino

The gluino doesn't mix with the other MSSM fields after EWSB and SUSY breaking because it is a colour octet. Therefore, it is a rather 'simple' compared to the other mass eigenstates which involve rotation matrices. The mass of the gluino is given by

$$M_{\tilde{g}} = |M_3| \tag{4.143}$$

Since the gluino is strongly interacting, it is one of the most important fields when searching for SUSY at Hadron colliders. However, also the obtained limit on the gluino mass depends strongly on the underlying assumptions, especially the LSP mass:



### 4.4.3 Chargino and Neutralino Sector

We turn now to the electroweakino sector which is formed by the electroweak gauginos and the Higgsinos. The relevant terms in the Lagrangian to understand the mixing are

$$\begin{aligned}-\mathcal{L}_{\tilde{\chi}} =& \underbrace{M_1\tilde{B}\tilde{B} + M_2\tilde{W}^a\tilde{W}^a}_{\text{soft-terms}} + \underbrace{\mu\tilde{H}_d\tilde{H}_u}_{\text{superpotential}} + \text{c.c.} \\ &+ \underbrace{\frac{1}{2}\sqrt{2}g_1\tilde{B}(\tilde{H}_dH_d^* - \tilde{H}_uH_u^*) + \frac{1}{2}\sqrt{2}g_2\sigma^a_{ij}\tilde{W}^a(\tilde{H}^i_dH^{j*}_d + \tilde{H}^i_uH^{j*}_u)}_{\text{gaugino-fermion-scalar interactions}} + \text{c.c.} \qquad (4.144) \\ =& M_1\tilde{B}\tilde{B} + M_2(\tilde{W}^1\tilde{W}^1 + \tilde{W}^2\tilde{W}^2 + \tilde{W}^3\tilde{W}^3) + \mu(\tilde{H}^0_d\tilde{H}^0_u - \tilde{H}^-_d\tilde{H}^+_u) + \text{c.c.} \\ &+ \frac{1}{2}\sqrt{2}g_1\tilde{B}(\tilde{H}^0_dH^{0*}_d + \tilde{H}^-_dH^{-*}_d) - \frac{1}{2}\sqrt{2}g_2\Big[\tilde{W}^1\left(\tilde{H}^0_dH^{-*}_d + \tilde{H}^-_dH^{0*}_d + \tilde{H}^0_uH^{+*}_u + \tilde{H}^+_uH^{0*}_u\right) \\ &+ i\tilde{W}^2\left(\tilde{H}^0_dH^{-*}_d - \tilde{H}^-_dH^{0*}_d - \tilde{H}^0_uH^{+*}_u + \tilde{H}^+_uH^{0*}_u\right) + \tilde{W}^3\left(\tilde{H}^0_dH^{0*}_d - \tilde{H}^-_dH^{-*}_d - \tilde{H}^0_uH^{u*}_d + \tilde{H}^+_uH^{+*}_u\right)\Big] + \text{c.c.}\end{aligned} \tag{4.145}$$

$\tilde{W}^1$ and $\tilde{W}^2$ get rotated similar to the $W$-boson to get electric eigenstates:

$$\tilde{W}^\pm = \frac{1}{\sqrt{2}}\left(\tilde{W}^1 \pm i\tilde{W}^2\right) \tag{4.146}$$

After inserting the Higgs VEVs, we finally have

$$\begin{aligned}-\mathcal{L}_{\tilde{\chi}} =& M_1\tilde{B}\tilde{B} + M_2(\tilde{W}^+\tilde{W}^- + \tilde{W}^3\tilde{W}^3) + \mu(\tilde{H}^0_d\tilde{H}^0_u - \tilde{H}^-_d\tilde{H}^+_u) + \text{c.c.} \\ &+ \frac{1}{2}g_1\tilde{B}\tilde{H}^0_dv_d + \frac{1}{2}g_2\left(\tilde{H}^0_d\tilde{W}^3v_d + \tilde{H}^0_u\tilde{W}^3v_u + \tilde{H}^-_d\tilde{W}^+v_d + v_u\tilde{H}^+_u\tilde{W}^-\right) + \text{c.c.}\end{aligned} \tag{4.147}$$

We want to write our Lagrangian in the form

$$-\mathcal{L}_{\tilde{\chi}} = \frac{1}{2}(\tilde{B}\tilde{W}^3\tilde{H}^0_d\tilde{H}^0_u)M_{\tilde{\chi}^0}\begin{pmatrix}\tilde{B}\\ \tilde{W}^3\\ \tilde{H}^0_d\\ \tilde{H}^0_u\end{pmatrix} + (\tilde{W}^+\tilde{H}^+_u)M_{\tilde{\chi}^\pm}\begin{pmatrix}\tilde{W}^-\\ \tilde{H}^-_d\end{pmatrix} \tag{4.148}$$

We see an important difference between the neutral and charged sector:

- In the neutral sector, the mass matrix is symmetric. We will need one matrix to diagonalise it. Therefore, the four gauge eigenstates $\{\tilde{B}, \tilde{W}^3, \tilde{H}^0_d, \tilde{H}^0_u\}$ will mix to four Majorana fermions. We call them *neutralinos* $\tilde{\chi}^0_1, \dots \tilde{\chi}^0_4$.
- In the charged sector, the mass matrix is not symmetric. We will need two matrices to diagonalise it. Therefore, the four $\{\tilde{W}^+, \tilde{H}^+_u, \tilde{W}^-, \tilde{H}^-_d\}$ will mix to two Dirac fermions. We call them *charginos* $\tilde{\chi}^\pm_1, \tilde{\chi}^\pm_2$.

