

# Einführung in Theoretische Teilchenphysik

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## Exercise Sheet 4

Submission: Mo, 20.11.17, 12:00

Discussion: Mon, 20.11.17 14:00 Room 11/12  
 Wed, 22.11.17 09:45 Room 10/1

### Exercise 1: Flavor-spin wave function of a hadron

To completely characterize the quantum state of hadrons, we need a product wave-function which factorizes into several pieces, each of them corresponding to separate Hilbert spaces: i) a **spatial** part, describing the relative location and motion of the quarks; ii) a **spin** part, representing the orientation of their spins; iii) a **flavor** part, indicating the quark type (e.g.  $u, d$ ); and the **color** part, which specifies the individual quark color charges.

$$|\Psi\rangle_{\text{hadron}} = |\Psi^{\text{space}}\rangle \otimes |\Psi^{\text{spin}}\rangle \otimes |\Psi^{\text{flavor}}\rangle \otimes |\Psi^{\text{color}}\rangle .$$

From **Pauli's Exclusion Principle** we know that the total wave function must be antisymmetric under the permutation of two quarks <sup>1</sup>.

For the spatial part, one assumes the lowest-lying hadronic states to be bound states of (anti)quarks with no relative angular momenta,  $\vec{L} = 0$ . The spatial wave function is therefore symmetric. The spin state can be either completely symmetric ( $j = s = \frac{3}{2}$ ) or of mixed symmetry ( $j = s = \frac{1}{2}$ ). Finally, due to color confinement, all hadron states are color singlets, hence  $|\Psi_{\text{color}}\rangle$  is completely antisymmetric.

- (a) Bearing all these ingredients in mind, convince yourself that the spin-flavor wave-function of the state  $\Delta^{++}$  from the baryon decuplet is given by (quite trivial!)

$$|\Psi\rangle_{\Delta^{++}}^{\text{flavor-spin}} = |uuu\rangle \otimes |\uparrow\uparrow\uparrow\rangle = |u \uparrow u \uparrow u \uparrow\rangle .$$

- (b) Using the same notation, write down the *normalized* spin-flavor wave function for  $\Delta^+$  ( $J = \frac{1}{2}$  uud bound state) with spin down  $m_j = -\frac{1}{2}$ . If it was feasible to pull one particle apart, what would be the probability that the first quark would be a d-quark with spin up?

Constructing  $|\Psi\rangle^{\text{flavor}} \otimes |\Psi\rangle^{\text{spin}}$  for states of the baryon octet is a little trickier, as we must combine states of mixed symmetry to make a completely symmetric combination. The general recipe is:

$$|\Psi\rangle^{\text{flavor}} \otimes |\Psi\rangle^{\text{spin}} = N \left\{ |\Psi_{12}^{\text{flavor}}\rangle |\Psi_{12}^{\text{spin}}\rangle + |\Psi_{13}^{\text{flavor}}\rangle |\Psi_{13}^{\text{spin}}\rangle + |\Psi_{23}^{\text{flavor}}\rangle |\Psi_{23}^{\text{spin}}\rangle \right\},$$

where  $\Psi_{ij}$  denote a WF with mixed symmetry, viz. antisymmetric under the  $i \leftrightarrow j$  quark-pair exchange. This way, the product wave function  $\Psi_{ij}^{\text{flavor}} \otimes \Psi_{ij}^{\text{spin}}$  is symmetric under such quark exchange.

- (c) Write down the *six* mixed symmetry spin- $\frac{1}{2}$  wave functions  $|\Psi_{ij}^{\text{spin}}\rangle$  for  $i, j = 1, 2, 3$ . Notice that exactly the same structure applies to the isospin- $\frac{1}{2}$  flavor wave function.

<sup>1</sup>Notice that we treat all quarks as identical particles, regardless of their spin, flavor or color. These degrees of freedom correspond to different possible states of a single type of particle.

