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Exercise 1: Lorentz boost - Rapidity

1+4+1 = 6 points

The aim of this exercise is to understand rapidity more closely, as it is commonly considered in collider experiments to describe final state particles in addition to their transverse momentum. Rapidity $\vec{\nu}$ is the vector associated with the boost operators $\vec{K}$. For this exercise we perform a boost along the y axis and assume that the boost operator K^2 is given by

$$K^2 = K_y = -i\left(t\frac{\partial}{\partial y} + y\frac{\partial}{\partial t}\right) .$$

(a) Calculate the effects of K_y, $(K_y)^2$, $(K_y)^3$ and $(K_y)^4$ on the four-vector $x^\mu = (t, x, y, z)^T$.

(b) Determine the finite Lorentz transformation $x^{\mu\prime} = \exp(i\nu K_y)\, x^\mu$, where ν is the rapidity, by using the results from part (a).
Hint: Expand the exponential, write the result as the sum of three vectors $(t, x, y, z)^T$, $(y, 0, t, 0)^T$ and $(t, 0, y, 0)^T$ and then identify sinh and cosh in the prefactors of the vectors.

With the boost parameters $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, the boost can be written as

$$x^{\mu\prime} = \begin{pmatrix} \gamma & 0 & \gamma\beta & 0 \\ 0 & 1 & 0 & 0 \\ \gamma\beta & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} x^\mu .$$

(c) Compare this alternative form of the boost with your result from (b). Show that the rapidity is given by

$$\nu = \frac{1}{2}\ln\frac{1+\beta}{1-\beta} .$$

Exercise 2: Noether theorem for two scalars

1+2+1 = 4 points

In this and the subsequent exercise we want to use Noether's theorem for fields, which allows to deduce important conserved currents if the Lagrangian is invariant under e.g. internal or spacetime symmetries. We consider the following Lagrangian of two real scalar fields ϕ_1 and ϕ_2

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_1)(\partial^\mu\phi_1) + \frac{1}{2}(\partial_\mu\phi_2)(\partial^\mu\phi_2) - \frac{m^2}{2}(\phi_1^2 + \phi_2^2) - \frac{\lambda}{4}(\phi_1^2 + \phi_2^2)^2 .$$

(a) Show that the Lagrangian is invariant under the internal transformation ($\theta \in \mathbb{R}$)

$$\begin{aligned} x_\mu &\longrightarrow x'_\mu = x_\mu \\ \phi_1 &\longrightarrow \phi'_1 = \phi_1\cos\theta - \phi_2\sin\theta \\ \phi_2 &\longrightarrow \phi'_2 = \phi_1\sin\theta + \phi_2\cos\theta . \end{aligned}$$

(b) Calculate the Noether current and the Noether charge for the transformation of the previous subexercise. They are defined through

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta\Phi_i \quad \text{and} \quad Q = \int d^3x j^0 \,,$$

where a sum over i is implied and $\delta\Phi_i$ is an infinitesimal transformation defined by $\phi_i' = \phi_i + \delta\phi_i \equiv \phi_i + \theta\delta\Phi_i$, e.g. $\delta\Phi_1 = -\phi_2$.

(c) Rewrite the Lagrangian in terms of a complex scalar field $\varphi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ and show that $\varphi \longrightarrow \varphi' = e^{i\theta}\varphi$ equals the transformation provided in subexercise (a).

Exercise 3: Noether theorem for N Dirac spinors **1+2+2 = 5 points**

We consider the following Lagrangian of N Dirac fields each with mass m

$$\mathcal{L} = \sum_{i=1}^{N} \overline{\psi}_i (i\partial\!\!\!/ - m)\psi_i = \sum_{i=1}^{N} \overline{\psi}(i\gamma_\mu \partial^\mu - m)\psi_i \,.$$

Therein we use $\overline{\psi} = \psi^\dagger \gamma^0$.

(a) Use the Euler-Lagrange equations to deduce the equations of motion.
Hint: Perform the derivatives with respect to ψ_i and $\overline{\psi}_i$ and treat both independently.

(b) Determine which of the following internal (i.e. $x_\mu \longrightarrow x'_\mu = x_\mu$) transformations leave the Lagrangian $\mathcal{L}$ invariant. In contrast to the previous subexercise 2 (a) consider infinitesimal transformations, i.e. check if in $\mathcal{L}' = \mathcal{L} + \delta\mathcal{L}$ the expression

$$\delta\mathcal{L} = \sum_{i=1}^{N} \delta\overline{\psi}_i (i\partial\!\!\!/ - m)\psi_i + \overline{\psi}_i (i\partial\!\!\!/ - m)\delta\psi_i$$

vanishes. The transformations are:

$$\begin{aligned}
&\text{(i)} && \psi \longrightarrow \psi_i' = e^{i\alpha}\psi_i \quad \text{U(1) transformation} \\
&\text{(ii)} && \psi \longrightarrow \psi_i' = e^{i\epsilon_a T^a_{ij}}\psi_j \quad \text{SU}(N) \text{ transformation} \\
&\text{(iii)} && \psi \longrightarrow \psi_i' = e^{i\alpha\gamma^5}\psi_i \\
&\text{(iv)} && \psi \longrightarrow \psi_i' = e^{i\epsilon_a T^a_{ij}\gamma^5}\psi_j \,.
\end{aligned}$$

In this exercise α and ϵ_a are real parameters and T^a are the hermitian, traceless SU(N) generators. A summation over a and j is implied in the above transformations. The transformations (iii) and (iv) are named chiral transformations. What is the effect of setting $m = 0$ in transformations (iii) and (iv)? *Hint:* First show e.g. for (i) $\delta\psi_i = i\alpha\psi_i \equiv \alpha\delta\Psi_i$ and $\delta\overline{\psi}_i = -i\alpha\overline{\psi}_i$. Pay attention to indices in (ii) and (iv) and to the fact that different vector spaces are combined.

(c) Deduce the Noether currents for the four transformations, being defined through

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i)} \delta\Psi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi}_i)} \delta\overline{\Psi}_i \,.$$

Hint: Note that for (ii) and (iv) j^μ_a carries an additional index, which stems from the definition of $\epsilon_a \delta\Psi^a_i = \delta\psi_i$.