#### 4.4.3.1 Neutralinos

The mass matrix of the neutralinos is given by

$$M_{\tilde{\chi}^0} = \begin{pmatrix} M_1 & 0 & -\frac{1}{2}g_1v_d & \frac{1}{2}g_1v_u \\ 0 & M_2 & \frac{1}{2}g_2v_d & -\frac{1}{2}g_2v_u \\ -\frac{1}{2}g_1v_d & \frac{1}{2}g_2v_d & 0 & -\mu \\ \frac{1}{2}g_1v_u & -\frac{1}{2}g_2v_u & -\mu & 0 \end{pmatrix} \tag{4.149}$$

This matrix is diagonalised by an unitary matrix $N$:

$$N^* M_{\tilde{\chi}^0} N^\dagger = \begin{pmatrix} m_{\tilde{\chi}_1^0} & 0 & 0 & 0 \\ 0 & m_{\tilde{\chi}_2^0} & 0 & 0 \\ 0 & 0 & m_{\tilde{\chi}_3^0} & 0 \\ 0 & 0 & 0 & m_{\tilde{\chi}_4^0} \end{pmatrix} \tag{4.150}$$

The ordering of the mass is that

$$|m_{\tilde{\chi}_1^0}| < |m_{\tilde{\chi}_2^0}| < |m_{\tilde{\chi}_3^0}| < |m_{\tilde{\chi}_4^0}| \tag{4.151}$$

holds. Note, that masses can become negative. One shouldn't confuse this with a tachyon because the sign is not physical: it can be absorbed in the rotation matrix $N$ by multiplying the corresponding line with $'i'$. $N$ rotates the four Weyl fermions to new states labelled $\lambda_i^0$. The relation between the gauge and mass eigenstates is

$$\tilde{B} = \sum_j N_{j1}^* \lambda_j^0 \tag{4.152}$$

$$\tilde{W}^0 = \sum_j N_{j2}^* \lambda_j^0 \tag{4.153}$$

$$\tilde{H}_d^0 = \sum_j N_{j3}^* \lambda_j^0 \tag{4.154}$$

$$\tilde{H}_u^0 = \sum_j N_{j4}^* \lambda_j^0 \tag{4.155}$$

From $\lambda_i^0$ we can build four Majorana fermions

$$\tilde{\chi}_i^0 = \begin{pmatrix} \lambda_i^0 \\ \lambda_i^{0*} \end{pmatrix} \tag{4.156}$$

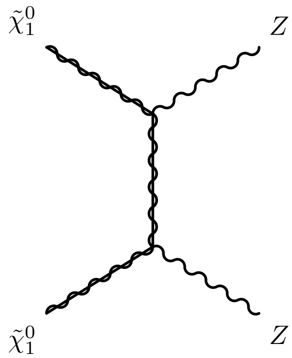
The neutralino play a very important role in SUSY models because the lightest neutralino is often the dark matter candidate. Depending on the hierarchy in the Lagrangian parameters, the nature of the lightest neutralino is different:

**a)** $M_1 < M_2,\mu$: Bino dark matter

**b)** $M_2 < M_1,\mu$: Wino dark matter

**c)** $\mu < M_1$, $M_2$: Higgsino dark matter

Although one has in all three cases neutralino dark matter, the properties of the dark matter particle can be quite different. This becomes already obvious when checking the neutralino–Z vertex. This vertex is important for the annihilation channel $\tilde{\chi}_1^0 \tilde{\chi}_1^0 \to ZZ$:

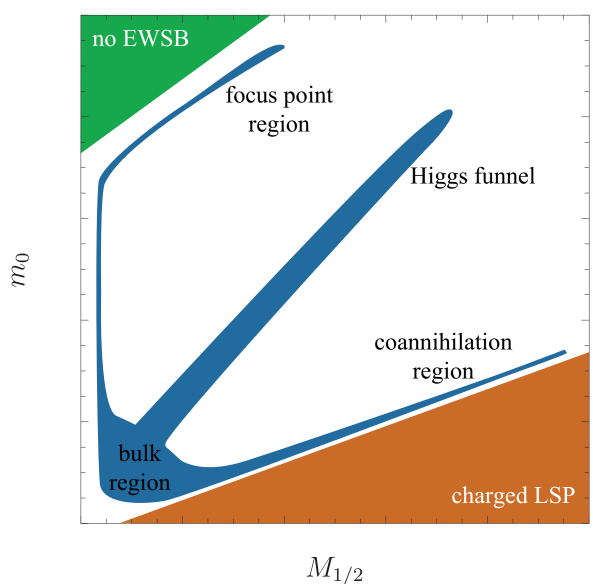



The expression for the neutralino Z-vertex in four component notation is:

$$c_{\tilde{\chi}_1^0\tilde{\chi}_1^0 Z} = -\frac{i}{2}(g_1 \sin\Theta_W + g_2 \cos\Theta_W)(N_{13}^* N_{13} - N_{14}^* N_{14})\gamma_\mu \begin{pmatrix} P_L \\ -P_R \end{pmatrix} \tag{4.157}$$

Thus, only a Higgsino LSP can annihilate in these final states. Bino and Wino LSPs need other channels which are usually less efficient. Therefore, it's much easier to obtain the correct relic density for a Higgsino LSP.

The dark matter properties in the CMSSM can be summarised in the following plot:




The different regions are

**a) Bulk region**: rather light sfermions trigger $\tilde{\chi}^0\tilde{\chi}^0 \to f\bar{f}$ via a $t$-channel exchange

**b) Co-annihilation region**: the stau is the NLSP and close in mass to the LSP. This opens new annihilation channels like $\tilde{\chi}^0\tilde{\tau}_1 \to Z\tau$

**c) Focus-point region**: in the focus point region, $\mu$ is small and the lightest neutralino is a Higgsino, i.e. $\tilde{\chi}^0\tilde{\chi}^0 \to ZZ/WW$ works very efficiently

**d) Higgs funnel**: there is a resonant annihilation $\tilde{\chi}^0\tilde{\chi}^0 \to A \to f\bar{f}$ for $m_A \simeq 2m_{\tilde{\chi}}$.

