Einführung in Theoretische Teilchenphysik

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Exercise Sheet 4

Submission: Mo, 20.11.17, 12:00
Discussion: Mon, 20.11.17 14:00 Room 11/12
            Wed, 22.11.17 09:45 Room 10/1

Exercise 1: Flavor-spin wave function of a hadron

To completely characterize the quantum state of hadrons, we need a product wave-function which factorizes into several pieces, each of them corresponding to separate Hilbert spaces: i) a spatial part, describing the relative location and motion of the quarks; ii) a spin part, representing the orientation of their spins; iii) a flavor part, indicating the quark type (e.g. u, d); and the color part, which specifies the individual quark color charges.

\[ |\Psi\rangle_{\text{hadron}} = |\Psi\rangle_{\text{space}} \otimes |\Psi\rangle_{\text{spin}} \otimes |\Psi\rangle_{\text{flavor}} \otimes |\Psi\rangle_{\text{color}}. \]

From Pauli’s Exclusion Principle we know that the total wave function must be antisymmetric under the permutation of two quarks \(^1\).

For the spatial part, one assumes the lowest-lying hadronic states to be bound states of (anti)quarks with no relative angular momenta, \(\vec{L} = 0\). The spatial wave function is therefore symmetric. The spin state can be either completely symmetric (\(j = s = \frac{3}{2}\)) or of mixed symmetry (\(j = s = \frac{1}{2}\)). Finally, due to color confinement, all hadron states are color singlets, hence \(|\Psi_{\text{color}}\rangle\) is completely antisymmetric.

(a) Bearing all these ingredients in mind, convince yourself that the spin-flavor wave-function of the state \(\Delta^{++}\) from the baryon decouplet is given by (quite trivial!)

\[ |\Psi\rangle_{\Delta^{++}}^{\text{flavor-spin}} = |uuu\rangle \otimes |\uparrow\uparrow\uparrow\rangle = |u \uparrow u \uparrow u \uparrow\rangle. \]

(b) Using the same notation, write down the normalized spin-flavor wave function for \(\Delta^+\) (\(J = \frac{1}{2}\) uud bound state) with spin down \(m_j = -\frac{1}{2}\). If it was feasible to pull one particle apart, what would be the probability that the first quark would be a d-quark with spin up?

Constructing \(|\Psi\rangle_{\text{flavor}} \otimes |\Psi\rangle_{\text{spin}}\) for states of the baryon octet is a little trickier, as we must combine states of mixed symmetry to make a completely symmetric combination. The general recipe is:

\[ |\Psi\rangle_{\text{flavor}} \otimes |\Psi\rangle_{\text{spin}} = N \left\{ |\Psi_{12}^{\text{flavor}}\rangle |\Psi_{12}^{\text{spin}}\rangle + |\Psi_{13}^{\text{flavor}}\rangle |\Psi_{13}^{\text{spin}}\rangle + |\Psi_{23}^{\text{flavor}}\rangle |\Psi_{23}^{\text{spin}}\rangle \right\}, \]

where \(\Psi_{ij}\) denote a WF with mixed symmetry, viz. antisymmetric under the \(i \leftrightarrow j\) quark-pair exchange. This way, the product wave function \(\Psi_{ij}^{\text{flavor}} \otimes \Psi_{ij}^{\text{spin}}\) is symmetric under such quark exchange.

(c) Write down the six mixed symmetry spin-\(\frac{1}{2}\) wave functions \(|\Psi_{ij}^{\text{spin}}\rangle\) for \(i, j = 1, 2, 3\). Notice that exactly the same structure applies to the isospin-\(\frac{1}{2}\) flavor wave function.