- (d) From the above result, show that the spin-flavor wave function of a proton with spin up can be written as:

$$|\Psi\rangle_p^{\text{flavor-spin}} = \frac{1}{3\sqrt{2}} \left[ 2|u \uparrow u \uparrow d \downarrow\rangle + 2|u \uparrow d \downarrow u \uparrow\rangle + 2|d \downarrow u \uparrow u \uparrow\rangle - |u \uparrow u \downarrow d \uparrow\rangle - |u \uparrow d \uparrow u \downarrow\rangle - |d \uparrow u \uparrow u \downarrow\rangle - |u \downarrow u \uparrow d \uparrow\rangle - |u \downarrow d \uparrow u \uparrow\rangle - |d \uparrow u \downarrow u \uparrow\rangle \right].$$

- (e) The interaction of a spin- $\frac{1}{2}$  particle with a classical magnetic field  $\vec{B}$  is governed by  $\hat{H}_{\text{Pauli}} = -\vec{\mu} \cdot \vec{B}$ , where the magnetic moment operator is given by the 3rd component projection for a spin-up state,  $\hat{\mu}_z = \frac{q\hat{S}_z}{m}$ . Show that the magnetic moment of the proton can be written in terms of the up and down-quark magnetic moments as  $\mu_p = \frac{1}{3}(4\mu_u - \mu_d)$ . By direct analogy, evaluate the neutron magnetic moment  $\mu_n$  and compare the ratio  $\frac{\mu_p}{\mu_n} \Big|_{\text{theory}}$  to the experimental measurement  $\frac{\mu_p}{\mu_n} \Big|_{\text{exp}} = -0.68497945(58)$ .

### Exercise 2: Poincaré Group: the Pauli-Lubanski Operator

The spin of a moving particle can be written in terms of the *Pauli-Lubanski pseudovector*

$$W^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma} \quad W_\mu = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} P^\sigma M^{\nu\rho}$$

where  $M^{\nu\rho} = i(x^\nu \partial^\rho - x^\rho \partial^\nu)$  denotes the relativistic angular momentum tensor operator, and  $P^\sigma = -i\partial^\sigma$  is the 4-momentum.

We define the generalized Levi-Civita symbol in four dimensions as:

$$\varepsilon_{\mu\nu\rho\sigma} = \begin{cases} 1 & \text{if } \{\mu, \nu, \rho, \sigma\} \text{ is an even permutation of } \{1, 2, 3, 0\} \\ -1 & \text{if } \{\mu, \nu, \rho, \sigma\} \text{ is an odd permutation of } \{1, 2, 3, 0\} \\ 0 & \text{otherwise} \end{cases},$$

with  $\varepsilon^{\mu\nu\rho\sigma} = g^{\mu\alpha} g^{\nu\beta} g^{\rho\gamma} g^{\sigma\delta} \varepsilon_{\alpha\beta\gamma\delta} = -\varepsilon_{\mu\nu\rho\sigma}$ .

Prove the following properties:

- The  $W^\mu$  components for a particle at rest are  $(0, -m\vec{J})$ ,
- $[M_{\alpha\beta}, P_\mu] = i(g_{\mu\beta}P_\alpha - g_{\mu\alpha}P_\beta)$ ,
- $W_\mu P^\mu = 0$ , and  $[W_\mu, P_\nu] = 0$ ,
- $W^2 = -\frac{1}{2}M_{\mu\nu}M^{\mu\nu}P^2 + M_{\mu\rho}M^{\nu\rho}P^\mu P_\nu$  with  $\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\mu\alpha\beta\gamma} = \begin{vmatrix} \delta_\nu^\alpha & \delta_\nu^\beta & \delta_\nu^\gamma \\ \delta_\rho^\alpha & \delta_\rho^\beta & \delta_\rho^\gamma \\ \delta_\sigma^\alpha & \delta_\sigma^\beta & \delta_\sigma^\gamma \end{vmatrix}$ ,
- $P^2$  and  $W^2$  are the Casimir operators of the Poincaré group, *i.e.*, that they commute with all its generators,  $[P^2, P_\mu] = [P^2, M_{\mu\nu}] = 0$  and  $[W^2, P_\mu] = [W^2, M_{\mu\nu}] = 0$ ,
- $W^2|\mathbf{p} = 0, m, j\rangle = -m^2 j(j+1)|\mathbf{p} = 0, m, j\rangle$  where  $|\mathbf{p} = 0, m, j\rangle$  is an eigenvector for a particle of mass  $m$ , momentum  $\mathbf{p}$ , and total angular momentum  $j$ .