#### 4.4.3.2 Charginos

In the charged sector, we have the following mass matrix

$$M_{\tilde{\chi}^\pm} = \begin{pmatrix} M_2 & \frac{1}{\sqrt{2}}g_2 v_u \\ \frac{1}{\sqrt{2}}g_2 v_d & \mu \end{pmatrix} \tag{4.158}$$

This matrix is obviously not symmetric in general, therefore one needs to rotation matrices $U$ and $V$ to diagonalise it

$$U^* M_{\tilde{\chi}^\pm} V^\dagger = \begin{pmatrix} m_{\tilde{\chi}_1^\pm} & 0 \\ 0 & m_{\tilde{\chi}_2^\pm} \end{pmatrix} \tag{4.159}$$

We assume again that the eigenstates are ordered by their mass

$$|m_{\tilde{\chi}_1^\pm}| < |m_{\tilde{\chi}_2^\pm}| \tag{4.160}$$

The two matrices $U$ and $V$ can be obtained with a so called singular value decomposition of the matrix $M_{\tilde{\chi}^\pm}$. However, in practice it is often easier to consider the squared matrices and use the relations:

$$U^* M_{\tilde{\chi}^\pm} M^\dagger_{\tilde{\chi}^\pm} U^\dagger = \begin{pmatrix} m^2_{\tilde{\chi}_1^\pm} & 0 \\ 0 & m^2_{\tilde{\chi}_2^\pm} \end{pmatrix} \tag{4.161}$$

$$V^* M^\dagger_{\tilde{\chi}^\pm} M_{\tilde{\chi}^\pm} V^\dagger = \begin{pmatrix} m^2_{\tilde{\chi}_1^\pm} & 0 \\ 0 & m^2_{\tilde{\chi}_2^\pm} \end{pmatrix} \tag{4.162}$$

The matrices $U$ and $V$ rotate the positive and negative charged fields separately:

$$\begin{pmatrix} \lambda_1^+ \\ \lambda_2^+ \end{pmatrix} = V \begin{pmatrix} \tilde{W}^+ \\ \tilde{H}_u^+ \end{pmatrix} \tag{4.163}$$

$$\begin{pmatrix} \lambda_1^- \\ \lambda_2^- \end{pmatrix} = U \begin{pmatrix} \tilde{W}^- \\ \tilde{H}_d^- \end{pmatrix} \tag{4.164}$$

$$\tag{4.165}$$

The Dirac fermions are built from the two Weyl fermion via

$$\tilde{\chi}_i^\pm = \begin{pmatrix} \lambda_i^+ \\ (\lambda_i^-)^* \end{pmatrix} \tag{4.166}$$

## 4.5 Higgs Physics

We turn now to the Higgs sector of the MSSM. The part of the Lagrangian which fixes the masses of the Higgs reads in terms of gauge eigenstates

$$\begin{aligned} -\mathcal{L}_H &= \underbrace{|\mu|^2[(|H_u^0|^2+|H_u^+|^2)+(|H_d^0|^2+|H_d^-|^2)]}_{\text{F-terms}} \\ &+ \underbrace{[B_\mu\,(H_u^+H_d^- - H_u^0H_d^0)+\text{c.c.}]+m_{H_u}^2(|H_u^0|^2+|H_u^+|^2)+m_{H_d}^2(|H_d^0|^2+|H_d^-|^2)}_{\text{Soft terms}} \\ &+ \underbrace{\frac{1}{8}(g^2+g'^2)(|H_u^0|^2+|H_u^+|^2-|H_d^0|^2-|H_d^-|^2)^2+\frac{1}{2}g^2|H_u^+H_d^{0*}+H_u^0H_d^{-*}|^2}_{\text{D}-terms} \end{aligned} \tag{4.167}$$

The only potentially complex parameters are $B_\mu$ and $\mu$. However, possible phases can be rotates away by a re-definition of the Higgs fields. Therefore, there is no CP violation at tree-level in the MSSM Higgs sector.
After EWSB, the neutral Higgs fields decompose as

$$H_i^0 = \frac{1}{\sqrt{2}}\left(\phi_i + i\sigma_i + v_i\right) \qquad i = d, u \tag{4.168}$$

with $\tan\beta = \frac{v_u}{v_d}$, $v = \sqrt{v_d^2+v_u^2} = 246$ GeV. There are three different parts of the Higgs sector

- **CP even**: $\phi_d$, $\phi_u$ mix to two eigenstates $h_1$, $h_2$
- **CP odd**: $\sigma_d$, $\sigma_u$ mix to two eigenstates $G$, $A$
- **charged**: $H_u^+$, $H_d^-$ mix to two eigenstates $G^\pm$, $H^\pm$

This categorisation only holds if one assumes that CP is not broken. While at tree-level any CP phase in the Higgs sector can be rotated away, one can have CP violation via loop corrections from all the other phases from soft-SUSY breaking. In that case $h_1$, $h_2$ and $G$, $A$ would further mix. However, we will always work with the assumption that CP is not violated.

### 4.5.1 CP odd sector

We start with the CP odd sector. We write the Lagrangian as

$$\mathcal{L}_A = (\sigma_d\sigma_u)m_A^2\begin{pmatrix}\sigma_d\\ \sigma_u\end{pmatrix} \tag{4.169}$$

with

$$m_A^2 = \begin{pmatrix} \frac{1}{8}\Big(g_1^2+g_2^2\Big)\Big(-v_u^2+v_d^2\Big)+m_{H_d}^2+|\mu|^2 & \Re\Big(B_\mu\Big) \\ \Re\Big(B_\mu\Big) & -\frac{1}{8}\Big(g_1^2+g_2^2\Big)\Big(-v_u^2+v_d^2\Big)+m_{H_u}^2+|\mu|^2 \end{pmatrix} \tag{4.170}$$

This matrix can be further simplified by using the so called *tadpole equations*: these are the conditions that one sits at the bottom from the potential:

$$\frac{\partial V}{\partial v_i} \equiv 0 \tag{4.171}$$

These equations read in our case

$$\frac{\partial V}{\partial v_d} = -v_u B_\mu + \frac{1}{8}\Big(g_1^2 + g_2^2\Big) v_d \Big(v_d^2 - v_u^2\Big) + v_d\Big(m_{H_d}^2 + |\mu|^2\Big) \tag{4.172}$$

$$\frac{\partial V}{\partial v_u} = -v_d B_\mu + \frac{1}{8}\Big(g_1^2 + g_2^2\Big) v_u \Big(v_u^2 - v_d^2\Big) + v_u\Big(m_{H_u}^2 + |\mu|^2\Big) \tag{4.173}$$

