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Exercise 1: Real corrections to $e^-\mu^+$ scattering — 1+2+2 = 5 points

We consider the scattering process of an electron with a heavy antilepton of charge +1, the muon μ^+. The Feynman rules for muons are identical with those of electrons, we just need to replace the mass m_e with the mass m_μ in propagators (and external spin sums). Mixed propagators $\langle 0|T\psi_e\bar{\psi}_\mu|0\rangle$ do not exist. The lowest-order process $e^-\mu^+ \to e^-\mu^+$ only has a single Feynman diagram involving a photon exchange between the lepton lines. We consider the process

$$e^-(p_1)\mu^+(p_2) \to e^-(p_1')\mu^+(p_2')\gamma(k)\,,$$

that involves the radiation of an additional photon. *Add-on:* This process is needed to cancel infrared divergences in higher-order corrections to the process $e^-\mu^+ \to e^-\mu^+$.

(a) Draw all four Feynman diagrams. Determine and label all momenta, that flow through internal propagators.

(b) Use the Feynman rules of QED to write down the amplitude. *Hint:* Keep the rather unpleasent long expression.

(c) The structure of the amplitude is such that $\mathcal{M} = \epsilon_\mu(k)\mathcal{M}^\mu$, where $\epsilon_\mu(k)$ is the polarization vector of the photon. Gauge invariance implies the so-called Ward identity, which claims that $k_\mu\mathcal{M}^\mu = 0$, where k_μ is the momentum of the photon. Show explicitly, that this relation holds for the amplitude in subexercise (b). *Hint:* Pair up the amplitudes of two of the diagrams and therein "massage" the numerator of the Feynman propagators such that you can use Dirac's equation, e.g. $(\not{p}_1 - m_e)u(p_1) = 0$ and $\bar{u}(p_1')(\not{p}_1' - m_e) = 0$. The remaining terms should cancel between the two diagrams.

Exercise 2: Trace identities in 4 dimensions — 2+3 = 5 points

In the evaluation of amplitudes for scattering processes involving fermions we need to determine traces over γ matrices, which come from fermionic propagators and spin sums of external fermions. Independent of any representation, the Dirac matrices γ^μ in 4 space-time dimensions, see also sheet 1, obey the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. The γ^μ can always be chosen to be unitary so that $(\gamma^\mu)^\dagger = (\gamma^\mu)^{-1}$ holds.

(a) Prove the following Dirac algebra relations by using the unitarity and anti-commutation relations of the Dirac matrices only (i.e. do not use the explicit representations of the Dirac matrices from sheet 1):

$$\begin{aligned}
(\gamma^\mu)^\dagger &= \gamma^0\gamma^\mu\gamma^0\,,\\
\gamma_\mu\gamma^\mu &= 4\cdot 1_{4\times 4}\,,\\
\gamma_\mu\gamma^\alpha\gamma^\mu &= -2\gamma^\alpha\,,\\
\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\mu &= 4g^{\alpha\beta}\,.
\end{aligned}$$

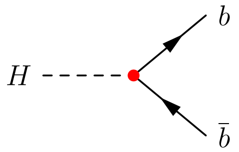
(b) Prove the following trace identities by using the anti-commutation relations of the Dirac matrices and the general properties of a matrix trace only:

$$\mathrm{tr}\,(\gamma^\mu) = 0\ , \qquad \mathrm{tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho) = 0\ ,$$
$$\mathrm{tr}\,(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}\ ,$$
$$\mathrm{tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)\ .$$

Add-on: Any combination of an odd number of γ matrices vanishes as long as γ^5 is not involved. Specific rules involving γ^5 can be found in the literature. Regularization in higher-order calculations can e.g. be performed by calculating in d rather than in 4 dimensions. In that case d-dimensional identities need to be used, see TTP2!

Exercise 3: Higgs-boson decay into bottom quarks **4+1 = 5 points**

We want to discuss a simple $1 \to 2$ decay, namely the decay of the Higgs boson of the Standard Model (SM) of particle physics into a pair of fermions. We consider the most dominant decay mode into two b-quarks, $H(p) \to b(p_1)\bar{b}(p_2)$, which is apart from a color factor identical to the calculation $H \to e^-e^+$. Though we have not dicussed the SM in all its details yet, we only need one Feynman rule for the vertex compared to the QED Feynman rules. It is given by



$$= -im_b\sqrt{\sqrt{2}G_F}\,,$$

and has in particular no Lorentz structure as the Higgs boson is a scalar particle (in contrast to the photon-electron-positron vertex, which is proportional to γ^μ). The Higgs boson has a mass of m_H, the bottom quark b and the corresponding anti-quark $\bar{b}$ have both a mass of $m_b = m_{\bar{b}}$. G_F is Fermi's constant.

(a) Determine the matrix element $i\mathcal{M}$ of the decay, which only involves spinors and the above vertex. Show that the partial decay width, which is defined by

$$d\Gamma(H \to b\bar{b}) = \frac{1}{2m_H} d\Phi_2 \sum_{\text{spins}} \sum_{\text{color}} |\mathcal{M}|^2$$

results in

$$\Gamma(H \to b\bar{b}) = \left(\frac{1}{16\pi m_H}\beta_b\right)\left(6m_b^2\sqrt{2}G_F m_H^2 \beta_b^2\right) \qquad \text{with} \qquad \beta_b = \sqrt{1 - \frac{4m_b^2}{m_H^2}}\ .$$

Hint: For the phase space you can use the result for $d\Phi_2$ from sheet 9. The sum over spins is carried out by the replacements $\sum_s u_s(p_1)\bar{u}_s(p_1) = \not{p}_1 + m_b$ and $\sum_s v_s(p_2)\bar{v}_s(p_2) = \not{p}_2 - m_b$. Then use the trace identities derived in exercise 2. Remember that $p^2 = m_H^2$, $p_i^2 = m_b^2$ and $2p_1 \cdot p_2 = p^2 - p_1^2 - p_2^2$. The sum over color implies a factor of 3, as it sums over the three colors (associated with the strong interaction, i.e. QCD) of the final state bottom quarks. The vertex has no color structure, as the Higgs boson is color neutral.

(b) Calculate the partial decay width for a Higgs mass of $m_H = 125\,\mathrm{GeV}$, the bottom-quark mass of $m_b = 4.8\,\mathrm{GeV}$ and $G_F = 1.17 \cdot 10^{-5}\,\mathrm{GeV}^{-2}$. Why does the Higgs boson not decay into a pair of top quarks with $m_t = 172\,\mathrm{GeV}$? Why are the decays into lighter quarks of less relevance than the decay into bottom quarks?