Theoretische Teilchenphysik 1

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Exercise sheet 11

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Exercise 25: Integration in D dimensions (P) (*2+5+3 = 10 points*)

Many processes beyond the tree level involve integrals which formally diverge. In order to regularize these divergences, we can use *dimensional regularization*. The idea is simple: we perform the integration in $D = 4 - 2\varepsilon$ instead of 4 dimensions. The term ε serves as the regulator and exposes the divergence when taking the limit $\varepsilon \to 0$. In this exercise, we want to compute some ingredients needed for a computation of a one-loop integral.

(a) Show that the area of a unit sphere in D dimensions is given by

$$\int d\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)} \ ,$$

where $\Gamma(x)$ denotes the Euler gamma function. Consider the following special values, $D \in \{1, 2, 3\}$, and check the consistency of your results.

(b) The Euler gamma function $\Gamma(x)$ has simple poles at the non-positive integers. By analytically continuing the recursion relation $\Gamma(x+1) = x\Gamma(x)$ for all values (apart from the non-positive integers), show that the expansions of $\Gamma(x)$ near the poles $x = \varepsilon \approx 0$ and $x = \varepsilon - 1 \approx -1$ (with ε being small) are given by

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma_{\mathrm{E}} + \frac{1}{2}\left[\gamma_{\mathrm{E}}^2 + \zeta(2)\right]\varepsilon + \mathcal{O}(\varepsilon^2) \qquad (\text{for } \varepsilon \to 0) \ ,$$

$$\Gamma(\varepsilon - 1) = -\frac{1}{\varepsilon} + \gamma_{\mathrm{E}} - 1 + \left[\gamma_{\mathrm{E}} - 1 - \frac{\gamma_{\mathrm{E}}^2}{2} - \frac{\zeta(2)}{2}\right]\varepsilon + \mathcal{O}(\varepsilon^2) \qquad (\text{for } \varepsilon \to 0) \ ,$$

where γ_{E} is the Euler-Mascheroni constant and $\zeta(x)$ is the Euler-Riemann ζ function. *Hint:* There are several ways to perform this proof. One way is to rewrite the recursion relation in such a way that the *digamma function* $\psi(x) \equiv \frac{\Gamma'(x)}{\Gamma(x)} = \frac{1}{\Gamma(x)}\frac{\partial}{\partial x}\Gamma(x)$ and the *trigamma function* $\psi_1(x) = \frac{\partial}{\partial x}\psi(x)$ explicitly appear in the recursion relation and to use their special values $\psi(1) = -\gamma_{\mathrm{E}}$ and $\psi_1(1) = \zeta(2)$ and the recurrence formulas $\psi(x+1) = \psi(x) + 1/x$ and $\psi_1(x+1) = \psi_1(x) - 1/x^2$. Another way is to use a specific representation of the gamma function, e.g. a representation over an infinite sum, and to expand this representation near the poles. If you decide for such a way, you have to proof the validity of the representation that you choose.

(c) Show that

$$\int \frac{d^D l_\mathrm{E}}{(2\pi)^D} \frac{1}{(l_\mathrm{E}^2 + \Delta)^n} = \frac{1}{(4\pi)^{D/2}} \frac{\Gamma\left(n - \frac{D}{2}\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{D}{2}} ,$$

$$\int \frac{d^D l_\mathrm{E}}{(2\pi)^D} \frac{l_\mathrm{E}^2}{(l_\mathrm{E}^2 + \Delta)^n} = \frac{1}{(4\pi)^{D/2}} \frac{D}{2} \frac{\Gamma\left(n - 1 - \frac{D}{2}\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-1-\frac{D}{2}} ,$$

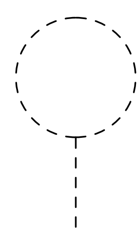
where $n \in \mathbb{N}\backslash\{0\}$, l_E denotes a Euclidean four-vector, i.e. $l_\mathrm{E}^2 = l_0^2 + l_1^2 + l_2^2 + \ldots$, and $\Delta > 0$.

Hint: Integrate in D-dimensional spherical coordinates and use the result from part (a). Additionally, use the definition of the *Euler beta function*

$$B(\alpha, \beta) \equiv \int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} .$$

Exercise 26: The one-loop one-point integral A_0 (P) *(1+2+1+1 = 5 points)*

In this exercise, we consider the simplest one-loop Feynman diagram, the *tadpole diagram*, shown in the figure below, in the framework of dimensional regularization.



We consider φ^3 theory for a scalar field φ with non-negative mass m. The trilinear self-coupling of the scalar field is given as $\lambda_{\varphi\varphi\varphi} = ig$ and the propagator of the massive scalar field with momentum p is simply $\frac{i}{p^2 - m^2}$.

(a) Show that the amplitude of the one-loop tadpole diagram in $D = 4 - 2\varepsilon$ dimensions is given by

$$i\mathcal{A} = -g(\mu^2)^{(4-D)/2} \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2 - m^2 + i\epsilon} .$$

where μ has the dimension of mass and $\epsilon \to 0^+$ shifts the root of the denominator into the complex plane.