We can use these equations to eliminate two parameters from the potential. Common choice are to solve the equations either with respect to $\mu$, $B_\mu$ or $m_{H_d}^2$, $m_{H_u}^2$. For the moment, we use the second option and find

$$m_{H_d}^2 = \frac{1}{v_d}\left(v_u B_\mu - \frac{1}{8}\Big(g_1^2 + g_2^2\Big) v_d \Big(v_d^2 - v_u^2\Big) - v_d|\mu|^2\right) \tag{4.174}$$

$$m_{H_u}^2 = \frac{1}{v_u}\left(v_d B_\mu - \frac{1}{8}\Big(g_1^2 + g_2^2\Big) v_u \Big(v_u^2 - v_d^2\Big) - v_u|\mu|^2\right) \tag{4.175}$$

When we insert this in $m_A^2$, the matrix becomes rather simple:

$$m_A^2 = \begin{pmatrix} \frac{v_u}{v_d} B_\mu & B_\mu \\ B_\mu & \frac{v_d}{v_u} B_\mu \end{pmatrix} \tag{4.176}$$

$$= B_\mu \begin{pmatrix} \tan\beta & 1 \\ 1 & 1/\tan\beta \end{pmatrix} \tag{4.177}$$

The eigenvalues of this matrix are

$$m_G^2 = 0 \tag{4.178}$$

$$m_A^2 = \frac{1 + \tan\beta^2}{\tan\beta} B_\mu \tag{4.179}$$

The state with zero mass is the Goldstone of the $Z$-boson because we have performed the calculation in Landau gauge. Also the matrix which brings the CP odd gauge to the mass eigenstates is completely fixed in terms of $\tan\beta$:

$$\begin{pmatrix} G \\ A \end{pmatrix} = \begin{pmatrix} -\cos\beta & \sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \sigma_d \\ \sigma_u \end{pmatrix} \tag{4.180}$$

### 4.5.2 CP even sector

We turn now to the CP even sector. As usual, we want to write

$$-\mathcal{L}_H = \frac{1}{2}(\phi_d \phi_u) M_H^2 \begin{pmatrix} \phi_d \\ \phi_u \end{pmatrix} \tag{4.181}$$

and find for the scalar mass matrix

$$M_H^2 = \begin{pmatrix} \frac{1}{8}\Big(g_1^2 + g_2^2\Big)\Big(3v_d^2 - v_u^2\Big) + m_{H_d}^2 + |\mu|^2 & -\frac{1}{4}\Big(g_1^2 + g_2^2\Big) v_d v_u - B_\mu \\ -\frac{1}{4}\Big(g_1^2 + g_2^2\Big) v_d v_u - B_\mu & -\frac{1}{8}\Big(g_1^2 + g_2^2\Big)\Big(-3v_u^2 + v_d^2\Big) + m_{H_u}^2 + |\mu|^2 \end{pmatrix} \tag{4.182}$$

After replacing $m_{H_d}^2$ and $m_{H_u}^2$ by the solutions of the tadpole equations, and trading $B_\mu$ for $M_A^2$, the matrix reads

$$M_H^2 = \begin{pmatrix} \frac{4M_A^2 \tan\beta^2 + (g_1^2+g_2^2)v^2}{4(\tan\beta^2+1)} & -\frac{\tan\beta\left((g_1^2+g_2^2)v^2+4M_A^2\right)}{4(\tan\beta^2+1)} \\ -\frac{\tan\beta\left((g_1^2+g_2^2)v^2+4M_A^2\right)}{4(\tan\beta^2+1)} & \frac{(g_1^2+g_2^2)\tan\beta^2 v^2+4M_A^2}{4(\tan\beta^2+1)} \end{pmatrix} \tag{4.183}$$

We can further simplify the matrix by using $M_Z^2 = \frac{1}{4}(g_1^2+g_2^2)v^2$

$$M_H^2 = \begin{pmatrix} \frac{M_A^2 \tan\beta^2 + M_Z^2}{\tan\beta^2+1} & -\frac{(M_A^2+M_Z^2)\tan\beta}{\tan\beta^2+1} \\ -\frac{(M_A^2+M_Z^2)\tan\beta}{\tan\beta^2+1} & \frac{M_Z^2\tan\beta^2+M_A^2}{\tan\beta^2+1} \end{pmatrix} \tag{4.184}$$

The eigenvalues of this matrix are

$$m_{h_{1,2}}^2 = \frac{1}{2}\left(M_A^2 + M_Z^2 \mp \sqrt{(M_A^2 - M_Z)^2 + 4M_Z^2 M_A^2 \sin^2 2\beta}\right) \tag{4.185}$$

In the so called decoupling limit, $M_A^2 \gg M_Z^2$ the eigenvalues become

$$m_h^2 = M_Z^2 \cos^2 2\beta \tag{4.186}$$
$$m_H^2 = M_A^2 \tag{4.187}$$

The tree-level mass of the light CP even Higgs is bounded in the MSSM by

$$m_h < M_Z \cos 2\beta \tag{4.188}$$

Thus, this model would be ruled out immediately if it is not possible to increase the Higgs mass. Therefore, we need to check the loop corrections.

