Exercises: Stefan Liebler (stefan.liebler@kit.edu) (Office 12/03 - Build. 30.23)

Exercise 1: The Lorentz group **3+3+2+1+2 = 11 points**

The inhomogeneous Lorentz or Poincaré group includes translations, associated with the generator $\exp(-ia^\mu P_\mu)$ with $P_\mu = i\partial_\mu$ and a vector a^μ, and Lorentz transformations, associated with the generator $\exp(-\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu})$ with $M_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu$ and a tensor $\omega^{\mu\nu}$. Therein $M_{\mu\nu}$ can additionally include a spin component.

(a) By using the generators $M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$ determine the Lie algebra of the SO(3, 1), i.e. prove that the $M_{\mu\nu}$ satisfy

$$[M_{\mu\nu},\, M_{\rho\sigma}] = i\left(g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} - g_{\nu\sigma}M_{\mu\rho} + g_{\mu\sigma}M_{\nu\rho}\right) .$$

(b) The generators $M_{\mu\nu} = -M_{\nu\mu}$ can be split into the three generators K^i of Lorentz boosts and the three generators J^i of rotations as follows

$$K^i = M^{0i}(K^i = -M_{0i}) \qquad \text{and} \qquad J^i = \frac{1}{2}\,\epsilon^{ijk}M_{jk}\,,$$

with ϵ^{ijk} being the Levi-Civita tensor (with $\epsilon^{123} \equiv +1$). Prove that the algebra of these generators is given by

$$\left[K^i, K^j\right] = -i\epsilon^{ijk}J^k\,, \quad \left[J^i, K^j\right] = i\epsilon^{ijk}K^k\,, \quad \left[J^i, J^j\right] = i\epsilon^{ijk}J^k\,,$$

and explain the physical meaning of each of these results.
Hint: In particular for subsequent exercises write $M_{\mu\nu}$ in matrix form.

(c) We define the operators

$$N^i = \frac{1}{2}(J^i + iK^i) \quad \text{and} \quad N^{i\dagger} = \frac{1}{2}(J^i - iK^i)\,.$$

Use the previous subexercise to show that

$$\left[N^i, N^{j\dagger}\right] = 0\,, \quad \left[N^i, N^j\right] = i\epsilon^{ijk}N^k\,, \quad \left[N^{i\dagger}, N^{j\dagger}\right] = i\epsilon^{ijk}N^{k\dagger}\,.$$

The operators N^i and $N^{i\dagger}$ thus both fulfill the Lie algebra of SO(3) (≅SU(2)).

(d) Investigate the effect of the parity operator $P = \mathrm{diag}(1, -1, -1, -1)$ on $M_{\mu\nu}$. How does P relate N^i and $N^{i\dagger}$? *Hint:* Notice that P is nothing else than the metric tensor. Then perform matrix multiplication using the explicit form of $M_{\mu\nu}$ and check what happens to K^i and J^i.

(e) We lastly define the Pauli-Lubanski vector for a particle with mass m and momentum P^μ through

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma}$$

with the total antisymmetric 4-dimensional Levi-Civita tensor $\epsilon^{\mu\nu\rho\sigma}$ (with $\epsilon^{0123} = +1$). Show that $P_\mu W^\mu = 0$ and $[W^\mu, P^\nu] = 0$. Show that in the rest frame of the particle,

$P_\nu = (m,0,0,0)^T$, the Casimir operator of the Poincaré group, being the square of the Pauli-Lubanski vector, takes the form

$$W^2 = W^\mu W_\mu = -m^2 \vec{J}^2 \qquad \text{with} \qquad \vec{J}^2 = \sum_i (J^i)^2 \,.$$

Hint: Remember the commutation relations of the Lorentz group $[P_\mu, P_\nu] = 0$ and $[M_{\mu\nu}, P_\rho] = -ig_{\mu\rho}P_\nu + ig_{\nu\rho}P_\mu$. *Add-on:* Since W^2 as well as $P^2 = P^\mu P_\mu$ are Lorentz-invariant by construction, they can be used with e.g. P^μ to form a complete set of observables under the Lorentz group which identifies a particle's behavior. For a massless particle we cannot find a Lorentz transformation into its rest frame. One can show that in this case $W^\mu = \lambda P^\mu$ with the helicity λ from sheet 2.

Exercise 2: Weyl and Dirac spinors **1+0.5+2.5 = 4 points**

We continue with the previous exercise and want to introduce and investigate the behavior of left- and right-chiral spinors under Lorentz transformations more closely. *Hint:* For (a) and (c) feel free to use a computer (e.g. for performing matrix multiplications). Pay attention to upper and lower indices. Then add a printout of the results to your solution.

(a) We now decompose the Lorentz transformation $M_{\mu\nu}$. Show that

$$U(\Lambda) = \exp\left(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right) = \underbrace{\exp\left(-i(\vec{\omega} - i\vec{\nu})\vec{N}\right)}_{=U_L(\Lambda)} \underbrace{\exp\left(-i(\vec{\omega} + i\vec{\nu})\vec{N}^\dagger\right)}_{=U_R(\Lambda)}$$

by identifying $\omega_{ij} = \epsilon_{ijk}\omega_k$ and $\omega_{0i} = \nu_i$ ($\omega_{\mu\nu} = -\omega_{\nu\mu}$).

(b) We identified N^i and $N^{i\dagger}$ to follow the Lie algebra of SU(2). A possible (2×2) matrix representation of the operators is thus given by the Pauli matrices $\frac{1}{2}\sigma_i$. Spin$-\frac{1}{2}$ fields that transform with $U_L(\Lambda)$, thus transforming in the $(1/2, 0)$ representation with $N^i = \frac{1}{2}\sigma_i$, $N^{i\dagger} = 0$, are named left-chiral Weyl spinors ψ_L, spin$-\frac{1}{2}$ fields that transform with $U_R(\Lambda)$ and thus are in the $(0, 1/2)$ representation with $N^i = 0$, $N^{i\dagger} = \frac{1}{2}\sigma_i$ are right-chiral Weyl spinors ψ_R. Of which dimensionality (in this matrix representation!) are thus the associated Weyl spinors? *Add-on:* Often ψ_L and ψ_R are named left- and right-handed rather than left- and right-chiral. Parity maps an element of the $(1/2, 0)$ representation to an element of $(0, 1/2)$ representation and thus into a different vector space. It is therefore sensible to consider the (reducible) representation of the Lorentz algebra $(1/2, 0)\bigoplus(0, 1/2)$, which results in the Dirac spinor.

(c) A Dirac spinor combines the two Weyl spinors ψ_L and ψ_R to a four-dimensional spinor and transforms (in the matrix representation of the previous subexercise) as follows

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \longmapsto \Psi' = \exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\Psi \,.$$

Therein we used $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ with the Dirac matrices in the chiral Weyl representation. Insert the definitions of $\omega_{\mu\nu}$ from (a) and the γ matrices in the Weyl representation to show that indeed the transformation is a (4×4) matrix, that decomposes as expected

$$-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu} = \begin{pmatrix} -\frac{i}{2}(\vec{\omega} - i\vec{\nu})\vec{\sigma} & 0 \\ 0 & -\frac{i}{2}(\vec{\omega} + i\vec{\nu})\vec{\sigma} \end{pmatrix} \,.$$

Add-on: The choice of a 4-dimensional representation for the Dirac spinor is not related to Minkowski space!