Prof. Dr. D. Zeppenfeld, Dr. S. Liebler

Release: 01.07.19
Submission: 08.07.19
Tutorial: 10.07.19

Exercises: Stefan Liebler (stefan.liebler@kit.edu) (Office 12/03 - Build. 30.23)

Exercise 1: Wick's theorem

1+1+3 = 5 points

As Wick's theorem is essential for the formulation of scattering amplitudes in terms of Feynman diagrams, we want to repeat a few necessary steps to derive it for a real scalar field ϕ.

(a) Argue why $\langle 0| {:}A{:} |0\rangle = 0$ for a normal-ordered operator.

(b) Verify that

$$\phi(x)\phi(y) = {:}\phi(x)\phi(y){:} + \left[\phi^+(x), \phi^-(y)\right]$$

in terms of $\phi^+(x) = \int d\tilde{p} a(p) e^{-ipx}$ and $\phi^-(x) = \int d\tilde{p} a^\dagger(p) e^{ipx}$. Use this to show explicitly that

$$T(\phi(x)\phi(y)) = {:}\phi(x)\phi(y){:} + f(x,y)\,,$$

where $f(x,y)$ is a c-number. Give the explicit form of $f(x,y)$ in terms of commutators and show that $f(x,y) = i\Delta_F(x,y)$.

(c) Write down Wick's theorem for three fields $T(\phi(x_1)\phi(x_2)\phi(x_3))$ and deduce the result for four fields. *Hint:* In contrast to the lecture do not take the vacuum expectation value. The vacuum expectation value lets all non-contracted combinations of fields vanish, see subexercise (a). *Add-on:* A common notation for a contraction in the literature is

$$\overbracket{\phi(x)\phi}(y) := \langle 0|T\phi(x)\phi(y)|0\rangle = i\Delta_F(x-y)\,.$$

Exercise 2: Feynman rules in ϕ^4 theory

3+1+4+2 = 10 points

As a practical application of Wick's theorem we want to derive the Feynman rules for ϕ^4 theory. We consider again ϕ^4 theory with the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4\,,$$

where the last term is the interaction term $\mathcal{L}_I$[2].

(a) We start with the propagator $\langle 0|T\phi(x)\phi(y)|0\rangle = i\Delta_F(x-y)$. The LSZ reduction formula[3] contains the vacuum expectation value

$$\langle 0|T\left\{\phi(x)\phi(y) + \phi(x)\phi(y)\frac{-i\lambda}{4!}\left[\int d^4z\phi^4(z)\right] + \dots\right\}|0\rangle\,,$$

[2] Note the different normalization 4! compared to previous exercises to simplify the Feynman rules!

[3] This exercise circumvents the additional factors that come with each external field, see lecture. They are part of the Feynman rules, that allow to construct the scattering amplitude!

where we expand the exponent with the interaction Lagrangian $\mathcal{L}_I$ to $\mathcal{O}(\lambda)$. The first term is nothing than the plain propagator $i\Delta_F(x-y)$. Use Wick's theorem to show that the second term results in

$$\begin{aligned}\langle 0|T\left\{\phi(x)\phi(y)\frac{-i\lambda}{4!}\int d^4z\phi^4(z)\right\}|0\rangle =&3\frac{\lambda}{4!}\Delta_F(x-y)\int d^4z\Delta_F(z-z)\Delta_F(z-z)\\ &+12\frac{\lambda}{4!}\int d^4z\Delta_F(x-z)\Delta_F(y-z)\Delta_F(z-z)\end{aligned}$$

Find a pictorial representation (i.e. Feynman diagram) for both the disconnected and connected contribution by drawing a line for each propagator and a dot for each point x, y and z. *Add-on:* The detailed evaluation of loop diagrams will be content of TTP2.

(b) The previous discussion was carried out in position space. However, Feynman rules turn out to be simpler in momentum space. For this purpose we use the well-known representation of the propagator in momentum space

$$i\Delta_F(x-y) = \langle 0|T\phi(x)\phi(y)|0\rangle = \int \frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}\frac{i}{p^2-m^2+i\epsilon},$$

such that we associate a momentum p (in general with direction) to each line in a Feynman diagram. Insert this representation of the propagator into the last term of the previous subexercise. Show that the integration over z results in momentum conservation at the vertex z. *Hint:* Recall $\int d^4z e^{-ikz} = (2\pi)^4\delta^{(4)}(k)$.

(c) We finally consider a real scattering process with two incoming particles with momenta p_1 and p_2 and two outgoing momenta p_3 and p_4. Split the vacuum expectation value given by

$$\langle 0|T\left\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\left[1+\frac{-i\lambda}{4!}\int d^4z\phi^4(z)\right]\right\}|0\rangle$$

and thus expanded up to $\mathcal{O}(\lambda)$ into combinations of propagators using Wick's theorem. Draw all Feynman diagrams, both connected and disconnected. *Add-on:* The external momenta are encoded in factors $\int d^4x_i e^{\pm ip_ix_i}$, see LSZ reduction formula, and need not to be considered further in this exercise.

(d) Use the previous subexercises to motivate that the Feynman rule for the vertex in combination with momentum conservation is given by $(-i\lambda)(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)$ in momentum space. *Add-on:* The combination of the external factors of the LSZ reduction formula together with vertex rules, momentum conservation at internal vertices, integration over unspecified momenta, see subexercise (b), and the propagators yields a set of Feynman rules, that allows to construct the scattering amplitude. Additionally each Feynman diagram comes with symmetry factors, as e.g. seen in subexercise (a).