Theoretische Teilchenphysik 1

Lecture: Prof. Dr. M. M. Mühlleitner
Exercises: Prof. Dr. M. M. Mühlleitner, P. Basler, M. Krause

Exercise sheet 6

Release: 21 May 2018
Submission: 28 May 2018
Tutorials: 30 May 2018

Lecture website: `https://www.itp.kit.edu/courses/ss2018/ttp1`

Exercise 15: Spinor algebra (*1+2+3+2+3 = 11 points*)

We want to examine the underlying algebra of fermionic structures. We consider the spinors

$$u(\vec{p},s) = \sqrt{p_0+m}\begin{pmatrix}\varphi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{p_0+m}\varphi_s\end{pmatrix} \qquad v(\vec{p},s) = \sqrt{p_0+m}\begin{pmatrix}\frac{\vec{\sigma}\cdot\vec{p}}{p_0+m}\varphi_s \\ \varphi_s\end{pmatrix}$$

where $s \in \{-\frac{1}{2}, \frac{1}{2}\}$, $\vec{\sigma}$ was given in exercise 1 and

$$\varphi_{\pm 1/2} = \begin{pmatrix}1\\0\end{pmatrix}\,, \qquad \varphi_{\mp 1/2} = \begin{pmatrix}0\\1\end{pmatrix}\,, \qquad p_0 = \sqrt{m^2+\vec{p}^2}\,,$$

where the upper sign applies for u and the lower sign for v.

(a) Prove that

$$(\vec{\sigma}\cdot\vec{p})\,(\vec{\sigma}\cdot\vec{p}) = \vec{p}^2 1_{2\times 2}\ .$$

(b) By using the explicit Dirac representation of the Dirac matrices (cf. exercise 1), prove that the spinors u and v are solutions to the Dirac equations

$$\begin{aligned}(\not{p}-m)u(\vec{p}) &= 0\,,\\ (\not{p}+m)v(\vec{p}) &= 0\,.\end{aligned}$$

(c) Prove the following orthogonality relations for the spinors u and v:

$$\begin{aligned}u^\dagger(\vec{p},s)u(\vec{p},s') &= 2p_0\delta_{ss'} \quad , \quad v^\dagger(\vec{p},s)v(\vec{p},s') = 2p_0\delta_{ss'}\ ,\\ u^\dagger(\vec{p},s)v(-\vec{p},s') &= 0 \quad , \quad v^\dagger(\vec{p},s)u(-\vec{p},s') = 0\ ,\\ \overline{u}(\vec{p},s)u(\vec{p},s') &= 2m\delta_{ss'} \quad , \quad \overline{v}(\vec{p},s)v(\vec{p},s') = -2m\delta_{ss'}\ ,\\ \overline{u}(\vec{p},s)v(\vec{p},s') &= 0 \quad , \quad \overline{v}(\vec{p},s)u(\vec{p},s') = 0\ .\end{aligned}$$

where $\overline{u} = u^\dagger\gamma^0$ and $\overline{v} = v^\dagger\gamma^0$.
Hint: Use γ^0 in the explicit Dirac representation.

(d) Prove the following completeness relations of the spinors u and v,

$$\sum_{s=-1/2}^{+1/2} u_\alpha(\vec{p},s)\bar{u}_\beta(\vec{p},s) = (\not{p}+m)_{\alpha\beta}\,,$$
$$\sum_{s=-1/2}^{+1/2} v_\alpha(\vec{p},s)\bar{v}_\beta(\vec{p},s) = (\not{p}-m)_{\alpha\beta}\,,$$

where α and β are indices in spin space.
Remark: These relations, also called *spin sums*, are very useful for practical calculations of fermionic scattering processes later on.

(e) Prove the *Gordon identities* for the spinor u,

$$0 = \bar{u}(p)[(p-q)^\mu + i\sigma^{\mu\nu}(p+q)_\nu]u(q)\,,$$
$$\bar{u}(p)\gamma^\mu u(q) = \frac{1}{2m}\bar{u}(p)[(p+q)^\mu + i\sigma^{\mu\nu}(p-q)_\nu]u(q)\,,$$
$$\bar{u}(p)\gamma^\mu\gamma^5 u(q) = \frac{1}{2m}\bar{u}(p)[(p-q)^\mu\gamma^5 + i\sigma^{\mu\nu}(p+q)_\nu\gamma^5]u(q)\,,$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu]$, purely algebraically, i.e. without using the explicit representation of u given above.
Hint: Use the Dirac equations.

Exercise 16: Quantization of the fermionic field (*3+3+3 = 9 points*)

We can now describe a fermionic field $\psi(x)$ as

$$\psi(x) = \sum_{s=-1/2}^{+1/2}\psi_s(x) = \int \mathrm{d}\tilde{k} \sum_{s=-1/2}^{+1/2}\left[a_s(\vec{k})u(\vec{k},s)e^{-ik\cdot x} + b_s^\dagger(\vec{k})v(\vec{k},s)e^{ik\cdot x}\right].$$

(a) The Lagrangian of a free fermionic field is invariant under a $U(1)$ transformation $\psi(x)\to e^{i\alpha(x)}\psi(x)$. The Noether charge Q associated to this symmetry is given by

$$Q = \int \mathrm{d^3x}\,\overline{\psi}\gamma^0\psi\,.$$

Express Q through the operators $a, a^\dagger, b, b^\dagger$ and interpret the results.

(b) By using the anti-commutators of $a, a^\dagger, b, b^\dagger$, calculate the anti-commutator $\{\psi_r(\vec{x},t),\psi_s^\dagger(\vec{y},t)\}$. What is the physical meaning of the fields $\psi^\dagger$ and ψ?

We now consider a coupling of fermions to a photon field. Through the minimal coupling by means of the covariant derivative $\not{D} = \not{\partial} + iq\not{A}$, the Dirac equation becomes

$$(i\not{D} - m)\psi = 0\,.$$

(c) Prove that in the non-relativistic limit, i.e. for $|\vec{p}| \ll m$, this equation transforms to the *Pauli equation* given by

$$i\partial_t \rho = \left[\left(\frac{\vec{\pi}^2}{2m} + q\varphi\right) - \vec{\mu}\cdot\vec{B}\right]\rho\,,$$

with the magnetic moment being $\vec{\mu} = g\frac{q}{2m}\vec{S}$, where $g = 2$ is the gyromagnetic factor, $\vec{B} = \vec{\nabla}\times\vec{A}$ is the magnetic field, $\vec{\pi} = \vec{p} - q\vec{A}$ is the canonical momentum and $\vec{S} = \frac{\vec{\sigma}}{2}$ is the spin operator.

Hint: Use the ansatz $\psi = e^{-ip\cdot x}\begin{pmatrix}\rho\\ \chi\end{pmatrix}$ for the spinor to split the Dirac equation into two equations for the up-type and down-type spinor ρ and χ, respectively. In the non-relativistic limit, argue why χ is heavily suppressed compared to ρ, which allows you to use $(i\partial_t - q\varphi)\chi \approx 0$ in the system of the two equations.