Lösungen Aufgabe 1

a) Nebenrechnung: $\alpha x^2 + ikx = \alpha\left(x^2 + \frac{ik}{\alpha}x\right) = \alpha\left[x^2 + 2\frac{ik}{2\alpha}x + \left(\frac{ik}{2\alpha}\right)^2 - \left(\frac{ik}{2\alpha}\right)^2\right]$

$$= \alpha\left[\left(x + i\frac{k}{2\alpha}\right)^2 + \frac{k^2}{4\alpha^2}\right] = \alpha\left(x + i\frac{k}{2\alpha}\right)^2 + \frac{k^2}{4\alpha}$$

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-(\alpha x^2 + ikx)} = \frac{e^{-k^2/4\alpha}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-\alpha\left(x + i\frac{k}{2\alpha}\right)^2}$$

$$= \frac{e^{-k^2/4\alpha}}{\sqrt{2\pi}}\int dy\, \frac{1}{\sqrt{\alpha}}\, e^{-\left(y + i\frac{k}{2\alpha}\right)^2} = \frac{e^{-k^2/4\alpha}}{\sqrt{2\pi}}\,\frac{1}{\sqrt{\alpha}}\sqrt{\pi} = \frac{1}{\sqrt{2\alpha}}\, e^{-k^2/4\alpha}$$

Subst: $y = x\sqrt{\alpha}$ ($dy/dx = \sqrt{\alpha}$) Hinweis

$\Rightarrow \hat{f}(k)$ wieder Gauss-Funktion!

b) i) $\psi(x,t) = \frac{1}{\sqrt{2\pi}}\int dk\, g(k)\, e^{i(kx - \omega(k)t)} = \frac{C}{\sqrt{2\pi}}\exp\left[-\frac{1}{2a^2}(k-k_0)^2 + ikx - i\frac{\hbar k^2}{2m}t\right]$

$u = k - k_0$, $\frac{du}{dk} = 1$

$$= \frac{C}{\sqrt{2\pi}}\int du\, \exp\left[-\frac{u^2}{2a^2} + i(u+k_0)x - i\frac{\hbar(u+k_0)^2 t}{2m}\right]$$

$$= \frac{C}{\sqrt{2\pi}}\int du\, \exp\left[-\frac{1}{2a^2}u^2 + iux + ik_0x - i\frac{\hbar k}{2m}u^2 - i\underbrace{\frac{\hbar k_0^2}{2m}}_{\omega_0}t - i\underbrace{\frac{\hbar k_0 t}{m}}_{v_g}u\right]$$

$$= \frac{C}{\sqrt{2\pi}}\int du\, \exp\left[-\frac{1}{2a^2}u^2 - i\frac{\hbar t}{2m}u^2 + iux - iv_g t u - ik_0 x - i\omega_0 t\right]$$

$$= \underbrace{\frac{C}{\sqrt{2\pi}}\, e^{i(k_0 x - \omega_0 t)}}_{\equiv \tilde{C}}\int du\, \exp\left[-\left(\frac{1}{2a^2} + i\frac{\hbar t}{2m}\right)u^2 + i(x - v_g t)u\right]$$

$$= \tilde{C}\int du\, \exp\left[-\underbrace{\frac{1 + i\frac{\hbar t a^2}{m}}{2a^2}}_{\equiv s}u^2 + i(x - v_g t)u\right.$$

$$= \tilde{C}\int du\, \exp\left[-s\left(u^2 - i\frac{x - v_g t}{s}u\right)\right]$$

$$= \tilde{C}\int du\, \exp\left[-s\left\{\left(u - i\frac{x - v_g t}{2s}\right)^2 + \frac{(x - v_g t)^2}{4s^2}\right\}\right]$$

$$= \tilde{C}\int du\, \exp\left[-s\left(u - i\frac{x - v_g t}{2s}\right)^2 - \frac{(x - v_g t)^2}{4s}\right]$$

$$= \tilde{C}\, e^{-\frac{(x - v_g t)^2}{4s}}\int du\, \exp\left[-s\left(u - i\frac{x - v_g t}{2s}\right)^2\right] \qquad y = u\sqrt{s},\quad dy/du = \sqrt{s}$$

$$= \tilde{C}\, e^{-\frac{(x - v_g t)^2}{4s}}\,\frac{1}{\sqrt{s}}\underbrace{\int dy\, e^{-\left(u - i\frac{x - v_g t}{2s^2}\right)^2}}_{\sqrt{\pi}}$$

$$= \frac{Ca}{\sqrt{1 + i\frac{\hbar t a^2}{m}}}\exp\left\{-\frac{a^2(x - v_g t)^2}{2\left(1 + i\frac{a^2\hbar t}{m}\right)}\right\} e^{i(k_0 x - \omega_0 t)}$$

ii) $$\Psi(x,t) = \frac{Ca}{\sqrt{1+i\frac{a^2\hbar t}{m}}} e^{i(k_0x-\omega_0 t)} \exp\left[-\frac{a^2(x-v_g t)^2\left(1-i\frac{a^2\hbar t}{m}\right)}{2\left(1+\frac{a^4\hbar^2t^2}{m^2}\right)}\right]$$

