Einführung in Theoretische Teilchenphysik

Lecture: PD Dr. S Gieseke – Exercises: Dr. D. López-Val, Dr. S. Patel, PD Dr. M. Rauch

Exercise Sheet 10

Submission: Mo, 15.01.18, 12:00

Discussion:	Mon, 15.01.18	14:00	Room 11/12
	Wed, 17.01.18	09:45	Room 10/1

Exercise 1: Gamma-Algebra – part 1

Gamma matrices play an important role when calculating amplitudes with fermions. One can derive several useful properties which are helpful in actual calculations and which we want to show in this exercise. These only need the Clifford-Algebra,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \cdot \mathbb{1}_4 \,,$$

not an explicit form of the γ matrices.
Show the following properties

$$\begin{aligned} \not{a}\not{a} \equiv a_\mu a_\nu \gamma^\mu \gamma^\nu &= a_\mu a^\mu \mathbb{1}_4 \,, \\ \gamma_\mu \gamma^\mu &= 4 \cdot \mathbb{1}_4 \,, \\ \gamma_\mu \gamma^\alpha \gamma^\mu &= -2\gamma^\alpha \,, \\ \gamma_\mu \gamma^\alpha \gamma^\beta \gamma^\mu &= 4g^{\alpha\beta} \cdot \mathbb{1}_4 \,, \end{aligned}$$

where a_μ is an arbitrary 4-vector, e.g. a momentum.

Exercise 2: Properties of Dirac Spinors

The Dirac spinors fulfil the following orthogonality

$$\begin{aligned} \bar{u}_s(\vec{p}) u_{s'}(\vec{p}) &= -\bar{v}_s(\vec{p}) v_{s'}(\vec{p}) = 2m\, \delta_{ss'} \,, \\ \bar{u}_s(\vec{p}) v_{s'}(\vec{p}) &= \bar{v}_s(\vec{p}) u_{s'}(\vec{p}) = 0 \,, \end{aligned}$$

and completeness relations

$$\sum_s \left(u_s(\vec{p}) \bar{u}_s(\vec{p}) - v_s(\vec{p}) \bar{v}_s(\vec{p}) \right) = 2m \,.$$

(a) Check the completeness relation by showing that when it is applied to the basis states $u_{s'}(\vec{p})$, $v_{s'}(\vec{p})$, $\bar{u}_{s'}(\vec{p})$ and $\bar{v}_{s'}(\vec{p})$ you obtain the correct result.

Now we define projection operators $\Lambda^\pm(\vec{p}) = \frac{\pm\not{p}+m}{2m}$, which project the states of positive and negative energy, respectively, out of an arbitrary state $f(\vec{p}) = \sum_s \alpha_s u_s(\vec{p}) + \beta_s v_s(\vec{p})$, $\alpha, \beta \in \mathbb{C}$.

(b) Show that $\Lambda^\pm(\vec{p})$ are indeed projectors:

$$\left(\Lambda^\pm(\vec{p})\right)^2 = \Lambda^\pm(\vec{p}) \,, \qquad \Lambda^+(\vec{p}) f(\vec{p}) = \sum_s \alpha_s u_s(\vec{p}) \,,$$

$$\Lambda^+(\vec{p}) + \Lambda^-(\vec{p}) = 1 \,, \qquad \Lambda^-(\vec{p}) f(\vec{p}) = \sum_s \beta_s v_s(\vec{p}) \,.$$

(c) Finally, using the previous results show that

$$\sum_s u_s(\vec{p})\bar{u}_s(\vec{p}) = \not{p} + m\,, \qquad\qquad \sum_s v_s(\vec{p})\bar{v}_s(\vec{p}) = \not{p} - m\,.$$

Exercise 3: Two Particle Phase Space

To calculate decay rates and cross sections we need an integration over the phase space of the particles in the final state. For a general process with two particles (momenta p_1, p_2, masses m_1, m_2) in the final state, this phase space integral is given by

$$\int \mathrm{d}\Phi_2 = \int \frac{\mathrm{d}^3 p_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3 p_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^{(4)}(q - p_1 - p_2)\,,$$

where q is the four-momentum of the incoming particles. This integral acts on the squared matrix element and a step function which represents the cuts on the final-state particles.
Show that one can rewrite the integral as

$$\int \mathrm{d}\Phi_2 = \int \mathrm{d}\Omega\, \frac{1}{32\pi^2 q^2} \lambda(q^2, m_1^2, m_2^2)\Theta(q_0)\Theta(q^2 - (m_1 + m_2)^2)$$

with the Källén function

$$\lambda(a^2, b^2, c^2) = \sqrt{a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2}$$

and the Heavyside step function Θ.
Thereby, $\mathrm{d}\Omega = \mathrm{d}(\cos\vartheta_1)\mathrm{d}\varphi_1$ is the integration over the solid angle of particle 1 in the centre-of-mass system. The function λ describes the momentum of both particles in the centre-of-mass frame, $|\vec{p}_1| = |\vec{p}_2| = \frac{\lambda(q^2, m_1^2, m_2^2)}{2\sqrt{q^2}}$.
Hints:

- Use the relation
$$\frac{\mathrm{d}^3 p}{2E} = \mathrm{d}^4 p\, \Theta(p_0)\delta(p^2 - m^2)\,.$$
- Work in the centre-of-mass frame of the two final-state particles. Justify this!