Theoretische Teilchenphysik 1

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Exercise sheet 7

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Exercise 17: Charge conjugation (P) (*2+2+1+2 = 7 points*)

The defining property of the charge conjugation matrix C is

$$C^{-1}\gamma^{\mu}C = -(\gamma^{\mu})^{\mathrm{T}} \tag{1}$$

for any representation of the Dirac matrices.

(a) Verify explicitly that in the Weyl representation, the matrix $C = i\gamma^0\gamma^2$ obeys Eq. (1). Show that in this representation, the charge conjugation operator obeys $C^{\dagger} = C^{\mathrm{T}} = C^{-1} = -C$.

(b) Show that for any arbitrary representation, i.e. only by using Eq. (1), that the following equations hold:

$$C^{-1}\gamma^5 C = (\gamma^5)^{\mathrm{T}}\,,$$
$$C^{-1}\sigma^{\mu\nu}C = -(\sigma^{\mu\nu})^{\mathrm{T}}\,,$$
$$C^{-1}(\gamma^{\mu}\gamma^5)C = (\gamma^{\mu}\gamma^5)^{\mathrm{T}}\,.$$

Hint: Use the following definition for γ^5: $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$.

(c) The charge conjugation *operator acting directly on the fermionic fields* is given by $\mathcal{C}$. From its definition via $\mathcal{C}\psi(x)\mathcal{C}^{\dagger} = C\overline{\psi}^{\mathrm{T}}(x)$, it is connected with the charge conjugation matrix C. By using this definition as well as $(\gamma^0)^{\dagger} = (\gamma^0)^{\mathrm{T}} = \gamma^0$, show that

$$\mathcal{C}\overline{\psi}(x)\mathcal{C}^{\dagger} = -\psi^{\mathrm{T}}(x)C^{-1}\,.$$

(d) Use the results of the previous parts to show that the fermionic bilinear covariants transform in the following way under charge conjugation,

$$\begin{aligned}
\mathcal{C}\,{:}\overline{\psi}(x)\psi(x){:}\,\mathcal{C}^\dagger &= {:}\overline{\psi}(x)\psi(x){:} \equiv S(x)\,,\\
\mathcal{C}\,{:}\overline{\psi}(x)\gamma^\mu\psi(x){:}\,\mathcal{C}^\dagger &= -{:}\overline{\psi}(x)\gamma^\mu\psi(x){:} \equiv -V^\mu(x)\,,\\
\mathcal{C}\,{:}\overline{\psi}(x)\sigma^{\mu\nu}\psi(x){:}\,\mathcal{C}^\dagger &= -{:}\overline{\psi}(x)\sigma^{\mu\nu}\psi(x){:} \equiv -T^{\mu\nu}(x)\,,\\
\mathcal{C}\,{:}\overline{\psi}(x)\gamma^\mu\gamma^5\psi(x){:}\,\mathcal{C}^\dagger &= {:}\overline{\psi}(x)\gamma^\mu\gamma^5\psi(x){:} \equiv A^\mu(x)\,,\\
\mathcal{C}\,{:}\overline{\psi}(x)\gamma^5\psi(x){:}\,\mathcal{C}^\dagger &= {:}\overline{\psi}(x)\gamma^5\psi(x){:} \equiv P(x)\,,
\end{aligned}$$

where the normal ordering, denoted by ${:}\mathcal{O}{:}$, is only stated here to indicate that the fermionic fields inside the bilinear covariants are quantized.
Hint: You can save some time if you first evaluate the generic bilinear transformation $\mathcal{C}\,{:}\overline{\psi}(x)\Gamma\psi(x){:}\,\mathcal{C}^\dagger$ by generalizing your results from the previous parts to any matrix $\Gamma \in \{1_{4\times 4}, \gamma^\mu, \sigma^{\mu\nu}, \gamma^\mu\gamma^5, \gamma^5\}$.

Exercise 18: Causality, spin-statistics theorem for fermions (P) (*2+2+2+3 = 9 points*)

We want to repeat the analysis on causality and the spin-statistics theorem from exercise 12, but this time, we consider fermionic fields. For this, consider quantized fields

$$\psi^+(x) \equiv \sum_s \int d\tilde{k}\, b_s(\vec{k})u_s(\vec{k})e^{ik\cdot x}\,, \quad \psi^-(x) \equiv \sum_s \int d\tilde{k}\, b_s^\dagger(\vec{k})v_s(\vec{k})e^{-ik\cdot x}\,.$$

