Theoretische Teilchenphysik 1

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Exercise sheet 5

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Exercise 12: Causality and the spin-statistics theorem (P) (*1+2+1 = 4 points*)

In exercise 11, you found that the Feynman propagator is non-zero for spacelike intervals, which seemed to be at odds with causality. However, in quantum field theories, causality is normally defined over the (anti)commutator of the fields rather than over the propagators. For this exercise, assume that we consider quantized fields of the form

$$\varphi^+(x) = \int d\tilde{k} e^{ik\cdot x} a(\vec{k}) \ , \quad \varphi^-(x) = \int d\tilde{k} e^{-ik\cdot x} a^\dagger(\vec{k}) \ .$$

The operators $a(\vec{k})$, $a^\dagger(\vec{k})$ shall obey the algebra

$$\begin{aligned} \left[a(\vec{k}), a(\vec{k}')\right]_\mp &= \left[a^\dagger(\vec{k}), a^\dagger(\vec{k}')\right]_\mp = 0 \,, \\ \left[a(\vec{k}), a^\dagger(\vec{k}')\right]_\mp &= (2\pi)^3 2\omega_k \delta(\vec{k} - \vec{k}') \,, \end{aligned}$$

where the index $-$ denotes the commutator, while the index $+$ denotes the anticommutator, and the index $\mp$ implies that we leave this choice open. Causality requires that for two field operators that are separated by a spacelike interval, we find

$$[\varphi(x), \varphi(y)]_\mp = 0 \ , \quad \big(\text{for } (x-y)^2 < 0\big) \ . \tag{1}$$

(a) Discuss either mathematically or geometrically (e.g. with help of a Minkowski diagram) why the vanishing (anti)commutator for spacelike intervals is an argument of causality, but not for timelike intervals, where it may be non-zero.

(b) Show that the fields $\varphi^+(x)$ and $\varphi^-(x)$ do not obey Eq. (1).
Hint: Calculate Eq. (1) for spacelike intervals $r^2 \equiv (x-y)^2 < 0$ for all combinations of $\varphi^+(x)$ and $\varphi^-(x)$. You can use the results from exercise 11.

In order to restore causality, we define new fields

$$\varphi_\lambda(x) = \varphi^+(x) + \lambda\varphi^-(x) \ , \quad \varphi_\lambda^\dagger(x) = \varphi^-(x) + \lambda^*\varphi^+(x) \ .$$

(c) Calculate Eq. (1) for all possible combinations of φ_λ and $\varphi_\lambda^\dagger(x)$. What is the value of λ if we require causality for both fields? Do you have to choose commutators or anticommutators to restore causality?

Remark: The connection between the spin of the fields and their algebra is called the spin-statistics theorem. In part (c), you will see that the requirement of causality automatically leads to the correct spin-statistics of scalar fields.

Exercise 13: Symmetry breaking (P) (*1+2+1+3+2+1 = 10 points*)

We first analyze the Lagrangian of φ^4 theory for real fields $\varphi(x)$, given by

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\varphi\right)\left(\partial^\mu\varphi\right) - V(\varphi) \ , \quad V(\varphi) = -\frac{1}{2}\mu^2\varphi^2 + \frac{\lambda}{4}\varphi^4 \ .$$

where μ^2 and λ are constants of the potential and $\lambda > 0$.

(a) Find the trivial extremum $\langle\varphi\rangle_1$ and the non-trivial extremum $v \equiv \langle\varphi\rangle_2$ of the potential $V(\varphi)$ with respect to the field φ. The non-trivial extremum v is called the *vacuum expectation value* of the field φ. What condition must μ^2 fulfill that the non-trivial extremum v actually exists in the potential $V(\varphi)$? In this case, is it a global minimum or maximum of the potential?

(b) The Lagrangian is invariant under the *discrete* symmetry $\varphi \to -\varphi$. We assume that the non-trivial minimum v exists. In this case, the field can condensate into this new minimum and it can be expanded about the vacuum expectation value as

$$\varphi(x) = v + \sigma(x) \ ,$$

where $\sigma(x)$ is a small perturbation of the field near v. Rewrite the Lagrangian in terms of v, λ and $\sigma(x)$. Express the mass of the field $\sigma(x)$ through the original parameters of the potential and then rewrite the Lagrangian in terms of this mass and v, only.

