Einführung in Theoretische Teilchenphysik

Lecture: PD. Dr. S Gieseke – Exercises: Dr. D. López-Val

Exercise Sheet 2

Submission: Mo, 6.11.17, 12:00

Discussion:	Mon, 6.11.17	14:00	Room 11/12
	Wed, 8.11.17	09:45	Room 10/1

WHEN HANDING IN, PLEASE MAKE SURE:

- your **NAME** and **GROUP** **(Mon/Wed)** are clearly indicated
- all sheets are **stapled** together
- you include the **question sheet** as **cover**

Exercise 1: Lie Algebra structure of $SU(N)$ and $SO(N)$

(a) Show that real $N \times N$ unimodular ($\det(R) = +1$) and orthogonal ($R^T\, R = \mathcal{I}_{N\times N}$) matrices (viz. elements of $SO(N)$) have $\frac{N(N-1)}{2}$ independent parameters.

(b) Show that complex $N \times N$ unimodular ($\det(U) = +1$) and hermitian ($U^\dagger = U$) matrices (viz. elements of $SU(N)$) have $N^2 - 1$ independent parameters.

(c) Prove that any finite transformation $U \in SU(N)$ may be written as a combination of infinitesimal transformations $U(\vartheta) = \exp\left(i\vartheta^a\, J_a\right)$ ($a = 1, 2, \ldots, N^2-1$), where the basis elements of the Lie algebra J_a are i) *hermitian* $J_a^\dagger J_a$; and ii) *traceless* matrices $\mathrm{Tr}(J_a) = 0$ ($\forall a$).

Hint Take the limit $\vartheta^a \to 0$ ($\forall a$). To check that $\mathrm{Tr}(J_a) = 0$, recall the identity $\det(e^A) = e^{\mathrm{Tr} A}$.

Hereafter we shall focus our attention to the $SU(2)$ case.

(d) Convince yourself that a generic $SU(2)$ transformation can be parameterized as

$$A = \begin{pmatrix} a_0 + ia_3 & a_2 + ia_1 \\ -a_2 + ia_1 & a_0 - ia_3 \end{pmatrix} = a_0 \,+ i\,\vec{a}\cdot\vec{\sigma} \tag{1}$$

where $(a_0, \vec{a}) = (a_0, a_1, a_2, a_3)$ are real-valued parameters satisfying $\sum_{i=0}^{3}\, a_i^2 = 1$, and the vector $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ contains the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad . \tag{2}$$

(e) Prove that the parameterization (1) is equivalent to $A = \exp\left[i\frac{\vartheta}{2}\,(\vec{\sigma}\cdot\hat{n})\right]$, where the rotation angle ϑ and the unit vector $\hat{n}$ are related to $(a_0, \vec{a})$. Write down these relations explicitly.

Exercise 2: A closer look at $SO(3)$

(a) Given a generic $SO(3)$ transformation represented in the form $R(\vartheta) = \exp(\vartheta_a\, K_a)$, show that the generators K_a are *real* and *antisymmetric* matrices; whereas, if represented as $R(\vartheta) = \exp(i\vartheta_a\, J_a)$, then J_a are *imaginary* and *hermitian*.

The following matrices implement independent rotations of angles ϑ_a $(a = 1, 2, 3)$ around the X, Y and Z axis in the 3-dimensional Euclidean space,

$$
\begin{aligned}
R_X(\vartheta_1) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta_1 & -\sin\vartheta_1 \\ 0 & \sin\vartheta_1 & \cos\vartheta_1 \end{pmatrix} \\
R_Y(\vartheta_2) &= \begin{pmatrix} \cos\vartheta_2 & 0 & -\sin\vartheta_2 \\ 0 & 1 & 0 \\ \sin\vartheta_2 & 0 & \cos\vartheta_2 \end{pmatrix} \\
R_Z(\vartheta_3) &= \begin{pmatrix} \cos\vartheta_3 & -\sin\vartheta_3 & 0 \\ \sin\vartheta_3 & \cos\vartheta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} .
\end{aligned}
\tag{1}
$$

(b) Taking the limit $R(\delta\vartheta_a) = 1 + i\delta\vartheta^a\, J_a + \mathcal{O}(\vartheta^2)$, obtain the matrix form of J_a $(a = 1, 2, 3)$. Check that your result is equivalent to the systematic recipe for computing the group generators,

$$
J_a = \frac{1}{i}\frac{\partial U(\vartheta_a)}{\partial\vartheta_a}\bigg|_{\vartheta_a=0} . \tag{2}
$$

(c) Using their explicit matrix form, check that J_a satisfy the commutation relation

$$
[J_a\, J_b] = f^c_{ab}\, J_c\, . \tag{3}
$$

Determine the structure constants of the Lie algebra f^c_{ab}.

Exercise 3: A closer look at $SU(3)$

(a) A generic finite transformation $S \in SU(3)$ in the fundamental representation may be written in terms of 8 real parameters (*why?*) and corresponding generators of the Lie algebra as

$$
S = \exp\left(-\frac{i}{2}\sum_{i=1}^{8}\alpha_i\lambda_i\right) \quad \text{with} \quad \mathrm{Tr}\left(\frac{\lambda_i}{2}\times\frac{\lambda_j}{2}\right) = \frac{1}{2}\delta_{ij} \tag{1}
$$

where λ_i denote the 3×3 Gell-Mann matrices. These implement the $SU(3)$ generators in the fundamental representation and, as such, are hermitian, traceless matrices satisfying:

$$
\boxed{\left[\frac{\lambda^a}{2}, \frac{\lambda^b}{2}\right] = i\, f^{ab}_c\, \frac{\lambda_c}{2}} \qquad \text{or} \qquad \boxed{[T^a\, T^b] = i\, f^{ab}_c\, T^c} . \tag{2}
$$

(b) Using Eq. (2), and recalling that the generators fulfill the Jacobi identity

$$
\boxed{[T^a\,, [T^b\,, T^c]] + [T^b\,, [T^c\,, T^a]] + [T^c\,, [T^a\,, T^b]] = 0} \tag{3}
$$

show that $\mathrm{Tr}\,(T^a\, T^b) = C(F)\,\delta^{ab}$, where $C(F)$ is a (real) numerical factor. Evaluate $C(F)$ explicitly, using e.g. the first Gell-Mann matrix,

$$
\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{4}
$$

Another important representation is the *adjoint* representation, for which the generators are given by the structure constants themselves:

$$
[F_i(A)]_{jk} \equiv -i\, f_{ijk} \; . \tag{5}
$$

(c) With the help of the identities (2) and (3), show that the adjoint generators $F(A)$ satisfy

$$\boxed{-(F_j\,F_i)_{mn} + (F_i\,F_j)_{mn} = i\,f_{ijk}\,(F_k)_{mn}} \qquad \text{and thereby} \qquad \boxed{[F_i, F_j]_{mn} = i\,f_{ijk}\,F_{kmn}}\,. \tag{6}$$

Notice that these are exactly the same relations satisfied by the $SU(3)$ generators in the fundamental representation T^a.

(d) Check that $\mathrm{Tr}[F_a(A)\,F_b(A)] = f_{ade}\,f_{bde} \equiv C(A)$, where again $C(A)$ is a real number.

(e) Verify that the quadratic Casimir of the group, $F^2 = F^a\,F_a$, commutes with every individual generator F_a.