$$= \frac{Ca}{\sqrt{1+i\frac{a^2\hbar t}{m}}} \exp\left[i\left(k_0x-\omega_0 t+\frac{\frac{a^4\hbar t}{m}(x-v_g t)^2}{2\left(1+\frac{a^4\hbar^2t^2}{m^2}\right)}\right)\right]\exp\left[-\frac{a^2(x-v_g t)^2}{2\left(1+\frac{a^4\hbar^2t^2}{m^2}\right)}\right]$$

$$|\Psi(x,t)|^2 = \Psi^*(x,t)\Psi(x,t) = \frac{|C|^2a^2}{\sqrt{\quad}}\exp\left[-\frac{a^2(x-v_g t)^2}{1+\frac{a^4\hbar^2t^2}{m^2}}\right]$$

Betragsquadrat von e^{ix}-Terme fallen weg ($e^{ix}e^{-ix}=1$)

$$\int dx\,|\Psi(x,t)|^2 = \frac{|C|^2a^2}{\sqrt{\quad}}\int dx\,\exp\left[-\frac{a^2(x-v_g t)^2}{1+\frac{a^4\hbar^2t^2}{m^2}}\right]$$

trägt nicht bei!

$$y = \sqrt{\frac{a^2}{1+\frac{a^4\hbar^2t^2}{m^2}}}\,x$$

$$= |C|^2 a\int dy\, e^{-y^2} = |C|^2a\sqrt{\pi} \overset{!}{=} 1$$

$$\Rightarrow |C| = \frac{1}{a\sqrt{\pi}}$$

$$\int dk\,|g(k)|^2 = \int dk\, C^2\exp\left(-\frac{(k-k_0)^2}{2a^2}2\right) = \int dk\, C^2 e^{-k^2/a^2}$$

$$y = \frac{k}{a}$$

$$= C^2a\int dy\,e^{-y^2} = \sqrt{\pi}\,a\,C^2 = 1 \Rightarrow C^2 = \frac{1}{\sqrt{\pi}\,a}$$

c) $$\langle x\rangle = \int dx\,|\Psi|^2x = \underbrace{\frac{C^2a^2}{\sqrt{1+\frac{\hbar^2a^4t^2}{m^2}}}}_{\tilde{C}}\int dx\,x\,\exp\left[-\frac{a^2(x-v_g t)^2}{1+\frac{a^4\hbar^2t^2}{m^2}}\right]$$

$$y = x - v_g t$$

$$= \tilde{C}\int dy\,(y+v_g t)\exp\left[-\frac{a^2y^2}{1+\frac{a^4\hbar^2t^2}{m^2}}\right]$$

kein Beitrag da ungerade Funktion!

$$= \tilde{C}\int dy\,\exp\left[-\underbrace{\frac{a^2y^2}{1+\frac{a^4\hbar^2t^2}{m^2}}}_{=u}\right] = C^2a\,v_g t\int du\,e^{-u^2} = v_g t$$

$\langle p\rangle$: $$p = \frac{\hbar}{i}\frac{\partial}{\partial x} \Rightarrow \frac{\partial}{\partial x}\Psi(x,t) = \Psi(x,t)\left[ik_0 - \frac{a^2(x-v_g t)}{1+i\frac{a^2\hbar t}{m}}\right]$$

$$\Rightarrow \Psi^*(x,t)\frac{\hbar}{i}\frac{\partial}{\partial x}\Psi(x,t) = |\Psi(x,t)|^2\left[\hbar k_0 + i\frac{\hbar a^2(x-v_g t)}{1+i\frac{a^2\hbar t}{m}}\right]$$

$$\Rightarrow \langle p\rangle = \int dx\,|\Psi(x,t)|^2\left[\hbar k_0 + i\frac{\hbar a^2(x-v_g t)}{1+i\frac{a^2\hbar t}{m}}\right]$$

$$= \underbrace{\frac{C^2a^2}{\sqrt{\quad}}\int dx\,\exp\left[-\frac{a^2(x-v_g t)^2}{1+\frac{a^4\hbar^2t^2}{m^2}}\right]}_{\text{bereits berechnet}}\left[\hbar k_0 + i\frac{\hbar a^2(x-v_g t)^2}{1+i\frac{a^2\hbar t}{m}}\right]$$

$$= C^2a\sqrt{\pi}\,\hbar k_0 = \hbar k_0$$

d) $(\Delta x)^2 = \langle (x-\langle x\rangle)^2\rangle = \langle x^2 + \langle x\rangle^2 - 2x\langle x\rangle\rangle = \langle x^2\rangle + \langle x\rangle^2 - 2\langle x\rangle^2$

$= \langle x^2\rangle - \langle x\rangle^2$

analog für Δp

$$\langle x^2\rangle = \int dx\, |\psi(x,t)|^2 x^2 = \underbrace{\frac{C^2 a^2}{\sqrt{1+\frac{a^4\hbar^2t^2}{m^2}}}}_{\tilde{C}} \int dx\, x^2 \exp\left[-\frac{a^2(x-v_g t)^2}{1+\frac{a^4\hbar^2t^2}{m^2}}\right]$$

