Exercises: Stefan Liebler (stefan.liebler@kit.edu) (Office 12/03 - Build. 30.23)

Exercise 1: Momentum operator and charge of the Dirac field **0 points**

The solution of the Dirac equation can be expanded in plain waves as follows

$$\psi(x) = \int d\tilde{p} \sum_{\lambda=\pm} \left[c_\lambda(p) u(p,\lambda) e^{-ip\cdot x} + d^\dagger_\lambda(p) v(p,\lambda) e^{ip\cdot x} \right] .$$

Therein $u(p,\lambda)$ and $v(p,\lambda)$ are Dirac spinors associated with positive and negative energies, respectively. They obey relations as shown on sheet 2, i.e. $u^\dagger(p,\lambda)u(p,\lambda') = 2\omega_p \delta_{\lambda\lambda'}$, $v^\dagger(p,\lambda)v(p,\lambda') = 2\omega_p \delta_{\lambda\lambda'}$, $u^\dagger(\tilde{p},\lambda)v(p,\lambda') = v^\dagger(\tilde{p},\lambda)u(p,\lambda') = 0$ with $\tilde{p} = (\omega_p, -\vec{p})^T$. A priori $c_\lambda^{(\dagger)}$ and $d_\lambda^{(\dagger)}$ are plain coefficients, which we assume to be not-nessesarily anti-commuting.

(a) Show that the componenents $T^{0\mu}$ of the energy-momentum tensor are given by $T^{0\mu} = \psi^\dagger i\partial^\mu \psi$. Express the four-momentum of the Dirac field

$$P^\mu = \int d^3x T^{0\mu}$$

in terms of $c_\lambda(p), c^\dagger_\lambda(p), d_\lambda(p)$ and $d^\dagger_\lambda(p)$.

(b) The charge of the Dirac field is given by

$$Q = \int d^3x \overline{\psi}(x)\gamma^0\psi(x) .$$

Express the charge again through the coefficients $c_\lambda(p), c^\dagger_\lambda(p), d_\lambda(p)$ and $d^\dagger_\lambda(p)$.

(c) For both subexercises (a) and (b) argue why having anti-commutation relations for d and $d^\dagger$ leads to physically sensible results.

Solution of exercise 1

(a) We start with the discussion of the energy-momentum tensor, which is defined by

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \partial^\nu \psi + (\partial^\nu \overline{\psi}) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi})} - g^{\mu\nu}\mathcal{L} .$$

The Lagrangian of the free Dirac field is given by

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - m)\psi ,$$

such that the second term of $T^{\mu\nu}$ does not contribute and we obtain

$$\begin{aligned} T^{\mu\nu} &= \overline{\psi} i\gamma^\mu \partial^\nu \psi + 0 - g^{\mu\nu}\overline{\psi}(i\gamma^\rho\partial_\rho - m)\psi \\ &= \overline{\psi} i\gamma^\mu \partial^\nu \psi . \end{aligned}$$

Therein we used the Dirac equation $(i\gamma^\rho\partial_\rho - m)\psi = 0$. For the four-momentum we need

$$T^{0\mu} = \overline{\psi} i\gamma^0\partial^\mu\psi = \psi^\dagger i\partial^\mu\psi\,,$$

where we used $(\gamma^0)^2 = 1_{4\times 4}$. We insert the expression into the four-momentum and expand the Dirac field following the instructions given in the exercise:

