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Information regarding the exercise course:

"Scheinkriterium": Criterion 2 on sheet 1 is changed as follows: If you obtain at least 40% of all points on the sum of the first 8 sheets (excluding sheet 6, thus effectively 7 sheets) you get a certificate for 8 ECTS points. If you obtain at least 40% of all points on the sum of all 12 sheets (excluding sheet 6, thus effectively 11 sheets) you get a certificate for 12 ECTS points. For now you do not have to register for one or the other. Depending on your performance you get a certificate for either 8 or 12 ECTS points. Criteria 1 and 3 on sheet 1 regarding registration and active participation in the exercise course remain identical in both cases.

Exercise 1: Chiral fermions — **2+2+2+2 = 8 points**

In this exercise we consider a fermion with mass m and four momentum $(E, p_x, p_y, p_z)^T$. The four-dimensional Dirac spinor can be decomposed into two two-dimensional Weyl spinors, which are defined through

$$\chi_+(p) = \frac{1}{\sqrt{2|\vec{p}|(|\vec{p}|+p_z)}}\begin{pmatrix}|\vec{p}|+p_z\\ p_x+ip_y\end{pmatrix} \quad\text{and}\quad \chi_-(p) = \frac{1}{\sqrt{2|\vec{p}|(|\vec{p}|+p_z)}}\begin{pmatrix}-p_x+ip_y\\ |\vec{p}|+p_z\end{pmatrix}.$$

(a) Show that χ_+ and χ_- are Weyl spinors with helicities $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively. Thus show that they fulfill the identity

$$\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\chi_\pm(p) = \pm\chi_\pm(p)\,.$$

The product of the Pauli matrices with the momentum defines the helicity operator and originates from $\vec{S}\cdot\vec{p}$, which measures the spin along the direction of flight. The helicities $-\frac{1}{2}$ and $+\frac{1}{2}$ correspond to the left-handed and right-handed fermion, respectively. *Add-on:* By construction helicity is not Lorentz-invariant. We define a Lorentz-invariant quantity named chirality through the projection operators on sheet 1 to be applied to Dirac spinors. Chirality corresponds to left- and right-handed representations of the Poincaré group, see later in the TTP1 course. Only for massless particles chirality equals helicity.

(b) The exact definition of the Dirac spinors depends on the chosen representation of the Dirac matrices. Confirm that in the chiral Weyl representation of the Dirac matrices (see sheet 1) the four-dimensional spinors

$$u(p,\lambda) = \begin{pmatrix}\sqrt{E-\lambda|\vec{p}|}\;\chi_\lambda(p)\\ \sqrt{E+\lambda|\vec{p}|}\;\chi_\lambda(p)\end{pmatrix} \quad\text{and}\quad v(p,\lambda) = \begin{pmatrix}-\lambda\sqrt{E+\lambda|\vec{p}|}\;\chi_{-\lambda}(p)\\ \lambda\sqrt{E-\lambda|\vec{p}|}\;\chi_{-\lambda}(p)\end{pmatrix}$$

are solutions of the Dirac equation for u- and v-spinors, respectively.
Hint: First write out the matrix form of $\not{p} = E\gamma^0 - p_i\gamma^i$.

(c) Confirm the relations $\bar{v}(p,\lambda)v(p,\lambda') = -2m\delta_{\lambda\lambda'}$ and $\bar{v}(p,\lambda)u(p,\lambda') = 0$.
Hint: First show $\chi_\lambda^\dagger(p)\chi_\lambda(p) = 1$ and $\chi_+^\dagger(p)\chi_-(p) = 0$ and remember $\bar{v} = v^\dagger\gamma^0$.

(d) Prove, without using explicit representations for the Dirac matrices, the Gordon identity, which is the relation

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{(p+p')^\mu}{2m} + i\frac{\sigma^{\mu\nu}q_\nu}{2m}\right]u(p)\,,$$

where $q = p' - p$ and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu]$. The relation holds for on-shell particles, such that you will need to use the Dirac equation. This decomposition is helpful to determine the Landé factor (g-factor) of a fermion, e.g. the electron.
Hint: Start from the right-hand side and first confirm $\sigma^{\mu\nu} = i(\gamma^\mu\gamma^\nu - g^{\mu\nu})$.