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\(^1\)Notice that we treat all quarks as identical particles, regardless of their spin, flavor or color. These degrees of freedom correspond to different possible states of a single type of particle.
(d) From the above result, show that the spin-flavor wave function of a proton with spin up can be written as:

\[ |\Psi_p^{\text{flavor-spin}}\rangle = \frac{1}{3\sqrt{2}} \left[ 2 |u \uparrow u \uparrow d \downarrow\rangle + 2 |u \uparrow d \downarrow u \uparrow\rangle + 2 |d \downarrow u \uparrow u \uparrow\rangle - |u \uparrow u \uparrow d \uparrow\rangle - |u \uparrow d \downarrow u \uparrow\rangle - |d \downarrow u \uparrow u \uparrow\rangle \right]. \]

(e) The interaction of a spin-\(\frac{1}{2}\) particle with a classical magnetic field \(\vec{B}\) is governed by \(\hat{H}_{\text{Pauli}} = -\vec{\mu} \cdot \vec{B}\), where the magnetic moment operator is given by the 3rd component projection, \(\mu_z = \frac{q}{2m} \vec{S}_z\). Show that the magnetic moment of the proton can be written in terms of the up and down-quark magnetic moments as 

\[ \mu_p = \frac{1}{3} (4\mu_u - \mu_d), \]

where \(\mu_u = \frac{2}{3} \left( \frac{g_{\mu}}{2m_u} \right)\) and \(\mu_d = -\frac{1}{3} \left( \frac{g_{\mu}}{2m_d} \right)\).

By direct analogy, evaluate the neutron magnetic moment \(\mu_n\) and compare the ratio \(\frac{\mu_n}{\mu_p}\) to the experimental measurement \(\mu_n^{\text{exp}} = -0.68497945(58)\).

**Hint:** For the numerical estimate, recall that \(m_u = m_d\) under the assumption of isospin invariance.

**Exercise 2: Poincaré Group: the Pauli-Lubanski Operator**

The spin of a moving particle can be written in terms of the *Pauli-Lubanski pseudovector* 

\[ W^\mu = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma} \]

\[ W_\mu = -\frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} P^\sigma M^{\nu\rho} \]

where \(M^{\mu\nu} = i(x^\nu \partial^\mu - x^\mu \partial^\nu)\) denotes the relativistic angular momentum tensor operator, and \(P^\mu = i\partial^\mu\) is the 4-momentum.

We define the generalized Levi-Civita symbol in four dimensions as:

\[ \varepsilon_{\mu\nu\rho\sigma} = \begin{cases} 
1 & \text{if } \{\mu, \nu, \rho, \sigma\} \text{ is an even permutation of } \{1, 2, 3, 0\} \\
-1 & \text{if } \{\mu, \nu, \rho, \sigma\} \text{ is an even permutation of } \{0, 1, 2, 3\} \\
0 & \text{otherwise}
\end{cases} \]

with \(\varepsilon^{\mu\nu\rho\sigma} = g^{\mu\alpha} g^{\nu\beta} g^{\rho\gamma} g^{\sigma\delta} \varepsilon_{\alpha\beta\gamma\delta} = -\varepsilon_{\mu\nu\rho\sigma}\).

Prove the following properties:

(a) The \(W^\mu\) components for a particle at rest are \((0, -m\vec{J})\),

(b) \([M_{\alpha\beta}, P_\mu] = i(g_{\mu\beta} P_\alpha - g_{\mu\alpha} P_\beta)\),

(c) \(W_\mu P^\mu = 0\), and \([W^\nu, P_\nu] = 0\),

(d) \(P^2\) and \(W^2\) are the Casimir operators of the Poincaré group, i.e., they commute with all its generators, \([P^2, P_\mu] = [P^2, M_{\mu\nu}] = 0\) and \([W^2, P_\mu] = [W^2, M_{\mu\nu}] = 0\) (you do not need to prove the last property, \([W^2, M_{\mu\nu}] = 0\), the calculation is really tedious.).

(e) Knowing (no proof required) that \(W^2 = -\frac{1}{2} M_{\mu\nu} M^{\mu\nu} P^2 + M_{\mu\nu} M^{\mu\nu} P^\sigma P_\sigma\), \(W^2|p = 0, m, j\rangle = -m^2 j(j+1)\) where \(|p = 0, m, j\rangle\) is an eigenvector for a particle of mass \(m\), momentum \(p\), and total angular momentum \(j\).