For the operators b_s and $b_s^\dagger$, we require the algebra

$$\begin{aligned}
\left[b_s(\vec{k}), b_{s'}(\vec{k}')\right]_\mp &= \left[b_s^\dagger(\vec{k}), b_{s'}^\dagger(\vec{k}')\right]_\mp = 0\,,\\
\left[b_s(\vec{k}), b_{s'}^\dagger(\vec{k}')\right]_\mp &= (2\pi)^3 2\omega_k \delta(\vec{k}-\vec{k}')\delta_{ss'}\,,
\end{aligned}$$

where as in exercise 12, the minus sign denotes the commutator, the plus sign denotes the anti-commutator, and $\mp$ indicates that we leave this choice open. Causality requires that for fermionic fields ψ with spinor indices α and β, we find

$$[\psi_\alpha(x), \psi_\beta(y)]_\mp = 0 \qquad \big(\text{for } (x-y)^2 < 0\big)\,. \tag{1}$$

(a) By using the charge conjugation matrix introduced in exercise 17, prove that

$$\begin{aligned}
C\overline{u}_s(\vec{k})^{\mathrm{T}} &= v_s(\vec{k})\,,\\
C\overline{v}_s(\vec{k})^{\mathrm{T}} &= u_s(\vec{k})\,.
\end{aligned}$$

Hint: Use the spinors and the charge conjugation matrix in explicit Weyl representation, given by

$$u_+(0) = \sqrt{2m}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \quad u_-(0) = \sqrt{2m}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \quad v_+(0) = \sqrt{2m}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}, \quad v_-(0) = \sqrt{2m}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}.$$

It suffices to show that the above relations hold for $\vec{k} = 0$. Due to the transformation of γ^μ under charge conjugation, the relation then follows for all $\vec{k}$.

(b) Show that the fields $\psi^+(x)$ and $\psi^-(x)$ do not obey Eq. (1).
Hint: Calculate Eq. (1) for spacelike intervals $r^2 \equiv (x-y)^2 < 0$ for all combinations of $\psi^+(x)$ and $\psi^-(x)$. Use the result from part (a) to eliminate the spinor v_s. You can use the results from exercises 11 and 15(d).

(c) Show that

$$\left[\psi^+_\alpha(x), \psi^-_\beta(y)\right]_\mp = -\left[\psi^+_\beta(y), \psi^-_\alpha(x)\right]_\mp \qquad \left(\text{for } (x-y)^2 < 0\right).$$

In order to restore causality, we define new fields

$$\psi(x) \equiv \psi^+(x) + \lambda\psi^-(x), \quad \psi^\dagger(x) \equiv \psi^-(x) + \lambda^*\psi^+(x).$$

(d) Calculate Eq. (1) for all possible combinations of $\psi(x)$ and $\overline{\psi}(x)$. What is the value of λ if we require causality for both fields for spacelike distances? Do you have to choose commutators or anticommutators to restore causality? Compare this result for the fermionic field to the result from exercise 12 for the scalar field.

Exercise 19: Polarization of a massive vector boson (P) (*1+1+1+1 = 4 points*)

We consider a vector boson with mass $M \neq 0$ and polarization vectors $\varepsilon^\mu_\lambda(k)$, where k^μ is its four-vector and λ denotes the three physical degrees of freedom for the polarization of the massive vector boson. The polarization vectors are normalized through the following relations:

$$\begin{aligned} k \cdot \varepsilon_\lambda(k) &= 0, \\ \varepsilon_\lambda(k) \cdot \varepsilon^*_{\lambda'}(k) &= -\delta_{\lambda\lambda'}. \end{aligned} \tag{1}$$

(a) Boost into the rest frame of the vector boson. By using the relations from Eq. (1), determine the form of the three polarization vectors under the assumption that the vector boson is linearly polarized in one longitudinal and two transversal modes.

(b) By using again the relations from Eq. (1) in the rest frame of the vector boson, determine the form of the polarization vectors if we now consider the vector boson to be longitudinally polarized in the z direction, but circularly polarized in the $x-y$ plane.

(c) By using Lorentz covariance, guess the form of the *completeness relation* $\Sigma_\lambda \varepsilon_\lambda^\mu(k)\varepsilon_\lambda^{*\nu}(k)$ of the massive vector boson and use Eq. (1) to determine the correct form of the polarization sum as given in the lecture.
Hint: Which tensors and four-vector combinations are compatible with $\varepsilon_\lambda^\mu(k)\varepsilon_\lambda^{*\nu}(k)$ to preserve Lorentz covariance? Express the completeness relation as a linear combination of these possible components and use Eq. (1) to determine the coefficients of these components in the rest frame of the vector boson.

(d) Show that the circularly polarized vector boson from part (b) fulfills the completeness relation from part (c) by inserting the polarization vectors explicitly for all μ and ν. You can again work in the rest frame of the vector boson.