Remark: Please note the difference between the dimension regulator ε and the shift of the contour ϵ!

(b) The one-loop one-point integral A_0 in D dimensions is defined as

$$A_0(m^2) \equiv (2\pi\mu)^{4-D} \int \frac{d^D l}{i\pi^2} \frac{1}{l^2 - m^2 + i\epsilon} \, .$$

Show that in $D = 4 - 2\varepsilon$ dimensions, up to $\mathcal{O}(\varepsilon)$, the integral gives the following result:

$$A_0(m^2) = \frac{m^2}{\varepsilon} + m^2 \left[1 - \ln\left(\frac{m^2}{Q^2}\right)\right] + m^2 \left[\frac{\zeta(2)}{2} + \frac{1}{2}\ln\left(\frac{m^2}{Q^2}\right)^2 - \ln\left(\frac{m^2}{Q^2}\right) + 1\right]\varepsilon \, ,$$

where $Q^2 \equiv 4\pi\mu^2 e^{-\gamma_{\rm E}}$ is the so-called *regularization scale.*
Hint: Perform a *Wick rotation*, i.e. rotate the timelike component l_0 of the four-vector l onto the imaginary axis in order to rotate l into Euclidean space. Then, use the results from exercise 25.

(c) Calculate the massless limit, i.e. $m^2 \to 0$, of your results from part (b).

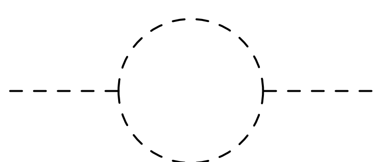
(d) Express the amplitude from part (a) through the integral A_0 and show that the full amplitude of the one-loop tadpole diagram, up to $\mathcal{O}(\varepsilon^0)$, is given by

$$i\mathcal{A} = -\frac{ig}{16\pi^2}\left\{\frac{m^2}{\varepsilon} + m^2\left(1 - \ln\left(\frac{m^2}{Q^2}\right)\right)\right\}$$

Remark: As you can see, the full amplitude diverges for $\varepsilon \to 0$, but due to dimensional regularization, the divergence could be isolated and *regulated.* As for many other diagrams, this so-called *UV divergence* now has to be treated with a *renormalization* procedure to give finite results. You will discuss renormalization in more detail in TTP 2.

Exercise 27: The one-loop two-point integral B_0 (P) (*1+1+2+1 = 5 points*)

Analogous to exercise 26, we again consider φ^3 theory with the Feynman rules as before. Now, we consider the following self-energy diagram with incoming momentum p:



(a) Assume that the two internal scalar particles have two different masses m_1 and m_2, but the trilinear coupling constant is still $\lambda_{\varphi\varphi\varphi}$ from exercise 26. Show that in dimensional regularization with $D = 4 - 2\varepsilon$, the amplitude of this process is then given by

$$i\mathcal{A} = g^2(\mu^2)^{(4-D)/2} \int \frac{d^D l}{(2\pi)^D} \frac{1}{[l^2 - m_1^2 + i\epsilon]\,[(l+p)^2 - m_2^2 + i\epsilon]} ,$$

where again $\epsilon \to 0^+$ is introduced to push the contour of the integral into the complex plane.

(b) By introducing a *Feynman parameter* u, show that the product of two denominators A and B can be written as

$$\frac{1}{AB} = \int_0^1 \frac{du}{[uA + (1-u)B]^2} .$$

(c) The one-loop two-point integral B_0 in D dimensions is defined as

$$B_0(p^2; m_1^2, m_2^2) \equiv (2\pi\mu)^{4-D} \int \frac{d^D l}{i\pi^2} \frac{1}{[l^2 - m^2 + i\epsilon]\,[(l+p)^2 - m^2 + i\epsilon]} .$$

Show that up to $\mathcal{O}(\varepsilon^0)$, the integral can be cast into the form

$$B_0(p^2; m_1^2, m_2^2) = \frac{1}{\varepsilon} - \int_0^1 du \left[\ln\left(\frac{\Delta_u}{Q^2}\right)\right] ,$$

where Δ_u is defined as

$$\Delta_u \equiv u^2 p^2 - u(p^2 + m_1^2 - m_2^2) + m_1^2 - i\epsilon .$$

Hint: Use the result from part (b) to rewrite the denominators. Perform a *Wick rotation* to switch the four-vector l to a Euclidean metric and then again use the results from exercise 25. Since ϵ is small, you can neglect products $u\epsilon$ so that ϵ appears only once in the rewritten denominator.
Remark: The two-point integral B_0 has an analytic solution in closed form. The derivation of this solution is beyond the scope of this exercise, but the explicit result up to $\mathcal{O}(\varepsilon^0)$ is given in the appendix of the lecture script.

(d) Express your result from part (a) through the integral B_0 and show that the amplitude is then given by

$$i\mathcal{A} = \frac{ig^2}{16\pi^2} B_0(p^2; m_1^2, m_2^2) .$$