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Exercise 1: Dilatation involving a real scalar field — 1+2+1+2 = 6 points

We consider again the Lagrangian of a real scalar field given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 - \frac{1}{4}\lambda\phi^4 \,.$$

(a) Derive the equation of motion for ϕ and the energy-momentum tensor $T^{\mu\nu}$ defined through

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}(\partial^\nu \phi) - g^{\mu\nu}\mathcal{L} \,.$$

(b) Show that the the action $S = \int d^4x \mathcal{L}$ is invariant under dilatations for $m = 0$, i.e. under the transformations

$$x'_\mu = e^{-\alpha}x_\mu \,, \qquad \phi'(x') = e^{\alpha}\phi(x) \,.$$

(c) Show that for $m = 0$ the Noether current for the dilatation given in the previous subexercise is given by

$$j^\mu = T^{\mu\alpha}x_\alpha + \frac{1}{2}\partial^\mu \phi^2 \,.$$

(d) Use the energy-momentum tensor and the equation of motion to show that in the massive case only the mass term breaks the invariance under dilatations, i.e. $\partial_\mu j^\mu = m^2\phi^2$.

Exercise 2: Quantization of the complex scalar field — 1+2+2 = 5 points

We close up with classical fields and move towards field operators, i.e. second quantization. Consider a complex scalar field ϕ, that can be split into the components

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \,, \qquad \phi^\dagger = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \,.$$

Both are quantized canonically, i.e.

$$[\phi_i(t,\vec{x}), \partial_0\phi_j(t,\vec{y})] = i\delta_{ij}\delta^{(3)}(\vec{x}-\vec{y}) \,,$$

which allows to introduce creation and annihilation operators as follows

$$\phi_i = \int d\tilde{k}(a_i(\vec{k})e^{-ik\cdot x} + a_i^\dagger(\vec{k})e^{ik\cdot x}) \quad \text{with} \quad d\tilde{k} = \frac{d^3k}{(2\pi)^3 2\omega_k}\,, \omega_k = \sqrt{\vec{k}^2 + m^2}\,.$$

(a) Express ϕ and $\phi^\dagger$ in terms of creation and annihilation operators using

$$a(\vec{k}) = \frac{1}{\sqrt{2}}(a_1(\vec{k}) + ia_2(\vec{k})) \quad \text{and} \quad b(\vec{k}) = \frac{1}{\sqrt{2}}(a_1(\vec{k}) - ia_2(\vec{k}))\,.$$

(b) Derive the commutation relations $[a(\vec{k}), a^\dagger(\vec{p})]$, $[b(\vec{k}), b^\dagger(\vec{p})]$ and $[a(\vec{k}), b^\dagger(\vec{p})]$ from the commutation relations of $a_i(\vec{k})$ and $a_i^\dagger(\vec{p})$.
Add-on: All other commutation relations of these operators vanish.

(c) We have shown on exercise sheet 4 that the complex scalar field is invariant under the transformation $\phi \longrightarrow e^{i\theta}\phi$. Show that the conserved charge can be written in the form

$$Q = \int d^3\vec{x} j^0(x) = \int d\tilde{k} i(a_1^\dagger(\vec{k})a_2(\vec{k}) - a_2^\dagger(\vec{k})a_1(\vec{k})) = \int d\tilde{k}(a^\dagger(\vec{k})a(\vec{k}) - b^\dagger(\vec{k})b(\vec{k}))\,.$$

For the charge the advantage of using the operators a and b is apparent: Interpret the two contributions as particle number operators N_a and N_b and provide a physical interpretation of the creation operators $a^\dagger(\vec{k})$ and $b^\dagger(\vec{k})$ acting on the vacuum.

Exercise 3: Vacuum fluctuations **1+3 = 4 points**

We consider a quantized, real scalar field with commutation relations for the creation and annihilation operators. At $t = 0$ we average the field within a sphere of radius R ($V = \frac{4\pi}{3}R^3$), i.e. we consider

$$\phi_R = \frac{1}{V}\int_{|\vec{x}|<R} d^3x\phi(x)\,.$$

(a) Show that the vacuum expectation value (VEV) of ϕ_R vanishes, i.e. $\langle 0|\phi_R|0\rangle = 0$.

(b) Derive $\langle 0|\phi_R^2|0\rangle$. Since the VEV of ϕ_R vanishes, but not the VEV of ϕ_R^2, the field ϕ cannot be constant within the sphere of constant radius R, but it has to fluctuate. Consider $m = 0$. Do you need to consider larger or smaller values of R to enlarge the fluctuations?
Hint: First show that the VEV of ϕ_R^2 is given by

$$\langle 0|\phi_R^2|0\rangle = \frac{1}{V^2}\int d\tilde{k}\left|\int_V d^3x e^{-ik\cdot x}\right|^2 \quad \text{and rewrite} \quad \int_V d^3x e^{i\vec{k}\cdot\vec{x}}$$

as one-dimensional integral over the radius of the sphere. Make use of the Bessel function

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)\,,$$

which enters the integral

$$I(a) = \int_0^\infty \frac{dy}{y\sqrt{a^2+y^2}}[J_{3/2}(y)]^2\,.$$

For $m = 0$ you will need $I(0) = \frac{1}{2\pi}$.