Theoretische Teilchenphysik 1

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Exercise sheet 2

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Exercise 3: SU(N) **representation (P)** (*3+1+2+1 = 7 points*)

The generators T^a of the fundamental representation of the SU(N) are given by

$$T^a_{ij}\,, \qquad a = 1, \dots, N^2-1\,, \qquad i,j = 1,\dots,N\,.$$

They are Hermitian, $T^{a\dagger} = T^a$, traceless, $\mathrm{Tr}(T^a) = 0$, and normalized through

$$\mathrm{Tr}\left(T^a T^b\right) = \frac{1}{2}\delta^{ab}\,.$$

They satisfy the commutation and anti-commutation relation

$$\left[T^a, T^b\right] = i f_{abc} T^c\,, \tag{1}$$

$$\left\{T^a, T^b\right\} = \frac{1}{N}\,\delta^{ab}\,\mathbb{1}_{N\times N} + d_{abc}T^c\,, \tag{2}$$

which defines the total antisymmetric structure constants f_{abc} and the total symmetric symbols d_{abc} of the SU(N). The commutation relation, Eq. (1), is satisfied for all SU(N) representations, whereas Eq. (2) only holds for the fundamental representation.
Every complex $N \times N$ matrix M can be decomposed into a linear combination of these N^2-1 generators, with coefficients c_0, c_a, as follows:

$$M = c_0 \mathbb{1}_{N\times N} + \sum_{a=1}^{N^2-1} c_a T^a\,. \tag{3}$$

(a) Show that the Fierz identity of the SU(N),

$$T^a_{ij}T^a_{kl} \equiv \sum_{a=1}^{N^2-1} T^a_{ij}T^a_{kl} = \frac{1}{2}\delta_{il}\delta_{jk} - \frac{1}{2N}\delta_{ij}\delta_{kl}\,, \tag{4}$$

is a result of combining the completeness relation, eq. (3), with the tracelessness of T^a.

(b) Show that, independent of any specific representation of SU(N), that

$$C_2 = T^a T^a \equiv \sum_{a=1}^{N^2-1} T^a T^a$$

is a Casimir invariant, i.e. that $[C_2, T^a] = 0$ holds for all generators T^a.

(c) By using the hermiticity of the generators, show that f_{abc} and d_{abc} are real.

(d) Calculate the value of C_2 in the fundamental representation.

Exercise 4: Lorentz group (*3+4 = 7 Points*)

(a) By using the generators

$$L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)\,,$$

determine the Lie algebra of the SO(3,1), i.e. prove that the $L_{\mu\nu}$ satisfy

$$[L_{\mu\nu},\, L_{\rho\sigma}] = i\,(g_{\nu\rho} L_{\mu\sigma} - g_{\mu\rho} L_{\nu\sigma} - g_{\nu\sigma} L_{\mu\rho} + g_{\mu\sigma} L_{\nu\rho})\ .$$

(b) The generators $M_{\mu\nu} = -M_{\nu\mu}$ satisfy the same Lie algebra given as the commutator in (a). By using the $M_{\mu\nu}$, the generators K^i of Lorentz boosts and the generators J^i of rotations are given by

$$K^i = M^{0i} \qquad \text{and} \qquad J^i = \frac{1}{2}\,\epsilon^{ijk} M_{jk}\,,$$

with ϵ^{ijk} being the Levi-Civita tensor (with $\epsilon^{123} \equiv +1$). Prove that the algebra of these generators is given by

$$\begin{aligned} \left[K^i, K^j\right] &= -i\epsilon^{ijk} J^k\,, \\ \left[J^i, K^j\right] &= i\epsilon^{ijk} K^k\,, \\ \left[J^i, J^j\right] &= i\epsilon^{ijk} J^k\,, \end{aligned}$$

and explain the physical meaning of each of these results.

Exercise 5: Lorentz boosts (P) (*1+4+1 = 6 Points*)

For this exercise assume that the boost operator K^2 is given by

$$K^2 = K_y = -i\left(t\frac{\partial}{\partial y} + y\frac{\partial}{\partial t}\right) .$$

(a) Calculate the effects of K_y and $(K_y)^2$ on the four-vector $x^\mu = (t, x, y, z)^T$.

(b) Determine the finite Lorentz transformation

$$x^{\mu\prime} = \exp\left(i\nu K_y\right) x^\mu ,$$

where ν is the *rapidity*, by using the results from part (a).

With the boost parameters $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, the boost can be written as

$$x^{\mu\prime} = \begin{pmatrix} \gamma & 0 & \gamma\beta & 0 \\ 0 & 1 & 0 & 0 \\ \gamma\beta & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} x^\mu .$$

(c) Compare this alternative form of the boost with your result from (b). Show that the rapidity is given by

$$\nu = \frac{1}{2}\ln\frac{1+\beta}{1-\beta} .$$