#### 4.5.2.1 Higgs couplings

The scalar mass and the gauge eigenstates are related by

$$\begin{pmatrix} h \\ H \end{pmatrix} = \begin{pmatrix} -\sin\alpha & \cos\alpha \\ \cos\alpha & \sin\alpha \end{pmatrix} \begin{pmatrix} \phi_d \\ \phi_u \end{pmatrix} \tag{4.189}$$

At tree-level, the following relations between $\alpha$ and $\beta$ exist:

$$\frac{\sin 2\alpha}{\sin 2\beta} = -\left(\frac{m_H^2 + m_h^2}{m_H^2 - m_h^2}\right), \qquad \frac{\tan 2\alpha}{\tan 2\beta} = \left(\frac{M_A^2 + M_Z}{M_A^2 - M_Z}\right), \tag{4.190}$$

Thus, for $M_A \gg m_h, M_Z$ both angles are related by $\alpha = \beta - \frac{1}{2}\pi$.
The mass matrices SM fermions are given by

$$M_d = \frac{1}{\sqrt{2}} Y_d v_d \tag{4.191}$$

$$M_u = \frac{1}{\sqrt{2}} Y_u v_u \tag{4.192}$$

$$M_l = \frac{1}{\sqrt{2}} Y_l v_d \tag{4.193}$$

In the limit of vanishing flavour mixing, we get

$$\begin{pmatrix} m_d \\ m_s \\ m_b \end{pmatrix} = \frac{1}{\sqrt{2}} v_d \begin{pmatrix} Y_d \\ Y_s \\ Y_b \end{pmatrix} \qquad \begin{pmatrix} m_u \\ m_c \\ m_t \end{pmatrix} = \frac{1}{\sqrt{2}} v_u \begin{pmatrix} Y_u \\ Y_c \\ Y_t \end{pmatrix} \qquad \begin{pmatrix} m_e \\ m_\mu \\ m_\tau \end{pmatrix} = \frac{1}{\sqrt{2}} v_d \begin{pmatrix} Y_e \\ Y_\mu \\ Y_\tau \end{pmatrix} \tag{4.194}$$

We can now check how the Higgs couplings are modified compared to the SM.

**a) Couplings to fermions**: under the assumption that only third generation Yukawa couplings are non-negligible, the Lagrangian for the gauge eigenstates is

$$\begin{aligned}
\mathcal{L} =& Y_t q_L t_R H_u + Y_b q_L b_R H_d + Y_\tau l e_R H_d + \text{c.c.} && (4.195)\\
=& Y_t t_L t_R H_u^0 + Y_b b_L b_R H_d^0 + Y_\tau e_L e_R H_d^0 + \cdots + \text{c.c.} && (4.196)\\
=& \frac{\sqrt{2} m_t}{v_u} t_L t_R H_u^0 + \frac{\sqrt{2} m_b}{v_d} b_L b_R H_d^0 + \frac{\sqrt{2} m_\tau}{v_d} e_L e_R H_d^0 + \cdots + \text{c.c.} && (4.197)\\
=& \frac{\sqrt{2} m_t}{v \sin\beta} t_L t_R H_u^0 + \frac{\sqrt{2} m_b}{v \cos\beta} b_L b_R H_d^0 + \frac{\sqrt{2} m_\tau}{v \cos\beta} e_L e_R H_d^0 + \cdots + \text{c.c.} && (4.198)\\
=& \frac{\sqrt{2} m_t}{v \sin\beta} t_L t_R h \cos\alpha + \frac{\sqrt{2} m_b}{v \cos\beta} b_L b_R h \sin\alpha + \frac{\sqrt{2} m_\tau}{v \cos\beta} e_L e_R h \sin\alpha + \cdots + \text{c.c.} && (4.199)\\
=& \underbrace{\frac{\sqrt{2} m_t}{v} t_L t_R h}_{c_{tth}^{\text{SM}}} \frac{\cos\alpha}{\sin\beta} + \underbrace{\frac{\sqrt{2} m_b}{v} b_L b_R h}_{c_{bbh}^{\text{SM}}} \frac{\sin\alpha}{\cos\beta} + \underbrace{\frac{\sqrt{2} m_\tau}{v} e_L e_R h}_{c_{\tau\tau h}^{\text{SM}}} \frac{\sin\alpha}{\cos\beta} + \cdots + \text{c.c.} && (4.200)
\end{aligned}$$

Similarly, one can derive the changes in the couplings compared to the SM for the other scalars. The results are

| | $u$ | $d$ | $l$ |
|---|---|---|---|
| $h$ | $\frac{\cos\alpha}{\sin\beta}$ | -$\frac{\sin\alpha}{\cos\beta}$ | -$\frac{\sin\alpha}{\cos\beta}$ |
| $H$ | $\frac{\sin\alpha}{\sin\beta}$ | $\frac{\cos\alpha}{\cos\beta}$ | $\frac{\cos\alpha}{\cos\beta}$ |
| $A$ | $\frac{1}{\tan\beta}$ | $\tan\beta$ | $\tan\beta$ |

**b) Couplings to Vectors**: we can repeat the exercise and consider the couplings to vector bosons

$$\begin{aligned}
\mathcal{L} =& \frac{1}{2} g_1^2 B_\mu B^\mu (|H_d|^2 - |H_u|^2) + \frac{1}{2} g_2^2 \sigma_{ij}^a W_\mu^a W^{\mu,a} (H_d^i H_d^{j*} - H_u^i H_u^{j*}) && (4.201)\\
=& \ldots && (4.202)\\
=& \underbrace{\frac{1}{2} v (g_2 \cos\Theta_W + g_1 \sin\Theta_W)^2 h Z_\mu Z^\mu}_{c_{hZZ}^{\text{SM}}} \sin(\alpha - \beta) + \underbrace{\frac{1}{2} v g_2^2 h W_\mu^- W^{+\mu}}_{c_{hWW}^{\text{SM}}} \sin(\alpha - \beta) && (4.203)
\end{aligned}$$

The overall changes in the Higgs-gauge boson couplings are

| | $Z$ | $W$ |
|---|---|---|
| $h$ | $\sin(\alpha - \beta)$ | $\sin(\alpha - \beta)$ |
| $H$ | $\cos(\alpha - \beta)$ | $\cos(\alpha - \beta)$ |

