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Exercise 1: Decay width of a muon - Squared matrix element $\boxed{1+2+2+4 = 9 \text{ points}}$

We want to calculate the partial decay width and therefore the lifetime of a muon with momentum p_1. The by far dominant decay mode of the muon is given as

$$\mu^-(p_1) \to \nu_\mu(p_2) + e^-(p_3) + \bar{\nu}_e(p_4)\,.$$

(a) Draw the Feynman diagram of the process. *Add-on:* It involves the coupling of a charged and massive W boson to a lepton and a neutrino, which emerges after electroweak symmetry breaking in the Standard Model of particle physics.

(b) Compose the matrix element and show that it is given by

$$i\mathcal{M} = \bar{u}(p_2)\frac{ig}{\sqrt{2}}\gamma^\alpha\frac{1-\gamma^5}{2}u(p_1)\frac{-ig_{\alpha\beta}}{(p_1-p_2)^2-m_W^2}\bar{u}(p_3)\frac{ig}{\sqrt{2}}\gamma^\beta\frac{1-\gamma^5}{2}v(p_4)\,.$$

Show that for typical momenta $m_\mu^2 \sim (p_1-p_2)^2 \ll m_W^2$, this expression simplifies to

$$i\mathcal{M} = -\,i\frac{G_F}{\sqrt{2}}\bar{u}(p_2)\gamma^\alpha\left(1-\gamma_5\right)u(p_1)\bar{u}(p_3)\gamma_\alpha\left(1-\gamma_5\right)v(p_4)\,.$$

G_F is the Fermi constant and is defined as $\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2}$.
Hint: You need the coupling of a W boson with a lepton and a neutrino and the propagator of a W boson with momentum q and mass m_W. They are given by

$$\frac{ig}{\sqrt{2}}\gamma^\alpha\frac{1-\gamma_5}{2} \qquad \text{and} \qquad \frac{-ig_{\alpha\beta}}{q^2-m_W^2} \qquad \text{, respectively.}$$

We dropped the spin indices of the fermions in the above formulas.
Add-on: In contrast to QED, electroweak theory is chiral, which explains the projection operator involving γ^5. It ensures that the W boson just couples to left-handed particles. The low-energy expansion of the W boson propagator is the prime example of an effective field theory, where the heavy degree (W boson) is integrated out and its contribution is condensed into an effective coupling G_F, that describes the direct coupling of 4 fermions.

(c) Calculate $\mathcal{M}^\dagger$ and simplify it as far as possible.

(d) Calculate

$$\overline{|\mathcal{M}|^2} = \frac{1}{2}\sum_{s_1,s_2,s_3,s_4}\mathcal{M}\mathcal{M}^\dagger\,,$$

where we averaged over the incoming spin combinations and summed over all possible outgoing spin combinations. As an intermediate step show that

$$\overline{|\mathcal{M}|^2} = \frac{G_F^2}{4}\mathrm{tr}\left[\not{p}_2\gamma^\alpha(1-\gamma_5)(\not{p}_1+m_\mu)\gamma^\beta(1-\gamma_5)\right]\mathrm{tr}\left[\not{p}_4\gamma_\beta(1-\gamma_5)(\not{p}_3+m_e)\gamma_\alpha(1-\gamma_5)\right]\,.$$

Finally show that

$$\overline{|\mathcal{M}|^2} = 64G_F^2(p_1\cdot p_4)(p_2\cdot p_3)\,.$$

Hint: Use the relations $(\gamma^5)^2 = 1$ and

$$\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^5) = 0\,, \quad \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = -4i\epsilon^{\mu\nu\rho\sigma}\,, \quad \epsilon^{\alpha\beta\nu\mu}\epsilon_{\alpha\beta}{}^{\rho\sigma} = -2\left(g^{\nu\rho}g^{\mu\sigma} - g^{\nu\sigma}g^{\mu\rho}\right)\,.$$

First interchange the γ matrices such, that there is only one γ^5 left in the trace. As the evaluation of the traces is tedious, we invite you to use a computer algebra program. You can use for example `Form`, `Reduce` or `FeynCalc`. You can alternatively install `FeynArts` and `FormCalc`, which allow you to write down the amplitude from a model file including the Feynman rules, i.e. it performs all steps from (a) to (d). The latter three codes need `Mathematica`.

Exercise 2: Decay width of a muon - Phase space **1+1+1+2+1 = 6 points**

We continue with the previous exercise and add the three-particle phase space. This allows to calculate the spectrum of the energy of the emitted electron and the partial decay width of the muon. The differential partial decay width is given by

$$d\Gamma = \frac{1}{2E_1}\overline{|\mathcal{M}|^2}d\Phi_3 \quad \text{with} \quad d\Phi_3 = \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}\frac{d^3\vec{p}_3}{(2\pi)^3 2E_3}\frac{d^3\vec{p}_4}{(2\pi)^3 2E_4}(2\pi)^4\delta^{(4)}(p_1 - p_2 - p_3 - p_4)\,.$$

In the following we work in the rest frame of the muon, i.e. $p_1 = (m, \vec{0})$, and assume all final state particles to be massless.

(a) Express $\overline{|\mathcal{M}|^2}$ in terms of m and E_4, where E_4 is the energy of $\bar{\nu}_e$.

(b) Perform the integration over $d^3\vec{p}_2$ in $d\Phi_3$ using the δ distribution. For this purpose use $\frac{d^3\vec{p}_2}{2E_2} = d^4p_2\theta(E_2)\delta(p_2^2)$ and show that

$$\delta(p_2^2) = \delta((p_1 - p_3 - p_4)^2) = \frac{1}{2E_3E_4}\delta\left(\cos\theta_4 - C\right) \quad \text{with} \quad C = 1 + \frac{m^2 - 2m(E_3 + E_4)}{2E_3E_4}\,,$$

where θ_4 denotes the angle between $\vec{p}_3$ and $\vec{p}_4$. Momentum conservation is now implicitly encoded through the previous δ distribution.

(c) Show that the three-particle phase space integrated over the angles associated with e^- and $\bar{\nu}_e$ is given by

$$d\Phi_3 = \frac{1}{4(2\pi)^3}dE_3dE_4\theta(E_2)\theta(1 - C)\theta(1 + C)\,.$$

Use the θ functions to show that the integration boundaries of E_3 and E_4 are for example given by $E_3 \in [0, \frac{m}{2}]$ and $E_4 \in [\frac{m}{2} - E_3, \frac{m}{2}]$.
Hint: Use $d^3\vec{p}_i = E_i^2dE_id\Omega_i$ with $d\Omega_i = d\phi_id\cos\theta_i$.

(d) Integrate over E_4 to obtain the energy spectrum of the electron $d\Gamma/dE_3$. Finally integrate over E_3 to show that

$$\Gamma = \frac{m^5G_F^2}{192\pi^3}\,.$$

(e) Finally calculate the life time given by $\tau = \hbar/\Gamma$ and compare with the experimental value. The numerical parameters are (see "Review of Particle Physics", http://pdg.lbl.gov)

$$\begin{aligned} m = m_\mu &= 0.1056583745\,\mathrm{GeV}\,, \\ G_F &= 1.1663787\cdot 10^{-5}\,\mathrm{GeV}^{-2}\,, \\ \hbar &= 6.582119514\cdot 10^{-25}\,\mathrm{GeVs}\,, \\ \tau_\mu &= 2.1969811\cdot 10^{-6}\,\mathrm{s}\,. \end{aligned}$$