$y = x - v_g t$

$$= \tilde{C}\int dy\,(y^2 + v_g^2 t^2)\exp\left[-\frac{a^2y^2}{1+\frac{a^4t^2\hbar^2}{m^2}}\right]$$

$$= C^2a^2\left[v_g^2t^2\sqrt{\pi} + \frac{1+\frac{a^4\hbar^2t^2}{m^2}}{2a^2}\sqrt{\pi}\right] = v_g^2t^2 + \frac{1}{2a^2}\left(1+\frac{a^4\hbar^2t^2}{m^2}\right)$$

wobei verwendet wurde, dass $\int dx\, x^2 e^{-ax^2} = \frac{a^{-3/2}\sqrt{\pi}}{2}$

$$\Rightarrow \Delta x = \frac{1}{\sqrt{2}a}\sqrt{1+\frac{a^4\hbar^2t^2}{m^2}}$$

$$\frac{\partial^2}{\partial x^2}\psi(x,t) = \frac{\partial}{\partial x}\psi(x,t)\left[-ik_0 - \frac{a^2(x-v_g t)}{1+i\frac{a^2\hbar t}{m}}\right]$$

$$= \psi(x,t)\left[ik_0 - \frac{a^2(x-v_g t)}{1+i\frac{a^2\hbar t}{m}}\right]^2 + \psi(x,t)\left[-\frac{a^2}{1+i\frac{a^2\hbar t}{m}}\right]$$

$$= \psi(x,t)\left[-k_0^2 + \frac{a^4(x-v_g t)^2}{\left(1+i\frac{a^2\hbar t}{m}\right)^2} - 2ik_0\frac{a^2(x-v_g t)}{1+i\frac{a^2\hbar t}{m}} - \frac{a^2}{1+i\frac{a^2\hbar t}{m}}\right]$$

$$\Rightarrow \langle p^2\rangle = \int dx\, \psi^*(x,t)\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2\psi(x,t)$$

$$= -\hbar^2\int dx\,|\psi(x,t)|^2\left[-k_0^2 - \frac{a^2}{1+i\frac{a^2\hbar t}{m}} + \frac{a^4(x-v_g t)^2}{\left(1+i\frac{a^2\hbar t}{m}\right)^2} \underbrace{- 2ik_0\frac{a^2(x-v_g t)}{1+i\frac{a^2\hbar t}{m}}}_{=0}\right]$$

$$= -\hbar^2\left\{\left(-k_0^2 - \frac{a^2}{1+i\frac{a^2\hbar t}{m}}\right) + \frac{C^2a^2}{\sqrt{1+\frac{\hbar^2a^4t^2}{m^2}}}\frac{a^4}{\left(1+i\frac{a^2\hbar t}{m}\right)}\int dy\, y^2\exp\left[\frac{-a^2y^2}{1+\frac{a^4\hbar^2t^2}{m^2}}\right]\right.$$

$$= \hbar^2k_0^2 + \frac{a^2\hbar^2}{1+i\frac{a^2\hbar t}{m}} - \frac{C^2a^2\hbar^2}{\sqrt{1+\frac{\hbar^2a^4t^2}{m^2}}}\frac{a^4}{\left(1+i\frac{a^2\hbar t}{m}\right)^2}\left(\frac{1+\frac{a^4\hbar^2t^2}{m^2}}{a^2}\right)^{3/2}\frac{\sqrt{\pi}}{2}$$

$$= \hbar^2k_0^2 + \frac{a^2\hbar^2}{1+i\frac{a^2\hbar t}{m}} - \frac{a^2\hbar^2}{2}\frac{1+\frac{a^4\hbar^2t^2}{m^2}}{\left(1+i\frac{a^2\hbar t}{m}\right)^2}$$

$$= \hbar^2k_0^2 + \frac{a^2\hbar^2}{2}$$

$$\Rightarrow \Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\hbar^2 k_0^2 + \frac{a^2\hbar^2}{2} - \hbar^2 k_0^2} = \frac{a\hbar}{\sqrt{2}}$$

$\Rightarrow$ $\langle p\rangle$ und Δp sind zeitlich konstant $\rightarrow$ Impulserhaltung

e) $|\Psi(x,t)|^2$ hat zeitabhängige Breite Δx

Für $t=0$: $\Delta x = \frac{1}{\sqrt{2}\,a}$ und $\Delta p = \frac{a\hbar}{\sqrt{2}}$ $\Rightarrow$ $\Delta x \cdot \Delta p = \frac{\hbar}{2} \neq 0$

$\Rightarrow$ Ort und Impuls können nicht gleichzeitig beliebig scharf gemessen werden!

$\Rightarrow$ Heisenberg'sche Unschärferelation

Allgemein für $t>0$ gilt $\Delta x \geq \Delta x(t=0)$

$$\Rightarrow \boxed{\Delta x \cdot \Delta p \geq \frac{\hbar}{2}}$$