In the decoupling limit, we find

$$\sin(\beta-\alpha) \quad \to \quad 1 \tag{4.204}$$
$$\cos(\beta-\alpha) \quad \to \quad 0 \tag{4.205}$$
$$-\frac{\sin\alpha}{\cos\beta} = \sin(\beta-\alpha) - \tan\beta\cos(\beta-\alpha) \quad \to \quad 1 \tag{4.206}$$
$$\frac{\cos\alpha}{\cos\beta} = \cos(\beta-\alpha) + \tan\beta\sin(\beta-\alpha) \quad \to \quad \tan\beta \tag{4.207}$$
$$\frac{\cos\alpha}{\cos\beta} = \sin(\beta-\alpha) + \cot\beta\cos(\beta-\alpha) \quad \to \quad 1 \tag{4.208}$$
$$\frac{\sin\alpha}{\sin\beta} = \cos(\beta-\alpha) - \cot\beta\sin(\beta-\alpha) \quad \to \quad 1/\tan\beta \tag{4.209}$$

So, $h$ has nearly the same couplings to SM fermions and gauge bosons as the Higgs boson of the SM without supersymmetry would have. Even if these tree-level relations get modified by radiative corrections, the light Higgs in the MSSM is very often *SM-like*, i.e. its couplings are nearly indistinguishable from the SM. This is very important because several couplings of the Higgs boson with a mass of 125 GeV to other particles have already been measured at the LHC. The overall result is that they are close to the SM expectations:

$$\begin{aligned}
\mu_{\gamma\gamma} &= 1.17^{+0.28}_{-0.26}\\
\mu_{ZZ^*} &= 1.46^{+0.40}_{-0.34}\\
\mu_{WW^*} &= 1.18^{+0.24}_{-0.21}\\
\mu_{\tau\tau} &= 1.44^{+0.42}_{-0.37}\\
\mu_{bb} &= 0.63^{+0.39}_{-0.37}\\
\mu_{\text{all}} &= 1.18^{+0.15}_{-0.14}
\end{aligned}$$

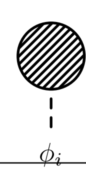
with

$$\mu_{XX} = \frac{\text{Coupling}(hXX)^{\text{experiment}}}{\text{Coupling}(hXX)^{\text{SM}}} \tag{4.210}$$

#### 4.5.2.2 Radiative Corrections to the Light Higgs mass

We have seen that the Higgs mass at tree-level has an upper limit of $M_Z$. Therefore, it's important to check if the radiative corrections are sufficient to push the mass to the desired value of 125 GeV. The calculation of the loop corrected Higgs mass involves two steps:

**a) 'Shifting the vacuum'**: there are corrections to the tadpole equations from diagrams of the form

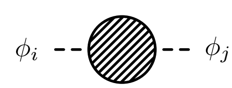


This causes shifts $\delta t_i$ to the tree-level conditions $T_i = \frac{\partial V}{\partial \phi_i}$

$$T_i + \delta t_i = 0 \tag{4.211}$$

Thus, the parameters which are obtained from the tadpole equations change their values once going to the loop level.

**b) 'Self-energies'**: once one is working at the loop-corrected minimum of the potential, the second step is to calculate the loop-corrected self-energies via diagrams of the form

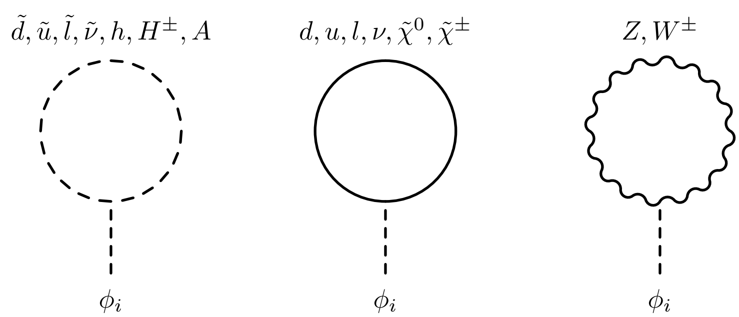


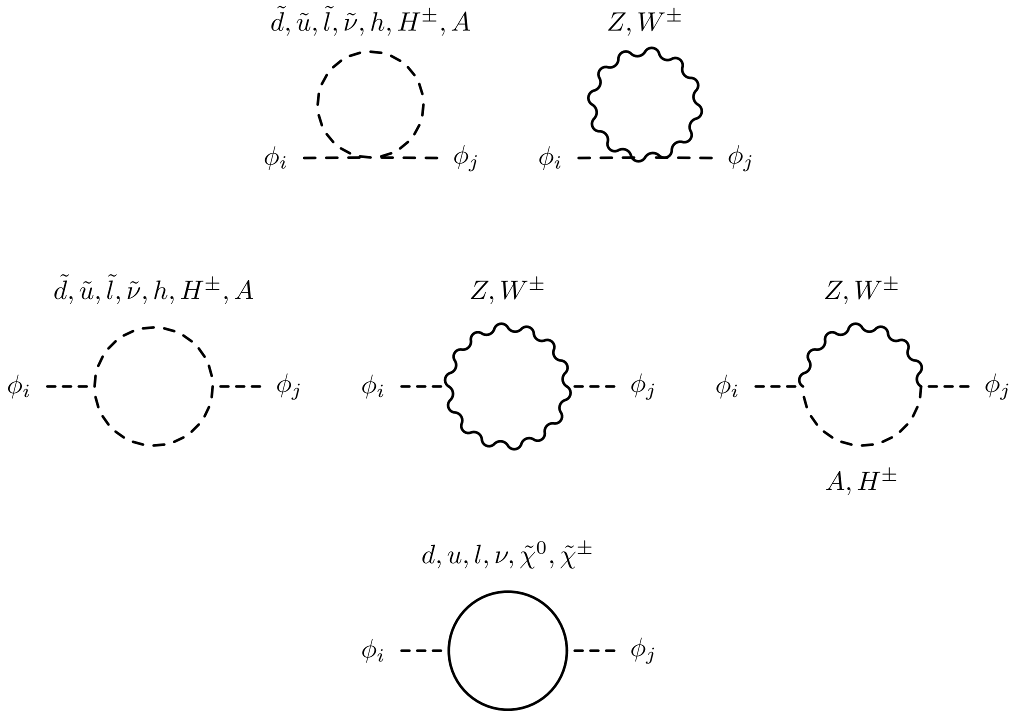

In general, the self-energy corrections are not diagonal. Therefore, it is convenient to work with external gauge eigenstates but mass eigenstates in the loop. Thus, one obtains the one-loop corrections to the mass matrix. The loop corrected masses are then the eigenvalues of the loop-corrected mass matrix $M^{\text{loop}}$ calculated as

$$M_{ij}^{\text{loop}} = M_{ij}^2 + \delta_{ij}\frac{\delta t_i}{v_i} + \Pi_{ij}(p^2) \tag{4.212}$$

