[Hints to solve Ex 2, part (e) of sheet 3
(useful tricks for relativistic kinematics)]

$\Rightarrow$ Key idea: use Lorentz transformations to switch between lab frame and center-of-mass frame

Lab frame S:

before collision

$\pi^+ \longrightarrow p$

(π^+: beam particle) (p: fixed target)

after collision

$p, \vec{k}_p^{\,lab}$

$\pi^+, \vec{k}_\pi^{\,lab}$

(1) Total 4-momentum

$$p_\nu^{Lab} = \Big(\underbrace{\sqrt{m_\pi^2 + \vec{p}_\pi^{\,2\,lab}}}_{E_\pi^{lab}},\ \vec{p}_\pi^{\,lab}\Big) + (m_p, \vec{0})$$

$$p_\nu p^\nu \Big|^{lab} = s \quad \text{(Lorentz-invariant, valid in all frames)}$$

$\Downarrow$

$$s = (E_\pi^{lab})^2 + m_p^2 - \vec{p}_\pi^{\,lab\,2} = m_\pi^2 + m_p^2 + 2 m_p E_\pi^{lab}$$
$$+ 2 E_\pi m_p$$

$$\Rightarrow \boxed{E_\pi^{lab} = \frac{s - m_p^2 - m_\pi^2}{2 m_p}} \quad [1]$$

Center of mass frame S'

after collision

$\longleftarrow \pi^+ \qquad p \longrightarrow$

(2) Total 4-momentum: $K_{CM,\nu} = \left(\sqrt{m_\pi + \vec{k}^2}, -\vec{k}\right) + \left(m_p + \vec{k}^2, \vec{k}\right)$

$$K_\nu^{CM} K^{\nu\,CM} = s = m_\pi^2 + k^2 + m_p^2 + k^2 + 2\sqrt{(k^2 + m_p^2)(k^2 + m_\pi^2)} \quad [2]$$

From [2], we get $\vec{k}_\pi = -\vec{k}_p \equiv$ momenta of π in the final state, CM frame

(3) We find the relative velocity ($\vec{u}$) between S and S':

$$\vec{p}_\pi^{\,lab} \longrightarrow \vec{p}_\pi^{\,CM} = \gamma(u)\left[\vec{p}_\pi^{\,lab} - \vec{u}\, E_\pi^{lab}\right] \quad \text{(known from [1])}$$

$$\vec{p}_p^{\,lab} = 0 \longrightarrow \vec{p}_p^{\,CM} = -m_p \vec{u}\, \gamma(u)$$

by construction, $\vec{p}_\pi^{\,CM} + \vec{p}_p^{\,CM} = 0 \Leftrightarrow u = \dfrac{\sqrt{(E_\pi^{lab})^2 - m_\pi^2}}{E_\pi^{lab} + m_p}$ [3]

(4) We use [3] to transform [2] back to the lab frame:

$$\vec{k}_\pi^{\,lab} = \gamma(u)\left[\vec{k}_\pi^{\,CM} + u\, E_\pi^{CM}\right] \quad [4]$$

~~Knowing that~~

(5) Back to S, we simply recall that

$$k_\pi^{lab} = \gamma(v_\pi^{lab})\, v_\pi^{lab}\, m_\pi$$

$\longrightarrow$ we got v_π^{lab} for the pion after the collision