$$\begin{aligned}
P^\mu &= \int d^3x T^{0\mu} = \int d^3x \psi^\dagger i\partial^\mu\psi \\
&= \int d^3x \int d\tilde{p} \left[\sum_{\lambda=\pm} c^\dagger_\lambda(p) u^\dagger(p,\lambda) e^{ip\cdot x} + d_\lambda(p) v^\dagger(p,\lambda) e^{-ip\cdot x}\right] \\
&\qquad \cdot i\partial^\mu \int d\tilde{q} \left[\sum_{\lambda'=\pm} c_{\lambda'}(q) u(q,\lambda') e^{-iq\cdot x} + d^\dagger_{\lambda'}(q) v(q,\lambda') e^{iq\cdot x}\right] \\
&= \int d^3x \int d\tilde{p} \int d\tilde{q} \sum_{\lambda=\pm}\sum_{\lambda'=\pm} \\
&\quad \cdot \Big(c^\dagger_\lambda(p) c_{\lambda'}(q) q^\mu u^\dagger(p,\lambda) u(q,\lambda') e^{i(p-q)\cdot x} + d_\lambda(p) d^\dagger_{\lambda'}(q)(-q^\mu) v^\dagger(p,\lambda) v(q,\lambda') e^{-i(p-q)\cdot x} \\
&\quad + c^\dagger_\lambda(p) d^\dagger_{\lambda'}(q)(-q^\mu) u^\dagger(p,\lambda) v(q,\lambda') e^{i(p+q)\cdot x} + d_\lambda(p) c_{\lambda'}(q) q^\mu v^\dagger(p,\lambda) u(q,\lambda') e^{-i(p+q)\cdot x}\Big)\,.
\end{aligned}$$

We now perform the integration over the three-dimensional space and therein identify the Fourier transform of the δ distribution

$$\int d^3x e^{is\cdot x} = e^{i\omega_s t}(2\pi)^3\delta^{(3)}(\vec{s})\,.$$

We thus obtain either $\vec{q} = \vec{p}$ or $\vec{q} = -\vec{p}$. In both cases we obtain $\omega_p = \omega_q$ as $\omega_q = \sqrt{|\vec{q}|^2 + m^2}$. Using the identities provided in the exercise, we are left with the first two terms with $\vec{p} = \vec{q}$ and thus obtain

$$\begin{aligned}
P^\mu &= \int d\tilde{p} \frac{1}{2\omega_p} \sum_{\lambda=\pm} \Big(c^\dagger_\lambda(p) c_\lambda(p) p^\mu 2\omega_p + d_\lambda(p) d^\dagger_\lambda(p)(-p^\mu) 2\omega_p + 0 + 0 \Big) \\
&= \int d\tilde{p} p^\mu \sum_{\lambda=\pm} \Big(c^\dagger_\lambda(p) c_\lambda(p) - d_\lambda(p) d^\dagger_\lambda(p) \Big)\,.
\end{aligned}$$

Before we continue with subexercise (b) we want to motivate the relations provided in the exercise for the Dirac spinors. We make use of the explicit form of the Dirac spinors in terms of Weyl spinors provided on sheet 2 and obtain

$$\begin{aligned}
u^\dagger(p,\lambda) u(p,\lambda') &= \sqrt{E - \lambda|\vec{p}|}\chi^\dagger_\lambda(p)\sqrt{E - \lambda'|\vec{p}|}\chi_{\lambda'}(p) + \sqrt{E + \lambda|\vec{p}|}\chi^\dagger_\lambda(p)\sqrt{E + \lambda'|\vec{p}|}\chi_{\lambda'}(p) \\
&= \Big(\sqrt{E - \lambda|\vec{p}|}\sqrt{E - \lambda'|\vec{p}|} + \sqrt{E + \lambda|\vec{p}|}\sqrt{E + \lambda'|\vec{p}|}\Big)\chi^\dagger_\lambda(p)\chi_{\lambda'}(p) \\
&= (E - \lambda|\vec{p}| + E + \lambda|\vec{p}|)\delta_{\lambda\lambda'} = 2E\delta_{\lambda\lambda'}\,.
\end{aligned}$$