Exercise 2: SU(N) **representations** **3+1+2+1 = 7 points**

The Lie groups SU(N) are of particular importance in particle physics as they for example form the gauge structures of the Weinberg-Salam model and quantum chromodynamics, which are associated with the Lie groups SU(2) and SU(3), respectively. Also SU(3) explains the flavour structure in the quark model (Eightfold Way). A representation of a group is a mapping on a set of matrices that fulfill the group axioms. A particularly important representation of SU(N) groups is the *fundamental* representation, an irreducible matrix representation of dimension N, which is formed by the generators

$$T^a_{ij}\,, \qquad a = 1, \ldots, N^2 - 1\,, \qquad i, j = 1, \ldots, N\,,$$

which are hermitian, i.e. $T^{a\dagger} = T^a$, and traceless, which implies $\mathrm{Tr}(T^a) = 0$. They are normalized through the relation $\mathrm{Tr}\left(T^a T^b\right) = \frac{1}{2}\delta^{ab}$. For $N = 2$ and 3 the generators of the *fundamental* representation are given by the Pauli matrices $T^a = \frac{\sigma^a}{2}$ for SU(2) and the Gell-Mann matrices $T^a = \frac{\lambda^a}{2}$ for SU(3), respectively. For the *fundamental* representation the generators fulfill the following commutation and anti-commutation relations

$$\left[T^a, T^b\right] = i f_{abc} T^c\,, \tag{1}$$

$$\left\{T^a, T^b\right\} = \frac{1}{N}\,\delta^{ab}\,1_{N\times N} + d_{abc}T^c\,, \tag{2}$$

which in turn defines the totally antisymmetric structure constants f_{abc} and the totally symmetric symbols d_{abc}. The commutation relation in Eq. (1) is actually valid for any representation of SU(N) and is the so-called Lie algebra of SU(N). [1] In the *fundamental* representation each complex ($N \times N$)-matrix M can be written in terms of the $N^2 - 1$ generators and the unit matrix in the form

$$M = c_0 1_{N\times N} + \sum_{a=1}^{N^2-1} c_a T^a\,. \tag{3}$$

(a) Show that the orthogonality relation in combination with Eq. (3) and the tracelessness of the generators in the *fundamental* representation yield the Fierz identity of SU(N) given by

$$T^a_{ij}T^a_{kl} \equiv \sum_{a=1}^{N^2-1} T^a_{ij}T^a_{kl} = \frac{1}{2}\delta_{il}\delta_{jk} - \frac{1}{2N}\delta_{ij}\delta_{kl}\,.$$

Hint: First show that $c_0 = \frac{1}{N}\mathrm{tr}(M)$ and $c_a = 2\mathrm{tr}(T^a M)$ using Eq. (3). Then insert these relations in Eq. (3) and express it with indices remembering $\mathrm{tr}(A) = A_{kl}\delta_{kl}$.

(b) Show that for any representation of SU(N)

$$C_2 = T^a T^a$$

with a sum over a is a Casimir invariant, i.e. $[C_2, T^a] = 0$ for all generators T^a.

(c) Show that the structure constants f_{abc} and d_{abc} are real by using the hermiticity of the generators of the *fundamental* representation. *Hint:* First show that $[T^a, T^b]^\dagger = [T^b, T^a]$ and $\{T^a, T^b\}^\dagger = \{T^a, T^b\}$.

(d) Calculate the value of C_2 in the fundamental representation. *Hint:* Use (a) and (b).

[1] Another matrix representation is given by the structure constants f_{abc} themselves, the *adjoint* representation. Also $\mathcal{T}^a = -T^{a*}$ (complex conjugated, not hermitian!) defines a set of ($N \times N$) matrices, that form the so-called $\overline{N}$ ("N–bar") representation of SU(N), which are of relevance for the behaviour of antiquarks under SU(3).