# Einführung in Theoretische Teilchenphysik

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## Exercise Sheet 4

Submission: Mo, 20.11.17, 12:00

| Discussion: | | | |
|---|---|---|---|
| | Mon, 20.11.17 | 14:00 | Room 11/12 |
| | Wed, 22.11.17 | 09:45 | Room 10/1 |

### Exercise 1: Flavor-spin wave function of a hadron

To completely characterize the quantum state of hadrons, we need a product wave-function which factorizes into several pieces, each of them corresponding to separate Hilbert spaces: i) a **spatial** part, describing the relative location and motion of the quarks; ii) a **spin** part, representing the orientation of their spins; iii) a **flavor** part, indicating the quark type (e.g. $u, d$); and the **color** part, which specifies the individual quark color charges.

$$|\Psi\rangle_{\text{hadron}} = |\Psi^{\text{space}}\rangle \otimes |\Psi^{\text{spin}}\rangle \otimes |\Psi^{\text{flavor}}\rangle \otimes |\Psi^{\text{color}}\rangle \ .$$

From **Pauli's Exclusion Principle** we know that the total wave function must be antisymmetric under the permutation of two quarks [1].

For the spatial part, one assumes the lowest-lying hadronic states to be bound states of (anti)quarks with no relative angular momenta, $\vec{L} = 0$. The spatial wave function is therefore symmetric. The spin state can be either completely symmetric ($j = s = \frac{3}{2}$) or of mixed symmetry ($j = s = \frac{1}{2}$). Finally, due to color confinement, all hadron states are color singlets, hence $|\Psi_{\text{color}}\rangle$ is completely antisymmetric.

(a) Bearing all these ingredients in mind, convince yourself that the spin-flavor wave-function of the state $\Delta^{++}$ from the baryon decouplet is given by (quite trivial!)

$$|\Psi\rangle^{\text{flavor-spin}}_{\Delta^{++}} = |uuu\rangle \otimes |\uparrow\uparrow\uparrow\rangle = |u\uparrow\, u\uparrow u\uparrow\rangle \ .$$

(b) Using the same notation, write down the *normalized* spin-flavor wave function for $\Delta^+$ ($J = \frac{1}{2}$ uud bound state) with spin down $m_j = -\frac{1}{2}$. If it was feasible to pull one particle apart, what would be the probability that the first quark would be a d-quark with spin up?

Constructing $|\Psi\rangle^{\text{flavor}} \otimes |\Psi\rangle^{\text{spin}}$ for states of the baryon octet is a little trickier, as we must combine states of mixed symmetry to make a completely symmetric combination. The general recipee is:

$$|\Psi\rangle^{\text{flavor}} \otimes |\Psi\rangle^{\text{spin}} = N \left\{ |\Psi^{\text{flavor}}_{12}\rangle\, |\Psi^{\text{spin}}_{12}\rangle + |\Psi^{\text{flavor}}_{13}\rangle\, |\Psi^{\text{spin}}_{13}\rangle + |\Psi^{\text{flavor}}_{23}\rangle\, |\Psi^{\text{spin}}_{23}\rangle \right\},$$

where $\Psi_{ij}$ denote a WF with mixed symmetry, viz. antisymmetric under the $i \leftrightarrow j$ quark-pair exchange. This way, the product wave function $\Psi^{\text{flavor}}_{ij} \otimes \Psi^{\text{spin}}_{ij}$ is symmetric under such quark exchange.

(c) Write down the *six* mixed symmetry spin-$\frac{1}{2}$ wave functions $|\Psi^{\text{spin}}_{ij}\rangle$ for $i, j = 1, 2, 3$. Notice that exactly the same structure applies to the isospin-$\frac{1}{2}$ flavor wave function.

