Theoretische Teilchenphysik 1

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Exercise sheet 12

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Lecture website: `https://www.itp.kit.edu/courses/ss2018/ttp1`

Exercise 28: Decay width of a muon (P) (*1+3+2+7+1+6 = 20 points*)

We want to calculate the decay width and therefore the lifetime of a muon with momentum p_1. The decay is given as

$$\mu^-(p_1) \to \nu_\mu(p_2) + e^-(p_3) + \bar{\nu}_e(p_4)\,.$$

(a) Draw the Feynman diagram of the process.

(b) Calculate the Matrix element and show that it can simplified to

$$\mathcal{M} = -\,i\frac{G_F}{\sqrt{2}}\bar{u}(p_2,s_2)\gamma^\alpha\left(1-\gamma_5\right)u(p_1,s_1)\bar{u}(p_3,s_3)\gamma_\alpha\left(1-\gamma_5\right)v(p_4,s_4)$$

if you apply the limit $q^2 \approx m_\mu^2 \ll m_W^2$, i.e. that the momentum $q = p_3 + p_4$ of the W boson is given at the muon mass. G_F is the Fermi constant and is defined as $\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2}$.

Hint The coupling of a W boson with a fermion and a neutrino is given as $\frac{ig}{\sqrt{2}}\gamma^\alpha\frac{1-\gamma_5}{2}$ and the propagator of a W boson with momentum q and mass m_W is given as $\frac{-ig_{\alpha\beta}}{q^2-m_W^2}$.

(c) Calculate $\mathcal{M}^\dagger$ and simplify it as far as possible.

(d) Calculate

$$\overline{|\mathcal{M}|^2} = \frac{1}{2}\sum_{s_1,s_2,s_3,s_4}\mathcal{M}\mathcal{M}^\dagger\,,$$

where we averaged over the incoming spin combinations and summed over all possible outgoing spin combinations. As an intermediate step show that

$$\overline{|\mathcal{M}|^2} = \frac{G_F^2}{4} \mathrm{tr}\left[\not{p}_2 \gamma^\alpha (1-\gamma_5)(\not{p}_1 + m_\mu)\gamma^\beta(1-\gamma_5)\right] \mathrm{tr}\left[\not{p}_4 \gamma_\beta (1-\gamma_5)(\not{p}_3 + m_e)\gamma_\alpha(1-\gamma_5)\right] .$$

Finally show that

$$\overline{|\mathcal{M}|^2} = 64 G_F^2 (p_1 \cdot p_4)(p_2 \cdot p_3) .$$

Hint Use the relation

$$\epsilon^{\alpha\beta\nu\mu} \epsilon_{\alpha\beta}{}^{\rho\sigma} = -2\left(g^{\nu\rho} g^{\mu\sigma} - g^{\nu\sigma} g^{\mu\rho}\right) .$$

(e) Simplify $\overline{|\mathcal{M}|^2}$ assuming you are in the rest frame of the muon and that the electron mass is negligible, i.e. $m_e \ll m_\mu$, and express it through m_μ and E_4.

(f) We now want to calculate a physical quantity, the decay width Gamma, which in the rest frame is given by

$$\mathrm{d}\Gamma = \frac{1}{2m_\mu} \overline{|\mathcal{M}|^2} \mathrm{d}\Phi_3 ,$$

where the three particle phase space element $\mathrm{d}\Phi_3$ is given by

$$\mathrm{d}\Phi_3 = \frac{1}{4(2\pi)^3} \mathrm{d}E_3 \mathrm{d}E_4 \Theta(m - (E_3 + E_4)) \Theta\left(E_3 + E_4 - \frac{m}{2}\right) \Theta\left((m - 2E_3)(m - 2E_4)\right) .$$

We replaced E_2 by the identity $m = E_2 + E_3 + E_4$.
As a first step calculate the energy spectrum of the emitted electron $\mathrm{d}\Gamma/\mathrm{d}E_3$. After that integrate $\mathrm{d}\Gamma/\mathrm{d}E_4$ to get Γ.
Finally calculate the life time given by

$$\tau = \frac{\hbar}{\Gamma}$$

and compare with the experimental value. The numerical parameters are

$$\begin{aligned} \tau_\mu &= 2.1969811 \cdot 10^{-6}\,\mathrm{s} , \\ m_\mu &= 0.1056583745\,\mathrm{GeV} , \\ G_F &= 1.1663787 \cdot 10^{-5}\,\mathrm{GeV}^{-2} , \\ \hbar &= 6.582119514 \cdot 10^{-25}\,\mathrm{GeVs} . \end{aligned}$$

Hint: Use the Θ functions in $\mathrm{d}\Phi_3$ to get the integration boundaries for E_3 and E_4 and express those of E_4 through E_3.