Therein and in the next relation we use $\chi^\dagger_\lambda(p)\chi_{\lambda'}(p) = \chi^\dagger_{-\lambda}(p)\chi_{-\lambda'}(p) = \delta_{\lambda\lambda'}$, see sheet 2. Similarly we obtain

$$\begin{aligned}
v^\dagger(p,\lambda) v(p,\lambda') &= (-\lambda)\sqrt{E + \lambda|\vec{p}|}\chi^\dagger_{-\lambda}(p)(-\lambda')\sqrt{E + \lambda'|\vec{p}|}\chi_{-\lambda'}(p) \\
&\qquad + \lambda\sqrt{E - \lambda|\vec{p}|}\chi^\dagger_{-\lambda}(p)\lambda'\sqrt{E - \lambda'|\vec{p}|}\chi_{-\lambda'}(p) \\
&= \lambda\lambda'(\sqrt{E + \lambda|\vec{p}|}\sqrt{E + \lambda'|\vec{p}|} + \sqrt{E - \lambda|\vec{p}|}\sqrt{E - \lambda'|\vec{p}|})\chi^\dagger_{-\lambda}(p)\chi_{-\lambda'}(p) \\
&= 2E\delta_{\lambda\lambda'}\,.
\end{aligned}$$

Next we want to calculate

$$\begin{aligned}u^\dagger(p,\lambda)v(\tilde{p},\lambda') &= \overline{u}(p\lambda)\gamma^0 v(\tilde{p},\lambda') = \frac{1}{m}\overline{u}(p,\lambda)m\gamma^0 v(\tilde{p},\lambda') = \frac{1}{m}\overline{u}(p,\lambda)p_\mu\gamma^\mu\gamma^0 v(\tilde{,}\lambda') \\ &= \frac{1}{m}\overline{u}(p,\lambda)\gamma^0(p_0\gamma^0 - p_1\gamma^1 - p_2\gamma^2 - p_3\gamma^3)v(\tilde{p},\lambda') \\ &= \frac{1}{m}\overline{u}(p,\lambda)\gamma^0\tilde{\not{p}}v(\tilde{p},\lambda') = -u^\dagger(p,\lambda)v(\tilde{p},\lambda')\,.\end{aligned}$$

We made use of the Dirac equations $(\tilde{\not{p}}+m)v(\tilde{p},\lambda') = 0$ and $\overline{u}(p,\lambda)(\not{p}-m) = 0$. We conclude that $u^\dagger(p,\lambda)v(\tilde{p},\lambda') = 0$. Alternatively this last relation can also be shown using the explicit form of the Dirac spinors in terms of Weyl spinors.

(b) As the conserved charge is given by

$$Q = \int d^3x\overline{\psi}(x)\gamma^0\psi(x) = \int d^3x\psi^\dagger(x)\psi(x)$$

the calculation is identical with the one of the previous subexercise setting $i\partial^\mu \to 1$. We therefore obtain

$$Q = \int d\tilde{p}\sum_{\lambda=\pm}\left(c_\lambda^\dagger(p)c_\lambda(p) + d_\lambda(p)d_\lambda^\dagger(p)\right)\,.$$

(c) We want both P^μ and Q to be normal ordered. Only if we demand anti-commuting relations in the form $\{d_\lambda(p), d^\dagger_{\lambda'}(p')\} = (2\pi)^3 2\omega_p\delta_{\lambda\lambda'}\delta(\vec{p}-\vec{p}')$ we get the expected signs for the four-momentum and the charge. As $c^\dagger_\lambda(p)$ generates fermions with charge $+1$ and $d^\dagger_\lambda(p)$ fermions with charge -1 we can define the particle number density operators $N^c_\lambda(p) = c^\dagger_\lambda(p)c_\lambda(p)$ and $N^d_\lambda(p) = d^\dagger_\lambda(p)d_\lambda(p)$. The four-momentum (neglecting an overall constant) is then proportional to $\sum_\lambda N^c_\lambda(p) + N^d_\lambda(p)$ and the charge is proportional to $\sum_\lambda N^c_\lambda(p) - N^d_\lambda(p)$.