Theoretische Teilchenphysik 1

Lecture: Prof. Dr. M. M. Mühlleitner
Exercises: Prof. Dr. M. M. Mühlleitner, P. Basler, M. Krause

Exercise sheet 3

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Exercise 6: Problems of the classical Klein-Gordon equation (P) (*1+1+1+1+1+1 = 6 points*)

We consider two *classical* scalar fields $\Phi(x)$ and $\Phi^*(x)$ with Lagrangian density

$$\mathcal{L}_S = (\partial_\mu \Phi(x))\,(\partial^\mu \Phi^*(x)) - m_S^2 \Phi(x)\Phi^*(x)\,,$$

where m_S is a free parameter and $\Phi(x)$ and $\Phi^*(x)$ are formally considered to be *independent* fields.

(a) By using the Euler-Lagrange equations, show that the equations of motion of the fields are given by the Klein-Gordon equations

$$\left(\Box + m_S^2\right) \Phi(x) = 0\,,$$
$$\left(\Box + m_S^2\right) \Phi^*(x) = 0\,,$$

where $\Box = \partial_\mu \partial^\mu$ is the d'Alembert operator.

(b) Make a plane wave ansatz $\Phi(x) \propto e^{ik_\mu x^\mu}$ with wave vector $k_\mu = (\omega, \vec{k})^T$ to solve the Klein-Gordon equation for Φ. Find a relation between ω, $\vec{k}$ and m_S.

(c) By inserting all solutions you found in (b) into the Schrödinger equation, find all possible energy states of your solutions. Explain the physical relevance of each energy state.

Quantum mechanically, the field Φ can be characterized by its four-current

$$j^\mu = \begin{pmatrix} \rho \\ \vec{j} \end{pmatrix} = \frac{i}{2m_S}\left[\Phi^* \left(\partial^\mu \Phi\right) - \left(\partial^\mu \Phi^*\right) \Phi\right]\,.$$

In the non-relativistic limit, ρ is often interpreted as the *probability density* of the scalar field, which necessitates $\rho \geq 0$ for the density.

(d) Prove that the four-current j^μ obeys a continuity equation, i.e. prove that

$$\partial_\mu j^\mu = 0\,.$$

(e) Calculate ρ and explain why, in contrast to the non-relativistic limit, the sign of ρ is indefinite in the covariant definition of the four-current j^μ.

(f) By considering all results you found in the previous parts, explain why the classical Klein-Gordon equation cannot describe physical particle states consistently.

Exercise 7: Lagrangian of a massive vector field (P) (*3+2+0.5+2.5+2 = 10 points*)

The Lagrangian of a massive free vector field $V^\mu(x)$ is given by

$$\mathcal{L}_V = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m_V^2}{2}V_\mu V^\mu$$

where $m_V \neq 0$ denotes the mass of the vector particle and $F^{\mu\nu} = \partial^\mu V^\nu - \partial^\nu V^\mu$ denotes the field-strength tensor.

(a) Calculate the equations of motion for V^μ, the so-called Proca equations.

(b) Using the equations of motion, prove that

$$\partial_\mu V^\mu = 0$$

(c) Use the results from (a) and (b) to show that all components of V^μ satisfy the Klein-Gordon equation separately and explain the physical meaning of this result.

A new Lagrangian $\mathcal{L} = \mathcal{L}_V + \mathcal{L}_D$ is given by adding a Dirac term

$$\mathcal{L}_D = \overline{\psi}(x)\left(i\not{D} - m_D\right)\psi(x)\,,$$

where the *covariant derivative* $D_\mu = \partial_\mu + iqV_\mu$ yields a coupling between the spinor ψ and the vectorfield V_μ.

(d) Consider ψ, $\overline{\psi}$ and V_μ as independent fields and calculate the new equations of motion for all three of them, separately.

(e) The vector current j^μ and axial vector current $j^{\mu 5}$ can be defined as

$$j^\mu = \overline{\psi}\gamma^\mu\psi \qquad j^{\mu 5} = \overline{\psi}\gamma^\mu\gamma^5\psi.$$

Consider the special case of $q = 0$, i.e. the fermion decouples from the vector boson. By using the equations of motion, prove that j^μ is a conserved quantity, whereas $j^{5\mu}$ is not conserved in general. In which special case is $j^{\mu 5}$ conserved, as well?

Exercise 8: Dimensional analysis (P) (*0.5+0.5+1+1+1 = 4 points*)

In natural units, i.e. with $\hbar = c = 1$, we can express all fields, coupling constants and parameters of the Lagrangians in dimensions of mass (or equivalently, momentum or energy):

$$[\text{mass}] = [\text{energy}] = [\text{momentum}] = 1 \ ,$$

where the bracket [...] denotes the dimension and 1 denotes that all three quantities have the same *mass dimension*.

(a) By using the formulae for the *de Broglie* wavelength and the energy of a photon, derive the mass dimensions of the quantities *length* and *time*.

In natural units and in $D-1$ space and 1 time dimensions, the action

$$S = \int d^D x \mathcal{L} = \int dt d^{D-1}\vec{x}\mathcal{L}$$

has no mass dimension, i.e. $[S] = 0$.

(b) From this, deduce the mass dimension of the Lagrangian density $\mathcal{L}$ in D space-time dimensions.

(c) By using your result from (b), analyze the Lagrangians $\mathcal{L}_S$, $\mathcal{L}_V$ and $\mathcal{L}_D$ given in exercises 6 and 7 and deduce the mass dimensions of the following fields quantities defined in these Lagrangians:

$$[\Phi] \ , \ [\psi] \ , \ [V^\mu] \ , \ [m_S] \ , \ [m_V] \ , \ [m_D] \ .$$

(d) Consider a new Lagrangian containing an additional coupling between the scalar field Φ and the spinors ψ,

$$\hat{\mathcal{L}} = \mathcal{L}_S + \frac{\lambda_{mn}}{m!n!} \left(\overline{\psi}\psi\right)^m \Phi^n \ ,$$

where $m, n \geq 0$, and determine the mass dimension of the coupling constant λ_{mn} by dimensional analysis.

(e) Using your previous results, consider the special case $D = 4$ of four physical space-time dimensions. Give the mass dimensions of all quantities that you derived before in this special case.

Remark: the dimensional analysis of this exercise is very useful for constructing and analyzing Lagrangians. Knowing the mass dimensions of $\mathcal{L}$ and of all fields allows for the construction of Lagrangians with all possible combinations of fields, masses and coupling constants.