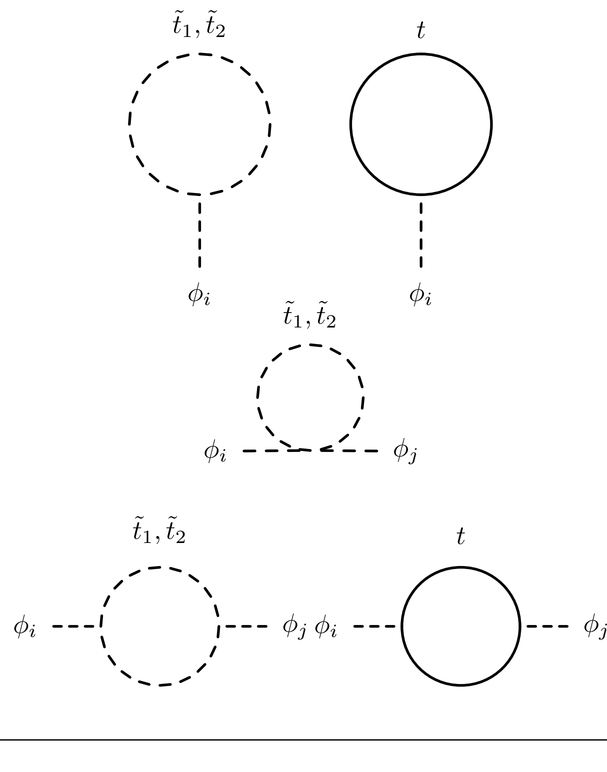
Here, we have assumed that the tadpole equatoin are solved for the soft-breaking masses: those appear always on the diagonal of the mass matrices and their change from the loop corrected tadpole is given by $\frac{\delta t_i}{v_i}$. $\Pi(p^2)$ is the self-energy matrix which depends on the external momenta. This demands usually an iterative procedure to get that the eigenvalues $m_i$ match the external momenta (on-shell condition). However, we consider here the simplified case $p^2 = 0$ at the one-loop level.

Since we have outlined the demanded procedure to obtain loop corrected masses, we can check which diagrams are needed to be calculated at the one-loop level in the MSSM. These are






Note, we have suppressed here the ghost diagrams which could be related to the vector boson contributions in Landau gauge. One can imagine, that calculation all of these diagrams becomes quite a piece of work. Therefore, we pick out those contributions which are the dominant ones. These are the diagrams with (s)tops:



One could now start and calculating all of these Feynman diagrams. However, since we are only interested in the $p^2 = 0$ approximation, a simpler approach exists: one can calculate the one-loop effective potential given by

$$\Delta V^{(1)} = \frac{1}{16\pi^2}\frac{1}{4}\sum_i^{\text{all fields}} r_i C_i (-1)^{2s_i}(2s_i+1)m_i^4\left(\log\frac{m_i^2}{Q^2}-\frac{3}{2}\right) \tag{4.213}$$

Here, $s_i$ is the spin of the particle, $C_i$ the colour factor and $r_i = 1$ for real bosons or Majorana particles, otherwise 2.
From which the necessary quantities are derived via

$$\delta t_i = \frac{\partial \Delta V}{\partial v_i} \tag{4.214}$$

$$\Pi_{ij} = \frac{\partial^2 \Delta V}{\partial v_i \partial v_j} \tag{4.215}$$

**a) top contributions**:
Since the top mass is given by $m_t = \frac{1}{\sqrt{2}}Y_t v_u$, we find

$$\Delta^{(t)}V = -\frac{3}{2}\frac{1}{16\pi^2}\frac{Y_t^4 v_u^4}{2}\left(\frac{Y_t^2 v_u^2}{2Q^2}-\frac{3}{2}\right) \tag{4.216}$$

and therefore

$$\delta^{(t)}t_u = -\frac{3}{32\pi^2}Y_t^4\left(-1+\log\frac{m_t^2}{Q^2}\right) \tag{4.217}$$

$$\Pi_{uu}^{(t)} = \frac{3}{32\pi^2}v_u^2 Y_t^4\left(1-\log\frac{m_t^2}{8Q^2}\right) \tag{4.218}$$

$$\rightarrow \frac{\delta^{(t)}}{v_u} - \Pi_{uu}^{(t)} = \frac{3}{32\pi^2}Y_t^4(-2\log\frac{m_t^2}{4Q^2}) \tag{4.219}$$

**b) stop contributions**:
Including only the contributions $\simeq Y_t^2$, the two stop masses squared are given by $m_{\tilde{t}_{1,2}}^2 = m_{\tilde{t}_{L,R}}^2 + \frac{1}{2}Y_t^2 v_u^2$. And therefore

$$\Delta^{(\tilde{t})}V = \frac{3}{4}\frac{1}{16\pi^2}\left((m_{\tilde{t}_L}^2+\frac{1}{2}Y_t^2v_u^2)^2\left(\log\frac{m_{\tilde{t}_L}^2+\frac{1}{2}Y_t^2v_u^2}{Q^2}-\frac{3}{2}\right)+(m_{\tilde{t}_R}^2+\frac{1}{2}Y_t^2v_u^2)^2\left(\log\frac{m_{\tilde{t}_R}^2+\frac{1}{2}Y_t^2v_u^2}{Q^2}-\frac{3}{2}\right)\right) \tag{4.220}$$

