Theoretische Teilchenphysik 1

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Exercise sheet 1

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Lecture website: `https://www.itp.kit.edu/courses/ss2018/ttp1`

Exercise 1: Gamma matrix representations (P) (*2+2+1+2+2+1 = 10 points*)

We consider the Dirac matrices γ^μ ($\mu = 0, ..., 3$) in the Weyl representation

$$\gamma^\mu_\chi = \left(\begin{pmatrix} 0 & 1_{2\times2} \\ 1_{2\times2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \right)$$

as well as in the Dirac representation

$$\gamma^\mu_D = \left(\begin{pmatrix} 1_{2\times2} & 0 \\ 0 & -1_{2\times2} \end{pmatrix}, \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \right) \quad ,$$

where $1_{2\times2}$ is the 2×2 unit matrix and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ with σ_i ($i = 1, 2, 3$) represent the Pauli matrices.

(a) Show that both representations obey the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ by inserting the explicit representations separately into the anti-commutation relations.

(b) The Weyl and Dirac representations are connected by a unitary transformation U such that

$$\gamma^\mu_\chi = U^\dagger \gamma^\mu_D U$$

holds. Up to an arbitrary phase, determine the unitary 4×4 matrix U explicitly.

(c) A fifth Dirac matrix, γ^5, can be defined by $\gamma_5 := i\gamma^0\gamma^1\gamma^2\gamma^3$. Show that the explicit forms of γ^5_D and γ^5_χ for both the Dirac and the Weyl representation, respectively, are given by

$$\gamma^5_D = \begin{pmatrix} 0 & 1_{2\times2} \\ 1_{2\times2} & 0 \end{pmatrix} \quad , \qquad \gamma^5_\chi = \begin{pmatrix} -1_{2\times2} & 0 \\ 0 & 1_{2\times2} \end{pmatrix} \quad .$$

(d) Show that $\{\gamma^5, \gamma^\mu\} = 0$ holds for both the Dirac and Weyl representation separately by inserting the Dirac matrices explicitly in these representations.

(e) We now define chirality projectors $\omega_\mp$ by

$$\omega_\mp := \frac{1_{4\times 4} \mp \gamma^5}{2} \ ,$$

where ω_- is the left-chiral projector and ω_+ is the right-chiral projector. By using the explicit representation of γ^5 in the Weyl basis only, show that the $\omega_\mp$ obey the following projector properties:

$$\omega_\mp^2 = \omega_\mp \ , \qquad \omega_-\omega_+ = \omega_+\omega_- = 0 \ .$$

(f) We now define a bispinor $\Psi := (\psi_L, \psi_R)$ in the Weyl basis, where ψ_L and ψ_R are left- and right-chiral spinors, respectively. Calculate the following bispinors by inserting the explicit representation of γ^5 in the Weyl basis:

$$\Psi_L := \omega_-\Psi \ , \qquad \Psi_R := \omega_+\Psi \ .$$

Interpret the result and explain why the Weyl basis is often called chiral basis.

Exercise 2: Dirac algebra in 4 **dimensions (P)** (*4+6 = 10 points*)

Independent of any representation, the Dirac matrices γ^μ and γ^5 in 4 space-time dimensions obey the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ as well as $\{\gamma^5, \gamma^\mu\} = 0$. The γ^μ can always be chosen to be unitary so that $(\gamma^\mu)^\dagger = (\gamma^\mu)^{-1}$ holds.

(a) Prove the following Dirac algebra relations by using the unitarity and anti-commutation relations of the Dirac matrices, only (i.e. do not use the explicit representations of the Dirac matrices from Exercise 1):

$$\begin{aligned}
(\gamma^\mu)^\dagger &= \gamma^0\gamma^\mu\gamma^0 \ , \\
\gamma^5\gamma^5 &= 1_{4\times 4} \ , \\
\gamma_\mu\gamma^\mu &= 4 \cdot 1_{4\times 4} \ , \\
\gamma_\mu\gamma^\alpha\gamma^\mu &= -2\gamma^\alpha \ , \\
\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\mu &= 4g^{\alpha\beta} \ .
\end{aligned}$$

(b) Prove the following trace identities by using the anti-commutation relations of the Dirac matrices and the general properties of a matrix trace, only:

$$\begin{aligned}
\mathrm{tr}\left(\gamma^\mu\right) &= 0 \ , \\
\mathrm{tr}\left(\gamma^5\right) &= 0 \ , \\
\mathrm{tr}\left(\gamma^\mu\gamma^\nu\gamma^5\right) &= 0 \ , \\
\mathrm{tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\right) &= 0 \ , \\
\mathrm{tr}\left(\gamma^\mu\gamma^\nu\right) &= 4g^{\mu\nu} \ , \\
\mathrm{tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) &= 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right) \ .
\end{aligned}$$