Exercises: Stefan Liebler (stefan.liebler@kit.edu) (Office 12/03 - Build. 30.23)

Exercise 1: Causality of the Green's function

2+5 = 7 points

The solution of the differential equation $(\Box_x + m^2)G(x-y) = -i\delta^{(4)}(x-y)$, i.e. the Green's function of the Klein-Gordon-equation, can be written in the form

$$\Delta(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i}{k^2 - m^2} \,.$$

(a) Performing the integration over k_0 the integrand has a pole at $k_0 = \pm\sqrt{\vec{k}^2 + m^2}$. There are four potential integration paths in the complex k_0 plane. The choice of the four options can be made explicit by infinitesimal shifts of the poles by $\pm i\epsilon$ (with $\epsilon > 0$). Which paths correspond to the following Green's functions?

$$\begin{aligned}
\Delta_{\text{ret}}(x-y) &= \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i}{(k_0 + i\epsilon)^2 - \vec{k}^2 - m^2} \\
\Delta_{\text{av}}(x-y) &= \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i}{(k_0 - i\epsilon)^2 - \vec{k}^2 - m^2} \\
\Delta_F(x-y) &= \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i}{k^2 - m^2 + i\epsilon} \\
\Delta_D(x-y) &= \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i}{k^2 - m^2 - i\epsilon}
\end{aligned}$$

Hint: Determine the poles in terms of $\omega_k = \sqrt{\vec{k}^2 + m^2}$ and ϵ and draw the four poles in the complex k_0 plane.

(b) Show that the four functions can be expressed through the real (quantized) fields ϕ and the vacuum state $|0\rangle$ as follows

$$\begin{aligned}
\Delta_{\text{ret}}(x-y) &= \theta(x_0 - y_0)\langle 0|[\phi(x), \phi(y)]|0\rangle \\
\Delta_{\text{av}}(x-y) &= -\theta(y_0 - x_0)\langle 0|[\phi(x), \phi(y)]|0\rangle \\
\Delta_F(x-y) &= \langle 0|T\phi(x)\phi(y)|0\rangle \\
\Delta_D(x-y) &= -\theta(x_0 - y_0)\langle 0|\phi(y)\phi(x)|0\rangle - \theta(y_0 - x_0)\langle 0|\phi(x)\phi(y)|0\rangle
\end{aligned}$$

using the time-ordering operator T given by

$$T\phi(x)\phi(y) = \begin{cases} \phi(x)\phi(y), & \text{if } x_0 > y_0 \\ \phi(y)\phi(x), & \text{if } x_0 < y_0 \end{cases} .$$

It moreover yields $\theta(t) = 1$ for $t \geq 0$, otherwise 0. What are the consequences for causality for the four Green's functions?
Hint: Perform the integrals over k_0. Split the integrals into $x_0 > y_0$ and $x_0 < y_0$ and close the contours at negative and positive imaginary parts of k_0, respectively, such that $e^{-ik_0(x_0 - y_0)} \to 0$ as soon as k_0 has an imaginary component. Use Cauchy's theorem. Lastly be reminded that $\langle 0|\phi(x)\phi(y)|0\rangle = \int d\tilde{k} e^{-ik\cdot(x-y)}$.

Exercise 2: Charge conjugation **2+2+2+2 = 8 points**

The defining property of the charge conjugation matrix C is

$$C^{-1}\gamma^\mu C = -(\gamma^\mu)^{\mathrm{T}} \tag{1}$$

for any representation of the Dirac matrices.

(a) Verify explicitly that in the Weyl representation, the matrix $C = i\gamma^0\gamma^2$ obeys Eq. (1). Show that in this representation, the charge conjugation operator obeys $C^\dagger = C^{\mathrm{T}} = C^{-1} = -C$.

(b) Show that for any arbitrary representation, i.e. only by using Eq. (1), that the following equations hold:

$$C^{-1}\gamma^5 C = (\gamma^5)^{\mathrm{T}}\,,$$
$$C^{-1}\sigma^{\mu\nu} C = -(\sigma^{\mu\nu})^{\mathrm{T}}\,,$$
$$C^{-1}(\gamma^\mu\gamma^5) C = (\gamma^\mu\gamma^5)^{\mathrm{T}}\,.$$

Hint: Use the following definition for γ^5: $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$.

(c) The charge conjugation *operator acting directly on the fermionic fields* is given by $\mathcal{C}$. From its definition via $\mathcal{C}\psi(x)\mathcal{C}^\dagger = C\overline{\psi}^{\mathrm{T}}(x)$, it is connected with the charge conjugation matrix C. By using this definition as well as $(\gamma^0)^\dagger = (\gamma^0)^{\mathrm{T}} = \gamma^0$, show that

$$\mathcal{C}\overline{\psi}(x)\mathcal{C}^\dagger = -\psi^{\mathrm{T}}(x)C^{-1}\,.$$

(d) Use the results of the previous parts to show that the fermionic bilinear covariants transform in the following way under charge conjugation,

$$\mathcal{C}:\overline{\psi}(x)\psi(x):\mathcal{C}^\dagger = :\overline{\psi}(x)\psi(x): \equiv S(x)\,,$$
$$\mathcal{C}:\overline{\psi}(x)\gamma^\mu\psi(x):\mathcal{C}^\dagger = -:\overline{\psi}(x)\gamma^\mu\psi(x): \equiv -V^\mu(x)\,,$$
$$\mathcal{C}:\overline{\psi}(x)\sigma^{\mu\nu}\psi(x):\mathcal{C}^\dagger = -:\overline{\psi}(x)\sigma^{\mu\nu}\psi(x): \equiv -T^{\mu\nu}(x)\,,$$
$$\mathcal{C}:\overline{\psi}(x)\gamma^\mu\gamma^5\psi(x):\mathcal{C}^\dagger = :\overline{\psi}(x)\gamma^\mu\gamma^5\psi(x): \equiv A^\mu(x)\,,$$
$$\mathcal{C}:\overline{\psi}(x)\gamma^5\psi(x):\mathcal{C}^\dagger = :\overline{\psi}(x)\gamma^5\psi(x): \equiv P(x)\,,$$

where the normal ordering, denoted by $:\mathcal{O}:$, is only stated here to indicate that the fermionic fields inside the bilinear covariants are quantized.
Hint: You can save some time if you first evaluate the generic bilinear transformation $\mathcal{C}:\overline{\psi}(x)\Gamma\psi(x):\mathcal{C}^\dagger$ by generalizing your results from the previous parts to any matrix $\Gamma \in \{1_{4\times4}, \gamma^\mu, \sigma^{\mu\nu}, \gamma^\mu\gamma^5, \gamma^5\}$.