Exercises: Stefan Liebler (stefan.liebler@kit.edu) (Office 12/03 - Build. 30.23)

Information regarding the exercise course:

"Scheinkriterium": In order to successfully pass "Theoretische Teilchenphysik I" we request you

- to register by writing an email to "stefan.liebler@kit.edu" until 30.04.2019 stating your name and "Matrikelnummer" as well as a sorted list of the three tutorials according to your preferences. Check the webpage of the course for the date of the three tutorials. The group assignment will be send to you by mail.
- to hand in hand-written solutions to the exercise sheets and obtain 40% of all points on the sum of 11 sheets. There will be 12 sheets but sheet 6 (Pfingstwoche) doesn't count. The exercise sheets are published each Monday on the webpage of the course, see bottom of the page. Please deliver your solution latest on Monday (one week after publication) before 14:00 to the corresponding mail box on the ground floor of the physics highrise. Two students can hand in together. We return your solutions in the tutorials on the following Wednesday. All dates for release, submission and the tutorials are printed on each sheet, see above. Lastly note that the 11 sheets will have around 15 points each.
- to actively participate in the tutorials and present (parts of) one exercise worth ~ 5 points at the blackboard.

Exercise 1: Classical electromagnetism $\boxed{\textbf{3+1+2 = 6 points}}$

We consider the action

$$S = \int d^4x \mathcal{L} \qquad \text{using the Lagrangian (density)} \quad \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu$$

with the Abelian field-strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ of the vector field A_μ. j^μ denotes the four-dimensional current $j^\mu = (\rho, \vec{j})$ in natural units.

(a) Use the Euler-Lagrange equations to derive the equations of motions for $A_\mu(x)$. For the vector field the Euler-Lagrange equations read

$$\partial_\rho \frac{\partial \mathcal{L}}{\partial(\partial_\rho A_\sigma)} - \frac{\partial \mathcal{L}}{\partial A_\sigma} = 0\,.$$

Show that the result $\partial_\mu F^{\mu\nu} = j^\mu$ is nothing but the (inhomogeneous) Maxwell equations. For this purpose put the equations in the standard form for the fields $\vec{E}$ and $\vec{B}$ by means of the identification $E_i = -F^{0i}$ and $\epsilon_{ijk} B_k = -F^{ij}$. *Hint:* For the derivatives first lower all indices by inserting the metric tensor and add δ's for catching the right indices.

(b) Prove from the previous subexercise the validity of the continuity equation $\partial_\mu j^\mu = 0$.

(c) The homogeneous Maxwell equations also have a covariant form, being $\partial_\mu \tilde{F}^{\mu\nu} = 0$ using the dual field-strength tensor $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ (with the total antisymmetric tensor $\epsilon^{\mu\nu\rho\sigma}$) or alternatively in form of the Bianchi identity, being $\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$. Insert the fields to check that the latter formula indeed reproduces the homogeneous Maxwell equations. Which combination of indices α, β and γ yields the homogeneous Maxwell equations?

Exercise 2: Gamma matrix representations $\boxed{\textbf{2+2+1+2+2 = 9 points}}$

For the representations of fermions we need to consider the Dirac matrices γ^μ ($\mu = 0, ..., 3$). This can be done in the (chiral) Weyl representation

$$\gamma_\chi^\mu = \left(\begin{pmatrix} 0 & 1_{2\times 2} \\ 1_{2\times 2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \right)$$

as well as in the Dirac representation

$$\gamma_D^\mu = \left(\begin{pmatrix} 1_{2\times 2} & 0 \\ 0 & -1_{2\times 2} \end{pmatrix}, \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \right) ,$$

where $1_{2\times 2}$ is the 2×2 unit matrix and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ with σ_i ($i = 1, 2, 3$) represent the Pauli matrices.

(a) Show that both representations obey the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ by inserting the explicit representations separately into the anti-commutation relations.
Hint: Remember $\{\sigma_i, \sigma_j\} = 2\delta_{ij} 1_{2\times 2}$ and show the relations for the three cases $\mu = \nu = 0$, $\mu = 0$, $\nu = i$ and $\mu = i, \nu = j$ separately.

(b) The Weyl and Dirac representations are connected by a unitary transformation U such that $\gamma_\chi^\mu = U^\dagger \gamma_D^\mu U$. Show that up to an arbitrary phase U is given by

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1_{2\times 2} & 1_{2\times 2} \\ -1_{2\times 2} & 1_{2\times 2} \end{pmatrix} .$$

(c) A fifth Dirac matrix, γ^5, can be defined by $\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3$. Show that the explicit forms of γ_D^5 and γ_χ^5 for both the Dirac and the Weyl representation, respectively, are given by

$$\gamma_D^5 = \begin{pmatrix} 0 & 1_{2\times 2} \\ 1_{2\times 2} & 0 \end{pmatrix} \quad , \qquad \gamma_\chi^5 = \begin{pmatrix} -1_{2\times 2} & 0 \\ 0 & 1_{2\times 2} \end{pmatrix} .$$

(d) Show that $\{\gamma^5, \gamma^\mu\} = 0$ holds for both the Dirac and Weyl representation separately by inserting the Dirac matrices explicitly in these representations.

(e) We now define chirality projectors $\omega_\mp$ by

$$\omega_\mp := \frac{1_{4\times 4} \mp \gamma^5}{2} ,$$

where ω_- is the left-chiral projector and ω_+ is the right-chiral projector. By using the explicit representation of γ^5 in the Weyl basis only, show that the $\omega_\mp$ obey the following projector properties:

$$\omega_\mp^2 = \omega_\mp , \qquad \omega_-\omega_+ = \omega_+\omega_- = 0 .$$

Note: In particle physics ω_- and ω_+ are often named P_L and P_R, respectively.