Derivation of cross section formula

$$S = 1 + iT$$

$$\rightarrow \langle f|S|p_1 p_2\rangle = i\langle f|T|p_1 p_2\rangle$$

$$= i(2\pi)^4\delta^4(p_f - p_1 - p_2)\underbrace{\langle f|\mathcal{T}|p_1 p_2\rangle}_{\mathcal{M}}$$

probability for $i \to f$ transition

$$W_{f\leftarrow i} = \int d\tilde{p}_1\, d\tilde{p}_2\, d\tilde{p}_1'\, d\tilde{p}_2'\; f_1^*(p_1) f_1(p_1') f_2^*(p_2) f_2(p_2')$$

$$* (2\pi)^4\delta^4(p_f - \bar{p}_1 - \bar{p}_2)\; \underbrace{\langle f|\mathcal{T}|p_1 p_2\rangle^* \langle f|\mathcal{T}|p_1' p_2'\rangle}_{\approx |\langle f|\mathcal{T}|\bar{p}_1 \bar{p}_2\rangle|^2}$$

$$* \underbrace{(2\pi)^4\delta^4(p_1' + p_2' - p_1 - p_2)}_{\int d^4x\, e^{ix\cdot(p_1+p_2-p_1'-p_2')}}$$

$$\approx \int d^4x\; \underbrace{\tilde{f}_1^*(x)\tilde{f}_1(x)|\tilde{f}_2(x)|^2(2\pi)^4\delta^4(p_f - \bar{p}_1 - \bar{p}_2)|\langle f|\mathcal{T}|\bar{p}_1\bar{p}_2\rangle|^2}_{\frac{d^2W_{f\leftarrow i}}{dV\,dT}}$$

transition probability/unit volume/unit time

$$\frac{d^2W_{f\leftarrow i}}{dV\,dT} = |\tilde{f}_1(x)|^2|\tilde{f}_2(x)|^2(2\pi)^4\delta^4(p_f - \bar{p}_1 - \bar{p}_2)|\langle f|\mathcal{T}|\bar{p}_1\bar{p}_2\rangle|^2$$

Differential cross section

$$d\sigma = \frac{\#\text{ scattering ev.}/\text{unit time}/\text{unit volume}}{\text{incident flux}\quad \#\text{ scatterers}/\text{unit volume}}$$

$$= \frac{d^2 W_{f\leftarrow i}/dV\,dT}{\text{incident flux} * \text{prob. density of scatterers}}$$

scatterer at rest: $\bar{p}_2 = (m_2, 0, 0, 0)$

scatterer density $= i\tilde{f}_1(x)^* \overleftrightarrow{\partial_0} \tilde{f}_1(x) \approx 2p_2^0 |\tilde{f}_2(x)|^2$

$|$incident flux$| = |i\tilde{f}_1(x) \overleftrightarrow{\nabla} \tilde{f}_1(x)| = 2|\vec{p}_1|\,|\tilde{f}_1(x)|^2$

$$\Rightarrow d\sigma = (2\pi)^4 \delta^4(P_f - \bar{p}_1 - \bar{p}_2) \underbrace{\frac{1}{4m_2|\vec{p}_1|}}_{\text{flux factor}} |\langle f|J|\bar{p}_1\bar{p}_2\rangle|^2$$

$$\text{flux factor} = \frac{1}{4m_2\sqrt{(p_1^0)^2 - m_1^2}}$$

$$= \frac{1}{4\sqrt{(p_1^0 m_2)^2 - m_1^2 m_2^2}} = \frac{1}{4\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}}$$