[1] Notice that we treat all quarks as identical particles, regardless of their spin, flavor or color. These degrees of freedom correspond to different possible states of a single type of particle

(d) From the above result, show that the spin-flavor wave function of a proton can be written as:

$$|\Psi\rangle_p^{\text{flavor-spin}} = \frac{1}{3\sqrt{2}}\Big[2\,|u\uparrow u\uparrow d\downarrow\rangle + 2\,|u\uparrow d\downarrow u\uparrow\rangle + 2\,|d\downarrow u\uparrow u\uparrow\rangle - |u\uparrow u\downarrow d\uparrow\rangle$$
$$- |u\uparrow d\uparrow u\downarrow\rangle - |d\uparrow u\uparrow u\downarrow\rangle - |u\downarrow u\uparrow d\uparrow\rangle - |u\downarrow d\uparrow u\uparrow\rangle - |d\downarrow u\downarrow u\uparrow\rangle\Big].$$

(e) The interaction of a spin-$\frac{1}{2}$ particle with a classical magnetic field $\vec{B}$ is governed by $\hat{\mathcal{H}}_{\text{pauli}} = -\vec{\mu}\cdot\vec{B}$, where the magnetic moment operator is given by the 3rd component projection for a spin-up state, $\hat{\mu}_z = \dfrac{q\hat{S}_z}{m}$. Show that the magnetic moment of the proton can be written in terms of the up and down-quark magnetic moments as $\mu_p = \frac{1}{3}\left(4\mu_u - \mu_d\right)$. By direct analogy, evaluate the neutron magnetic moment $\mu_n$ and compare the ratio $\left.\frac{\mu_p}{\mu_n}\right|_{\text{theory}}$ to the experimental measurement $\left.\frac{\mu_p}{\mu_n}\right|_{\text{exp}} = -0.68497945(58)$.

**Exercise 2: Poincaré Group: the Pauli-Lubanski Operator**

The spin of a moving particle can be written in terms of the *Pauli-Lubanski pseudovector*

$$W^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma} \qquad\qquad W_\mu = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}P^\sigma M^{\nu\rho}$$

where $M^{\nu\rho} = i(x^\nu\partial^\rho - x^\rho\partial^\nu)$ denotes the relativistic angular momentum tensor operator, and $P^\sigma = -i\partial^\sigma$ is the 4-momentum.
We define the generalized Levi-Civita symbol in four dimensions as:

$$\varepsilon_{\mu\nu\rho\sigma} = \begin{cases} 1 \text{ if } \{\mu,\nu,\rho,\sigma\} \text{ is an even permutation of } \{1,2,3,0\} \\ -1 \text{ if } \{\mu,\nu,\rho,\sigma\} \text{ is an even permutation of } \{0,1,2,3\} \\ 0 \text{ otherwise} \end{cases},$$

with $\varepsilon^{\mu\nu\rho\sigma} = g^{\mu\alpha}g^{\nu\beta}g^{\rho\gamma}g^{\sigma\delta}\varepsilon_{\alpha\beta\gamma\delta} = -\varepsilon_{\mu\nu\rho\sigma}$.

Prove the following properties:

(a) The $W^\mu$ components for a particle at rest are $(0, -m\vec{J})$,

(b) $[M_{\alpha\beta}, P_\mu] = i(g_{\mu\beta}P_\alpha - g_{\mu\alpha}P_\beta)$,

(c) $W_\mu P^\mu = 0$, and $[W_\mu, P_\nu] = 0$,

(d) $W^2 = -\frac{1}{2}M_{\mu\nu}M^{\mu\nu}P^2 + M_{\mu\rho}M^{\nu\rho}P^\mu P_\nu$ with $\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\mu\alpha\beta\gamma} = \begin{vmatrix} \delta_\nu^\alpha & \delta_\nu^\beta & \delta_\nu^\gamma \\ \delta_\rho^\alpha & \delta_\rho^\beta & \delta_\rho^\gamma \\ \delta_\sigma^\alpha & \delta_\sigma^\beta & \delta_\sigma^\gamma \end{vmatrix}$,

(e) $P^2$ and $W^2$ are the Casimir operators of the Poincaré group, *i.e.*, that they commute with all its generators, $[P^2, P_\mu] = [P^2, M_{\mu\nu}] = 0$ and $[W^2, P_\mu] = [W^2, M_{\mu\nu}] = 0$,

(f) $W^2|\mathbf{p} = 0, m, j\rangle = -m^2 j(j+1)|\mathbf{p} = 0, m, j\rangle$ where $|\mathbf{p} = 0, m, j\rangle$ is an eigenvector for a particle of mass $m$, momentum $\mathbf{p}$, and total angular momentum $j$.