We get

$$\delta^{(\tilde{t})}t_u = \frac{3}{64\pi^2}Y_t^2 v_u^2\Big[-2(m_{\tilde{t}_R}^2 + m_{\tilde{t}_L}^2 + v_u^2 Y_t^2) + (2m_{\tilde{t}_L}^2 + v_u^2 Y_t^2)\log\frac{m_{\tilde{t}_L}^2 + \frac{1}{2}v_u^2 Y_t^2}{Q^2} + (2m_{\tilde{t}_R}^2 + v_u^2 Y_t^2)\log\frac{m_{\tilde{t}_R}^2 + \frac{1}{2}v_u^2 Y_t^2}{Q^2}\Big] \tag{4.221}$$

$$\Pi_{uu}^{(\tilde{t})} = \frac{3}{64\pi^2}Y_t^2 v_u^2\Big[-2(m_{\tilde{t}_R}^2 + m_{\tilde{t}_L}^2 + v_u^2 Y_t^2) + (2m_{\tilde{t}_L}^2 + 3v_u^2 Y_t^2)\log\frac{m_{\tilde{t}_L}^2 + \frac{1}{2}v_u^2 Y_t^2}{Q^2} + (2m_{\tilde{t}_R}^2 + 3v_u^2 Y_t^2)\log\frac{m_{\tilde{t}_R}^2 + \frac{1}{2}v_u^2 Y_t^2}{Q^2}\Big] \tag{4.222}$$

$$\rightarrow \frac{\delta^{(\tilde{t})}}{v_u} - \Pi_{uu}^{(\tilde{t})} = -\frac{3}{32\pi^2}v_u^2 Y_t^4\left(\log\frac{m_{\tilde{t}_R}^2 + \frac{1}{2}v_u^2 Y_t^2}{Q^2} + \log\frac{m_{\tilde{t}_L}^2 + \frac{1}{2}v_u^2 Y_t^2}{Q^2}\right) \tag{4.223}$$

The sum of both contributions is

$$\frac{\delta t_u}{v_u} - \Pi_{uu} = -\frac{3}{32\pi^2}v_u^2 Y_t^4\left(-2\log\frac{v_u^2 Y_t^2}{Q^2} + \log\frac{m_{\tilde{t}_R}^2 + \frac{1}{2}v_u^2 Y_t^2}{Q^2} + \log\frac{m_{\tilde{t}_L}^2 + \frac{1}{2}v_u^2 Y_t^2}{Q^2}\right) \tag{4.224}$$

We see that in the limit of unbroken SUSY, $m_{\tilde{t}_{L,R}}^2 \rightarrow 0$ the contributions would cancel exactly. This is the famous solution to the the hierarchy problem. Even with broken SUSY, there is only a logarithmic dependence but not a quadratic one.

## 4.6 Fine-Tuning

We want to discuss here the so called 'fine-tuning' problem of the MSSM. One can understand this problem by starting from the tadpole equations which we have derived above

$$\frac{\partial V}{\partial v_d} = -v_u B_\mu + \frac{1}{8}\left(g_1^2 + g_2^2\right)v_d\left(v_d^2 - v_u^2\right) + v_d\left(m_{H_d}^2 + |\mu|^2\right) \tag{4.225}$$

$$\frac{\partial V}{\partial v_u} = -v_d B_\mu + \frac{1}{8}\left(g_1^2 + g_2^2\right)v_u\left(v_u^2 - v_d^2\right) + v_u\left(m_{H_u}^2 + |\mu|^2\right) \tag{4.226}$$

In the decoupling limit ($B_\mu \sim M_A^2 \rightarrow \infty$) and for large $\tan\beta$, these relations simplify to

$$\frac{\partial V}{\partial v_d} = 0 \tag{4.227}$$

$$\frac{\partial V}{\partial v_u} = v(m_{H_u}^2 + \mu^2\frac{1}{8}(g1^2 + g2^2)v^2) \tag{4.228}$$

which is often presented in the form

$$\frac{1}{2}M_Z^2 = -\mu^2 - m_{H_u}^2\ . \tag{4.229}$$

This makes the origin of the (little) hierarchy problem within the MSSM apparent: the r.h.s. contains terms which are naturally $\mathcal{O}(M_{\text{SUSY}})$, the SUSY breaking scale. Thus, in order to obtain the measured value of $M_Z$ there must be a cancellation between these terms which demands a certain level of tuning.

There are different measures to quantify the amount of fine-tuning $\Delta_{FT}$. A widely used one is the sensitivity measure

$$\Delta \equiv \max \mathrm{Abs}\big[\Delta_p\big], \qquad \Delta_p \equiv \frac{\partial \ln v^2}{\partial \ln p} = \frac{p}{v^2}\frac{\partial v^2}{\partial p} \ . \tag{4.230}$$

Here, $p$ are the independent parameters of the model, and the quantity $\Delta^{-1}$ gives a measure of the accuracy to which independent parameters must be tuned to get the correct electroweak breaking scale. Applying this measure to eq. (4.229), one finds

$$\frac{\partial \ln v^2}{\partial \ln p_i} = \frac{\partial \ln M_Z^2}{\ln p_i} = 2\frac{p_i^2}{M_Z^2}\left(-\frac{\partial \mu^2}{\partial p_i^2} - \frac{\partial m_{h_u}^2}{\partial p_i^2}\right) \tag{4.231}$$

Using $p^2 = \{\mu^2, m_{H_u^2}\}$ the very naive estimate for the fine-tuning is found to be

$$\Delta_\mu = -\frac{2\mu^2}{M_Z^2}\,, \qquad \Delta_{m_{h_u}^2} = -\frac{2m_{h_u}^2}{M_Z^2} \ . \tag{4.232}$$

Thus, a small FT needs moderately small $|\mu|$ and $|m_{H_u}^2|$ at the low scale. Therefore, if these parameters are pushed to larger values by instance from the negative collider searches, this renders SUSY a more and more fine-tuned model. This tuning is much lower than the one in the SM, but at some point the MSSM might no longer a 'natural' extension of the SM. This has increased the interest in non-minimal models in which the amount of tuning can be reduced compared to the MSSM.