

# Theoretische Teilchenphysik 1

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## Exercise sheet 6

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### Exercise 15: Spinor algebra (P)

(1+2+3+2+3 = 11 points)

We want to examine the underlying algebra of fermionic structures. We consider the spinors

$$u(\vec{p}, s) = \sqrt{p_0 + m} \begin{pmatrix} \varphi_s \\ \frac{\vec{\sigma} \cdot \vec{p}}{p_0 + m} \varphi_s \end{pmatrix} \quad v(\vec{p}, s) = \sqrt{p_0 + m} \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{p}}{p_0 + m} \varphi_s \\ \varphi_s \end{pmatrix}$$

where  $s \in \{-\frac{1}{2}, \frac{1}{2}\}$ ,  $\vec{\sigma}$  was given in exercise 1 and

$$\varphi_{+1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \varphi_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad p_0 = \sqrt{m^2 + \vec{p}^2}.$$

(a) Prove that

$$(\vec{\sigma} \cdot \vec{p}) (\vec{\sigma} \cdot \vec{p}) = \vec{p}^2 1_{2 \times 2}.$$

(b) By using the explicit Dirac representation of the Dirac matrices (cf. exercise 1), prove that the spinors  $u$  and  $v$  are solutions to the Dirac equations

$$\begin{aligned} (\not{p} - m)u(\vec{p}) &= 0, \\ (\not{p} + m)v(\vec{p}) &= 0. \end{aligned}$$

(c) Prove the following orthogonality relations for the spinors  $u$  and  $v$ :

$$\begin{aligned} u^\dagger(\vec{p}, s)u(\vec{p}, s') &= 2p_0\delta_{ss'} \quad , \quad v^\dagger(\vec{p}, s)v(\vec{p}, s') = 2p_0\delta_{ss'} \quad , \\ u^\dagger(\vec{p}, s)v(-\vec{p}, s') &= 0 \quad , \quad v^\dagger(\vec{p}, s)u(-\vec{p}, s') = 0 \quad , \\ \bar{u}(\vec{p}, s)u(\vec{p}, s') &= 2m\delta_{ss'} \quad , \quad \bar{v}(\vec{p}, s)v(\vec{p}, s') = -2m\delta_{ss'} \quad , \\ \bar{u}(\vec{p}, s)v(\vec{p}, s') &= 0 \quad , \quad \bar{v}(\vec{p}, s)u(\vec{p}, s') = 0 \quad . \end{aligned}$$

where  $\bar{u} = u^\dagger \gamma^0$  and  $\bar{v} = v^\dagger \gamma^0$ .*Hint:* Use  $\gamma^0$  in the explicit Dirac representation.

(d) Prove the following completeness relations of the spinors  $u$  and  $v$ ,

$$\sum_{s=-1/2}^{+1/2} u_{\alpha}(\vec{p}, s) \bar{u}_{\beta}(\vec{p}, s) = (\not{p} + m)_{\alpha\beta},$$

$$\sum_{s=-1/2}^{+1/2} v_{\alpha}(\vec{p}, s) \bar{v}_{\beta}(\vec{p}, s) = (\not{p} - m)_{\alpha\beta},$$

where  $\alpha$  and  $\beta$  are indices in spin space.

*Remark:* These relations, also called *spin sums*, are very useful for practical calculations of fermionic scattering processes later on.

(e) Prove the *Gordon identities* for the spinor  $u$ ,

$$0 = \bar{u}(p)[(p - q)^{\mu} + i\sigma^{\mu\nu}(p + q)_{\nu}]u(q),$$

$$\bar{u}(p)\gamma^{\mu}u(q) = \frac{1}{2m}\bar{u}(p)[(p + q)^{\mu} + i\sigma^{\mu\nu}(p - q)_{\nu}]u(q),$$

$$\bar{u}(p)\gamma^{\mu}\gamma^5u(q) = \frac{1}{2m}\bar{u}(p)[(p - q)^{\mu}\gamma^5 + i\sigma^{\mu\nu}(p + q)_{\nu}\gamma^5]u(q),$$

where  $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^{\mu}, \gamma^{\nu}]$ , purely algebraically, i.e. without using the explicit representation of  $u$  given above.

*Hint:* Use the Dirac equations.

**Exercise 16: Quantization of the fermionic field (P)** (3+3+3 = 9 points)

We can now describe a fermionic field  $\psi(x)$  as

$$\psi(x) = \sum_{s=-1/2}^{+1/2} \psi_s(x) = \int d\tilde{k} \sum_{s=-1/2}^{+1/2} \left[ a_s(\vec{k})u(\vec{k}, s)e^{-ik \cdot x} + b_s^{\dagger}(\vec{k})v(\vec{k}, s)e^{ik \cdot x} \right].$$

(a) The Lagrangian of a free fermionic field is invariant under a  $U(1)$  transformation  $\psi(x) \rightarrow e^{i\alpha(x)}\psi(x)$ . The Noether charge  $Q$  associated to this symmetry is given by

$$Q = \int d^3x \bar{\psi}\gamma^0\psi.$$

Express  $Q$  through the operators  $a, a^{\dagger}, b, b^{\dagger}$  and interpret the results.

(b) By using the anti-commutators of  $a, a^{\dagger}, b, b^{\dagger}$ , calculate the anti-commutator  $\{\psi_r(\vec{x}, t), \psi_s^{\dagger}(\vec{y}, t)\}$ . What is the physical meaning of the fields  $\psi^{\dagger}$  and  $\psi$ ?

We now consider a coupling of fermions to a photon field. Through the minimal coupling by means of the covariant derivative  $\mathcal{D} = \not{\partial} + iq\not{A}$ , the Dirac equation becomes

$$(i\mathcal{D} - m)\psi = 0.$$

- (c) Prove that in the non-relativistic limit, i.e. for  $|\vec{p}| \ll m$ , this equation transforms to the *Pauli equation* given by

$$i\partial_t\rho = \left[ \left( \frac{\vec{\pi}^2}{2m} + q\varphi \right) - \vec{\mu} \cdot \vec{B} \right] \rho,$$

with the magnetic moment being  $\vec{\mu} = g\frac{q}{2m}\vec{S}$ , where  $g = 2$  is the gyromagnetic factor,  $\vec{B} = \vec{\nabla} \times \vec{A}$  is the magnetic field,  $\vec{\pi} = \vec{p} - q\vec{A}$  is the canonical momentum and  $\vec{S} = \frac{\vec{\sigma}}{2}$  is the spin operator.

*Hint:* Use the ansatz  $\psi = e^{-ip \cdot x} \begin{pmatrix} \rho \\ \chi \end{pmatrix}$  for the spinor to split the Dirac equation into two equations for the up-type and down-type spinor  $\rho$  and  $\chi$ , respectively. In the non-relativistic limit, argue why  $\chi$  is heavily suppressed compared to  $\rho$ , which allows you to use  $(i\partial_t - q\varphi)\chi \approx 0$  in the system of the two equations.