Theoretische Teilchenphysik 1

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Exercise sheet 13

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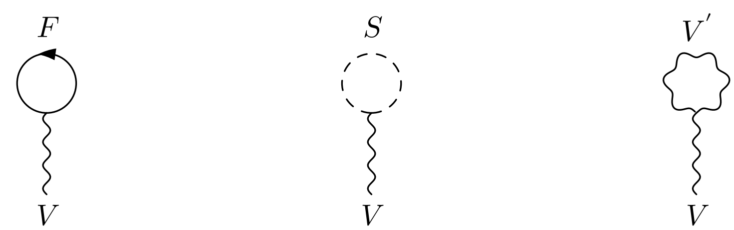
Exercise 29: Furry's theorem (P) (*4 + 2 = 6 bonus points*)

We consider the following tadpole diagrams of a vector boson V:



The generic Feynman rules for the couplings of the vector boson with the fermions F_i, scalars S_i and (other) vector bosons $V_{i,\mu}$ are given by

$$\begin{aligned}
\lambda(V_\mu, F_1, F_2) &= c_- \gamma_\mu \omega_- + c_+ \gamma_\mu \omega_+ \ , \\
\lambda(V_\mu, S(p_1), S(p_2)) &= c_S (p_1 - p_2)_\mu \ , \\
\lambda(V_{1,\mu}(p_1), V_{2,\nu}(p_2), V_{3,\rho}(p_3)) &= c_V \left[g_{\mu\nu}(p_2 - p_1)_\rho + g_{\nu\rho}(p_3 - p_2)_\mu + g_{\rho\mu}(p_1 - p_3)_\nu \right] \ ,
\end{aligned}$$

where c_S, c_V and $c_\mp$ are coupling constants and for all momentum-dependent Feynman rules, the momenta p_i are all flowing **into** the vertex by convention.

(a) Calculate all tadpole diagrams shown in the Feynman diagrams above and show that each of these diagrams vanishes separately.
Hint: Use a symmetry argument to show that the one-loop integrals must vanish.

(b) If we consider QED where the vector boson is the photon, $V = \gamma$, our result from (a) is a special case of *Furry's theorem* which states that Feynman diagrams consisting of an odd number of external photons must vanish. Prove Furry's theorem by using a symmetry argument in the Lagrangian of QED.
Hint: in QED, the electromagnetic current j^μ is odd under charge conjugation C, i.e. $C j^\mu C^\dagger = -j^\mu$, while the vacuum is invariant, $C\,|0\rangle = |0\rangle$.

Exercise 30: The two-point integral B_0, continued *(7 + 7 = 14 bonus points)*

In exercise 27(c), you have shown that the one-loop two-point integral B_0 can be written as

$$B_0(p^2; m_1^2, m_2^2) = \frac{1}{\varepsilon} - \int_0^1 du \left[\ln\left(\frac{\Delta_u}{Q^2}\right)\right] ,$$

where the *regularization scale* was given by $Q^2 = 4\pi\mu^2 e^{-\gamma_{\mathrm{E}}}$ and

$$\Delta_u \equiv u^2 p^2 - u\left(p^2 + m_1^2 - m_2^2\right) + m_1^2 - i\epsilon .$$

As it was mentioned in exercise 27, the remaining integration over the logarithm can be performed and a solution of the integral in closed form can be derived. In this exercise, we want to perform this final integration and find the analytic solution of the integral.

(a) Show that the analytic solution of the B_0 integral up to $\mathcal{O}(\varepsilon^0)$ is given by

$$B_0(p^2; m_1^2, m_2^2) = \frac{1}{\varepsilon} + 2 - \ln\left(\frac{p^2}{Q^2}\right) + \sum_{i=1}^{2}\left[x_i \ln\left(\frac{x_i - 1}{x_i}\right) - \ln\left(1 - x_i\right)\right] ,$$

where the roots $x_{1/2}$ are given by

$$x_{1/2} \equiv \frac{p^2 + m_1^2 - m_2^2}{2p^2} \pm \frac{\sqrt{(p^2 - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2 + 4i\epsilon p^2}}{2p^2} .$$

Remark: Note that depending on the kinematic configuration of the integral, i.e. depending on the values of p^2, m_1^2 and m_2^2, the argument of the logarithm can become negative. In these cases, the shift $\epsilon \to 0^+$ of the integration contour gives a unique prescription which side of the branch cut of the logarithm you have to choose, so that the imaginary part of B_0 is uniquely defined.

Hint: The logarithm in the integral has a branch cut along the negative real axis. With this convention, you can write the logarithm of a product of two complex numbers $a, b \in \mathbb{C}$ as

$$\ln(ab) = \ln a + \ln b + \eta(a,b) ,$$
$$\eta(a,b) \equiv 2\pi i\left[\Theta\left(-\mathrm{Im}(a)\right)\Theta\left(-\mathrm{Im}(b)\right)\Theta\left(\mathrm{Im}(ab)\right) - \Theta\left(\mathrm{Im}(a)\right)\Theta\left(\mathrm{Im}(b)\right)\Theta\left(-\mathrm{Im}(ab)\right)\right] .$$

As an important consequence, you can use the following relations without a proof:

$$\ln(ab) = \ln a + \ln b , \quad \text{(if } \mathrm{Im}(a) \text{ and } \mathrm{Im}(b) \text{ have different sign)} ,$$
$$\ln\left(\frac{a}{b}\right) = \ln a - \ln b , \quad \text{(if } \mathrm{Im}(a) \text{ and } \mathrm{Im}(b) \text{ have the same sign)} .$$

(b) For some special kinematic configurations, the solution of the integral simplifies significantly. Prove the following results for some of these special kinematic configurations:

$$
\begin{aligned}
B_0(p^2;0,0) &= \frac{1}{\varepsilon} + 2 - \ln\left(\frac{-p^2 - i\epsilon}{Q^2}\right) \qquad (\text{for } p^2 \neq 0)\ , \\
B_0(0;m^2,m^2) &= \frac{1}{\varepsilon} - \ln\left(\frac{m^2}{Q^2}\right) \qquad (\text{for } m^2 \neq 0)\ , \\
B_0(0;0,m^2) &= \frac{1}{\varepsilon} + 1 - \ln\left(\frac{m^2}{Q^2}\right) \qquad (\text{for } m^2 \neq 0)\ .
\end{aligned}
$$

Hint: Use the appropriate Taylor expansion in $x_{1/2}$.