Exercises: Stefan Liebler (stefan.liebler@kit.edu) (Office 12/03 - Build. 30.23)

Motivation:
You got 40% of all points on all sheets? You think it's too hot outside to actually work on the exercise sheets just for the sake of increasing your knowledge in the marvellous field of particle physics? Get 70% of the points on one of the remaining sheets (9, 10, 11 or 12) and earn a reward: Get ice cream in the last lecture in July! Amazing!

Exercise 1: Interaction picture **3+2 = 5 points**

This exercise is a recap of "Theorie E". In quantum mechanics the time evolution of a state in the Schrödinger picture is described by the Schrödinger equation

$$i\frac{d}{dt}|\psi(t)\rangle^S = H|\psi(t)\rangle^S$$

with the Hamilton operator H, whereas an operator O^S itself is constant in time. For the interaction picture we split the Hamilton operator into two parts, being

$$H = H_0 + H_I\,,$$

where H_0 describes free particles and H_I the interactions among them. The unitary time evolution operators $U_0(t,t_0)$ defined by $U_0(t,t_0) = e^{-iH_0(t-t_0)}$ yields the time evolution of the states in the Schrödinger picture in case $H_I = 0$. It is used to define the transformation into the interaction picture through

$$|\psi(t)\rangle^I = U_0^\dagger(t,t_0)|\psi(t)\rangle^S\,.$$

(a) Which form does the operator $O^I(t)$ take in the interaction picture, such that the matrix element

$$^I\langle\psi(t)|O^I(t)|\psi(t)\rangle^I = {}^S\langle\psi(t)|O^S|\psi(t)\rangle^S$$

is identical in both pictures? Determine the form of the operator H_0 in the interaction picture. Deduce $i\frac{d}{dt}O^I(t)$. Show that

$$i\frac{d}{dt}|\psi(t)\rangle^I = H_I^I(t)|\psi(t)\rangle^I\,,$$

where $H_I^I(t)$ denotes the operator H_I in the interaction picture. *Add-on:* The time evolution of $O^I(t)$ was your starting point in the lecture. The solution of the last equation yields $|\psi(t)\rangle^I = U_I(t,t_0)|\psi(t_0)\rangle^I$ with $U_I(t,t_0) = T\exp\left(-i\int_{t_0}^t dt' H_I^I(t')\right)$.

(b) Consider the Lagrangian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$ with

$$\mathcal{L}_0 = \frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) - \frac{m^2}{2}\phi^2\,, \qquad \mathcal{L}_I = -\frac{\lambda}{4}\phi^4\,.$$

Determine H_0 and H_I. Which differential equation describes the time evolution of the field **operators** $\phi^I(t)$ in the interaction picture? Remember the Fourier representation of $\phi^I(t)$ and $: H_0 := \int d\tilde{k} k_0 a^\dagger a$ to confirm that they are a solution of this equation.

Exercise 2: Two-particle phase space **2+4 = 6 points**

For the calculation of decay rates and cross sections, we need to integrate over the phase space of the particles in the final state. For a generic process with two particles with momenta p_1 and p_2 and masses m_1 and m_2 in the final state, this phase space integral is given by

$$\int d\Phi_2 = \int \frac{d^3p_1}{(2\pi)^3 2E_1} \int \frac{d^3p_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^{(4)}(q - p_1 - p_2) \ ,$$

where q is the total four-momentum of all incoming particles. This integral is performed over the absolute squared of the matrix element as well as over some additional Heaviside step functions Θ to implement the proper momentum cuts for the particles in the final state.

(a) Show that in the center-of-mass frame of the two final-state particles, the absolute value of the incoming three-momenta is given by

$$|\vec{p}_1| = |\vec{p}_2| = \frac{\lambda(q^2, m_1^2, m_2^2)}{2\sqrt{q^2}} \ ,$$

where λ is the *Källén function* given by

$$\lambda(a^2, b^2, c^2) \equiv \sqrt{a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2} \ .$$

(b) Show that in the center-of-mass frame of the two final-state particles, the phase space integral can be written as

$$\int d\Phi_2 = \int d\Omega \frac{1}{32\pi^2 q^2} \lambda(q^2, m_1^2, m_2^2) \Theta(q_0) \Theta(q^2 - (m_1 + m_2)^2) \ ,$$

where $d\Omega \equiv d(\cos\vartheta_1) d\phi_1$ is the integration over the solid angle of particle 1 in the center-of-mass frame. *Hint:* Use the meanwhile well-known relation

$$\frac{d^3p}{2E} = d^4p \Theta(p_0) \delta(p^2 - m^2) \ .$$

Exercise 3: Mandelstam variables **2+2 = 4 points**

We consider a scattering process of two particles into two particles. The two initial-state particles have four momenta p_1 and p_2 and masses m_1 and m_2, the two final-state particles have four momenta p_1' and p_2' and masses m_1' and m_2'. All four particles are on-shell, i.e. $p_i^2 = m_i^2$ and $(p_i')^2 = (m_i')^2$. We define the Mandelstam variables

$$s = (p_1 + p_2)^2, \qquad t = (p_1 - p_1')^2, \qquad u = (p_1 - p_2')^2 \ .$$

(a) Show that

$$s + t + u = \sum_{i=1,2} m_i^2 + (m_i')^2$$

by using four-momentum conservation.

(b) Assume $m_i = m$ and $m_i' = m'$. Determine the explicit expressions of s, t and u in the center-of-mass frame of the two colliding particles in terms of the energy of the colliding particles and the angle θ of the outgoing particles with respect to the initial ones. *Hint:* Set e.g. $p_1 = (E_{\text{CM}}/2, p\vec{e}_z)$, $p_2 = (E_{\text{CM}}/2, -p\vec{e}_z)$, $p_1' = (E_{\text{CM}}/2, \vec{k})$ and $p_2' = (E_{\text{CM}}/2, -\vec{k})$. The angle between $\vec{p}_1$ and $\vec{p'}_1$ is θ.