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Exercise 1: Vacuum of the Gupta-Bleuler photon — 2+4 = 6 points

In the Gupta-Bleuler formalism of the free photon field the most general vacuum state reads

$$|\phi\rangle = \sum_{n=0}^{\infty} C_n |\phi_n\rangle\,.$$

The states $|\phi_n\rangle$ do not include transverse photons, but exactly n scalar and longitudinal photons. The additional condition

$$(a_3(k) - a_0(k))|\phi_n\rangle = 0$$

makes them physical states. We moreover choose $|\phi_0\rangle = |0\rangle$.

(a) Show that the most general form of $|\phi_1\rangle$ is given by

$$|\phi_1\rangle = \int d\tilde{q} f(q) \left(a_3^\dagger(q) - a_0^\dagger(q) \right) |0\rangle\,.$$

Hint: Make the ansatz $|\phi_1\rangle = \int d\tilde{q} \sum_{r=0,3} a_r^\dagger(q) f_r(q)|0\rangle$.

(b) Show that the expectation value of the photon field in the above general vacuum state corresponds to a gauge fixing, i.e.

$$\langle\phi|A_\mu(x)|\phi\rangle = \partial_\mu \Lambda(x)\,,$$

where the function $\Lambda(x)$ using the explicit polarization vectors $\varepsilon_0^\mu(k) = n^\mu$ and $\varepsilon_3^\mu(k) = \frac{k^\mu}{k\cdot n} - n^\mu$ is given by

$$\Lambda(x) = \int \frac{d\tilde{k}}{k\cdot n} 2\mathrm{Re}\left(i C_0^* C_1 e^{-ik\cdot x} f(k) \right)\,.$$

Therein $f(k)$ is identical to the one in subexercise (a). The function $\Lambda(x)$ fulfills $\Box\Lambda(x) = 0$ and can be chosen arbitrarily through the choice of the corresponding vacuum state $|\phi\rangle$.
Hint: First show that $\langle\phi_n|NA_\mu(x)|\phi_{n-1}\rangle = \langle\phi_n|A_\mu(x)|\phi_{n-1}\rangle$ with

$$N = \int d\tilde{k}(a_3^\dagger(k)a_3(k) - a_0^\dagger(k)a_0(k))$$

counting longitudinal and scalar photons. Thus it yields $\langle\phi_n|A_\mu(x)|\phi_{n-1}\rangle = 0$ for $n \neq 1$.

Exercise 2: Massive vector boson **2+3+4 = 9 points**

We consider a vector boson with mass $m \neq 0$, which enters the Lagrangian density as follows

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

(a) Derive the equation of motion for A^μ, that is known as Proca equation. Show that this equation necessarily implies $\partial_\mu A^\mu = 0$ and that, once this condition is imposed, it is equivalent to a set of massive Klein-Gordon equations, namely one for each of the non-vanishing components of A^μ.

(b) We introduce a Fourier decomposition for the massive vector boson in analogy to the photon field, i.e.

$$A^\mu(x) = \int d\tilde{k} \sum_{r=0}^{3} \left(\varepsilon_r^\mu(k) a_r(k) e^{-ik\cdot x} + \varepsilon_r^{\mu *}(k) a_r^\dagger(k) e^{ik\cdot x} \right) .$$

A priori, this includes four polarization vectors $\varepsilon_r^\mu(k)$. Due to $\partial_\mu A^\mu = 0$ and thus $\sum_r k_\mu \varepsilon_r^\mu(k) a_r(k) = 0$ only three polarization vectors are physical. Show that a convenient basis for these polarization vectors, in the reference frame with $\vec{k} = (0, 0, |\vec{k}|)$, is given by

$$\varepsilon_1 = (0, 1, 0, 0)\,, \qquad \varepsilon_2 = (0, 0, 1, 0)\,, \qquad \varepsilon_3 = \frac{1}{m}(|\vec{k}|, 0, 0, \omega_k)$$

for the three physical polarization vectors, which are orthogonal to the unphysical polarization vector $\varepsilon_0^\mu = k^\mu/m$. The physical polarization vectors obey the orthonormality condition $\varepsilon_r^\mu(k)\varepsilon_{\mu s}^*(k) = -\delta_{rs}$. Using these explicit expressions show that the completeness relation of the physical polarization vectors reads

$$\sum_{r=1}^{3} \varepsilon_r^\mu(k) \varepsilon_r^{\nu *}(k) = -g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2} .$$

Hint: In the rest frame of the particle, $k' = (m, 0, 0, 0)$, we e.g. choose $\varepsilon_0' = (1, \vec{0})$ and the three unit vectors $\varepsilon_i' = (0, \vec{e}_i)$. Then the vectors ε_i' automatically fulfill $k'_\mu \varepsilon_i'^\mu = 0$. Boost from the rest frame into the above reference frame. *Add-on:* We showed the completeness relation in a special frame, but it is actually Lorentz-covariant.

(c) We now impose standard bosonic commutation relations for the surviving operators. They read

$$[a_r(k), a_s(k')] = [a_r^\dagger(k), a_s^\dagger(k')] = 0,$$
$$[a_r(k), a_s^\dagger(k')] = \delta_{rs} 2\omega_k (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}') .$$

Verify that the propagator of the massive vector boson takes the form

$$\langle 0|T A^\mu(x) A^\nu(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2} \right) e^{-ik(x-y)} .$$

Add-on: As the photon has only two rather than three physical degrees of freedom, the limit $m \to 0$ of this propagator is not well-defined and does not yield the photon propagator.