Theoretische Teilchenphysik 1

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Exercise sheet 4

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Exercise 9: Application of the Noether theorem (P) (*1+1+2 = 4 points*)

(a) Show that the Lagrangian of two real scalar fields $\varphi_{1,2}$

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\varphi_1\right)\left(\partial^\mu\varphi_1\right) + \frac{1}{2}\left(\partial_\mu\varphi_2\right)\left(\partial^\mu\varphi_2\right) - \frac{m^2}{2}\left(\varphi_1^2+\varphi_2^2\right) - \frac{\lambda}{4}\left(\varphi_1^2+\varphi_2^2\right)^2$$

is invariant under the transformation ($\vartheta = \text{const} \in \mathbb{R}$)

$$\begin{aligned} x_\mu &\to x'_\mu = x_\mu \\ \varphi_1 &\to \varphi'_1 = \varphi_1\cos\vartheta + \varphi_2\sin\vartheta \\ \varphi_2 &\to \varphi'_2 = -\varphi_1\sin\vartheta + \varphi_2\cos\vartheta \end{aligned}$$

(b) Calculate the Noether current density and the Noether charge.

(c) Calculate the Energy-momentum-tensor $T^{\mu\nu}$ through application of the transformation

$$\begin{aligned} x_\mu &\to x'_\mu = x_\mu + \epsilon_\mu \ , \\ \varphi_{1,2} &\to \varphi'_{1,2} = \varphi_{1,2} \, . \end{aligned}$$

Exercise 10: Complex scalar field (*1+2+3+2 = 8 points*)

We consider the Lagrangian of a complex scalar field φ, given by

$$\mathcal{L} = (\partial_\mu \varphi)^* (\partial^\mu \varphi) - m^2 \varphi^* \varphi \,.$$

It is a common practice in particle physics to express a complex field by two real fields

$$\varphi(\vec{x}) = \frac{1}{\sqrt{2}} (\varphi_1(\vec{x}) + i\varphi_2(\vec{x})) \ , \qquad \varphi(\vec{x})^* = \frac{1}{\sqrt{2}} (\varphi_1(\vec{x}) - i\varphi_2(\vec{x})) \ .$$

(a) Express $\mathcal{L}$ through φ_1 and φ_2. Why do we need the factor $\frac{1}{\sqrt{2}}$?

(b) The Fourier transformation of the real fields φ_i is given by

$$\varphi_i(\vec{x}) = \int \mathrm{d}\tilde{k} \left[a_i(k) \exp(-ikx) + a_i^\dagger(k) \exp(ikx) \right]$$

where $\mathrm{d}\tilde{k} = \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3 2\omega_k}$ and $a_i, a_i^\dagger$ satisfy the commutation relations defined in the lecture. With this we can define

$$a(k) = \frac{1}{\sqrt{2}} (a_1(k) + ia_2(k)) \ , \qquad b(k) = \frac{1}{\sqrt{2}} (a_1(k) - ia_2(k))$$

as well as $a^\dagger$ and $b^\dagger$. Express $\varphi(\vec{x})$ and $\varphi^*(\vec{x})$ through $a, a^\dagger, b, b^\dagger$. Interpret these new operators physically.

(c) Derive the commutators of $a, a^\dagger, b, b^\dagger$ through a_i and $a_i^\dagger$.

(d) The Lagrangian, expressed through the fields φ_1 and φ_2, is invariant under the same transformation as in exercise 9 (a). Calculate the Noether charge Q for this transformation and express it through a_i as well as a and b. Interpret the results.

Exercise 11: Calculation of the Feynman propagator (P) (*1.5+2+2+1.5+1 = 8 points*)

The Feynman propagator in position space is given by

$$i\Delta_F(r) = \int d^4k \frac{1}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} e^{-ik \cdot r} \ ,$$

where $\varepsilon > 0$ is small, $m > 0$ is the mass of a particle and k and r are four-vectors. As you will discuss in the lecture soon, the propagator, mathematically a special type of Green's function, represents the probability amplitude of a particle to *propagate* from one space-time point to another, separated by the four-vector r. In this exercise, we will evaluate the propagator explicitly.

(a) By performing the integration of the time component k^0 of the four-vector k, show that the propagator takes the form

$$i\Delta_F(r) = \int d^3\vec{k}\frac{1}{2\omega_k(2\pi)^3}e^{i\vec{k}\cdot\vec{r}}e^{-i|r^0|\omega_k} \ ,$$

where $\omega_k = \sqrt{\vec{k}^2 + m^2 - i\varepsilon}$ and r^0 denotes the time component of r.
Hint: Find the complex poles of the integrand and shift the integral to the complex plane. Choose the contours of integration according to the poles you find and use the residue theorem to evaluate the integral.

(b) Assume that the distance r is timelike, i.e. $r^2 > 0$. Evaluate the propagator and express it through the Bessel function of the third kind (also called *Hankel function*)

$$H_1^{(1)}(a) = \frac{-2ia}{\pi}\int_{1-i\varepsilon}^{\infty-i\varepsilon} dte^{-iat}\sqrt{t^2-1} \ , \quad (a>0) \ .$$

Hint: Choose a reference frame for the timelike vector r such that the integration becomes simpler.

(c) Assume that the distance r is spacelike, i.e. $r^2 < 0$. Evaluate the propagator and express it through the modified Hankel function

$$K_1^{(1)}(a) = \frac{1}{2i}\int_{-\infty}^{\infty} dt\frac{te^{iat}}{\sqrt{t^2+1-i\varepsilon}} \ , \quad (a>0) \ .$$

Hint: Choose a reference frame for the timelike vector r such that the integration becomes simpler.

(d) Assume that the distance r is lightlike, i.e. $r^2 = 0$. Evaluate the propagator in this case.
Hint: You will find that the integral diverges in the regime of high $|\vec{k}|$, i.e. in the so-called *ultraviolet regime*. Simplify the integration by assuming $|\vec{k}| \gg m$ and express the divergence through the delta distribution.

(e) Combine your results from the previous parts and show that the Feynman propagator takes the form

$$i\Delta_F(r) = \Theta(r^2)\frac{im}{8\pi\sqrt{r^2}}H_1^{(1)}(m\sqrt{r^2}) + \Theta(-r^2)\frac{m}{4\pi^2\sqrt{-r^2}}K_1^{(1)}(m\sqrt{-r^2}) + \frac{-i}{4\pi}\delta(r^2) \ .$$

Discuss the behaviour of the Feynman propagator for large spacetime distances in the case of spacelike and timelike r, separately. Interpret this result physically.
Hint: You can use without proof that for large arguments, the Hankel functions behave like $H_1^{(1)}(a) \sim e^{ia}$ and $K_1^{(1)}(a) \sim H_1^{(1)}(ia)$.