We now consider a complex scalar field $\varphi(x)$ which couples both to itself and to a vector field $A_\mu(x)$, described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \left(D_\mu\varphi\right)\left(D^\mu\varphi\right)^* - V(\varphi) \ , \quad V(\varphi) = -\mu^2\varphi^*\varphi + \frac{\lambda}{2}\left(\varphi^*\varphi\right)^2 \ ,$$

where $D_\mu = \partial_\mu + igA_\mu$.

(c) Analogous to part (a), find the trivial and non-trivial extrema of the potential. Show that the Lagrangian is invariant under the local *continuous* $U(1)$ gauge transformation

$$\varphi(x) \longrightarrow e^{i\alpha(x)}\varphi(x) \ , \quad A_\mu(x) \longrightarrow A_\mu(x) - \frac{1}{g}\partial_\mu\alpha(x) \ .$$

What would happen with this symmetry if we would naively add a mass term $m_A^2 A_\mu A^\mu$ for the gauge field by hand to the Lagrangian? Interpret this finding.

(d) Expand the field φ about its non-trivial extremum v,

$$\varphi(x) = v + \frac{h(x) + iG(x)}{\sqrt{2}} \ ,$$

where $h(x)$ and $G(x)$ are perturbations of the field near v. Why do we need two fields now in contrast to part (b), where one field $h(x)$ was sufficient in the expansion? Express the potential $V(\varphi)$ in terms of v, λ, $h(x)$ and $G(x)$ and identify the masses of the fields $h(x)$ and $G(x)$.

(e) Rewrite the kinetic term $(D_\mu\varphi)\,(D^\mu\varphi)^\dagger$ of the Lagrangian in terms of the new fields, where you can omit terms cubic and quartic in the fields $A_\mu(x)$, $h(x)$ and $G(x)$ as well as all mixing terms between these fields. Identify the effective mass term of the gauge field $A_\mu(x)$. Interpret your findings by comparing this to the result you found in part (c).

(f) Use the gauge freedom of the field $\varphi(x)$ as given in part (c) to remove the massless Goldstone field $G(x)$ from the Lagrangian. This special choice of gauge is called *unitary gauge*. Express the Lagrangian in this special case. Interpret your findings.

Exercise 14: Transformation of the covariant derivative (*6 points*)

The covariant derivative

$$D_\mu = \partial_\mu + igA_\mu = \partial_\mu + igA_\mu^a T^a \ ,$$

is explicitly dependent on the chosen representation of the generators T^a of the gauge group. In this exercise, we consider the transformation of the covariant derivative and of the gauge field,

$$D'_\mu = UD_\mu U^{-1} \ , \qquad\qquad A'_\mu = UA_\mu U^{-1} - \frac{i}{g}U(\partial_\mu U^{-1})$$

where representation matrices $U = \exp(i\vartheta^a T^a)$ are given in the *fundamental* representation. With this, prove that the covariant derivative transforms like

$$D'_\mu = VD_\mu V^{-1}$$

for *any arbitray* representation V and calculate the transformation explicitly.
Hint: Use the Baker–Hausdorff formula

$$e^B A e^{-B} = \sum_{n=0}^{\infty} \frac{1}{n!} A_n \ ,$$

where $A_n = [B, A_{n-1}], A_0 = A$ *and* $A = A_\mu$ *respectively* $A = \partial_\mu$ *and* $B = i\vartheta^a T^a$.

By starting with

$$D'_\mu = V\left(\partial_\mu + igA_\mu\right)V^{-1} = \partial_\mu + V(\partial_\mu V^{-1}) + igVA_\mu V^{-1} \ ,$$

you can transform the right-hand side in such a way that the transformation of the gauge fields A_μ *in the adjoint representation, with* $(T^a_{adj})_{bc} = (-if^a)_{bc}$*, appears explicitly. The structure constants* f^{abc} *are the same is introduced in exercise 3.*