KIT, ITP **SS 2018**

Theoretische Teilchenphysik 1

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Exercise sheet 9

Release: 11 June 2018
Submission: 18 June 2018
Tutorials: 20 June 2018

Lecture website: `https://www.itp.kit.edu/courses/ss2018/ttp1`

Exercise 22: Vacuum fluctuations (P) (*2+6+2 = 10 points*)

We consider a quantized real Klein-Gordon field $\varphi(\vec{x})$. At time $t = 0$, the field is averaged over a sphere of radius R and volume V:

$$\varphi_R = \frac{1}{V} \int_V d^3x \, \varphi(\vec{x}) \ .$$

(a) Show that the vacuum expectation value (vev) of φ_R vanishes, i.e. show that

$$\langle 0 | \, \varphi_R \, | 0 \rangle = 0 \ .$$

(b) Calculate the quantity

$$\langle 0 | \, \varphi_R^2 \, | 0 \rangle$$

and argue that it is non-vanishing.
Hint: For the Bessel function

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right)$$

and its integral

$$I(a) \equiv \int_0^\infty dy \, \frac{1}{y\sqrt{a^2 + y^2}} \left[J_{3/2}(y) \right]^2 \qquad (a \in \mathbb{R}) \ ,$$

you can use that for $a = 0$, the integral gives

$$I(0) = \frac{1}{2\pi} \ .$$

(c) Consider now the case of a massless field, i.e. $m = 0$. Does the radius of the sphere R has to be enlarged or diminished to enlarge the fluctuations? Provide a physical interpretation of your results.
Remark: Due to the fact that the vev of φ_R vanishes, but the vev of φ_R^2 is non-zero, the field itself cannot be constant in the vacuum. It *fluctuates* within the sphere of radius R.

Exercise 23: Two-particle phase space (P) *(3+7 = 10 points)*

For the calculation of decay rates and cross sections, we need to integrate over the phase space of the particles in the final state. For a generic process with two particles with momenta p_1 and p_2 and masses m_1 and m_2 in the final state, this phase space integral is given by

$$\int d\Phi_2 = \int \frac{d^3p_1}{(2\pi)^3 2E_1} \int \frac{d^3p_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^{(4)}(q - p_1 - p_2) \ ,$$

where q is the total four-momentum of all incoming particles. This integral is performed over the absolute squared of the matrix element as well as over some additional Heaviside step functions Θ to implement the proper momentum cuts for the particles in the final state.

(a) Show that in the center-of-mass frame of the two final-state particles, the absolute value of the incoming three-momenta is given by

$$|\vec{p}_1| = |\vec{p}_2| = \frac{\lambda(q^2, m_1^2, m_2^2)}{2\sqrt{q^2}} \ ,$$

where λ is the *Källén function* given by

$$\lambda(a^2, b^2, c^2) \equiv \sqrt{a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2} \ .$$

(b) Show that in the center-of-mass frame of the two final-state particles, the phase space integral can be written as

$$\int d\Phi_2 = \int d\Omega \frac{1}{32\pi^2 q^2} \lambda(q^2, m_1^2, m_2^2) \Theta(q_0) \Theta(q^2 - (m_1^2 + m_2^2)^2) \ ,$$

where $d\Omega \equiv d(\cos\vartheta_1) d\varphi_1$ is the integration over the solid angle of particle 1 in the center-of-mass frame.
Hint: Prove and use the relation

$$\frac{d^3p}{2E} = d^4p\,\Theta(p_0)\delta(p^2 - m^2) \ .$$