Einführung in Theoretische Teilchenphysik

Lecture: PD. Dr. S Gieseke – Exercises: Dr. D. López-Val, Dr. S. Patel, PD. Dr. M. Rauch

Exercise Sheet 6

Submission: Mo, 04.12.17, 12:00

Discussion:	Mon, 04.12.17	14:00	Room 11/12
	Wed, 06.12.17	09:45	Room 10/1

Exercise 1: Lagrangian for a generic spin-1/2 field

Consider the Lagrangian describing a spin-1/2 field ψ,

$$\mathcal{L} = \psi^\dagger \left[i\hbar \frac{\partial}{\partial_t} - c\vec{\alpha}\cdot\vec{p} - \beta\, mc^2 \right] \psi \,,$$

along with the so-called *Weyl representation* for $\vec{\alpha}, \beta$,

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}; \qquad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix},$$

and the γ matrices

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix},$$

which satisfy the Clifford Algebra $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$.

(a) Show that the Lagrangian $\mathcal{L}$ can be rewritten in terms of the above γ matrices as

$$\mathcal{L} = \bar{\psi}\, \left[i\hbar c \gamma^\mu \partial_\mu - mc^2 \right] \psi \qquad \text{mit} \qquad \bar{\psi} \equiv \psi^\dagger\, \gamma^0 \,.$$

(b) Applying the Euler-Lagrange equations to your result, obtain the covariant form of the Dirac Equation:

$$(\mathrm{i}\gamma^\mu \partial_\mu - m)\, \psi(x) = 0 \,.$$

(c) The general expression for the stress-energy tensor in field theory is

$$T^\mu_\nu = \frac{\delta\, \mathcal{L}}{\delta(\partial_\mu \psi)} (\partial_\nu\, \psi) + \frac{\delta\, \mathcal{L}}{\delta(\partial_\mu \bar{\psi})} (\partial_\nu\, \bar{\psi}) - \mathcal{L}\, \delta^\mu_\nu \,.$$

Prove that, for a Dirac field, the above expression takes the form

$$T^\mu_\nu = i\hbar c \bar{\psi}\, (\gamma^\mu \partial_\nu - \delta^\mu_\nu \gamma^\sigma \partial_\sigma)\psi + \delta^\mu_\nu mc^2 \bar{\psi}\psi \,.$$

Exercise 2: Lagrangian formulation of Classical Electrodynamics

Classical Electrodynamics is governed by the Lagrangian

$$\mathcal{L} = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - J^\mu A_\mu \tag{1}$$

written in terms of the field-strength tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and a source term J^μ.

(a) Apply the gauge transformation $A^\mu(x) \to A'^\mu(x) = A^\mu(x) + \partial^\mu f(x)$ and show that the transformed Lagrangian $\mathcal{L}'$ leads to the same equations of motion for $A^\mu(x)$.

(b) Show how the Maxwell's Equations (in Heavyside-Lorentz units $c = \epsilon_0 = \mu_0 = 1$)

$$\vec{\nabla} \cdot \vec{E} = 0 \, ; \qquad \vec{\nabla} \times \vec{B} = \vec{J} + \frac{\partial \vec{E}}{\partial t} \tag{2}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \, ; \qquad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \, . \tag{3}$$

may be obtained from

$$\partial_\mu F^{\mu\nu} = J^\nu \, , \quad \partial_\mu \tilde{F}^{\mu\nu} \equiv \partial_\mu \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = 0$$

Prove that, in the particular case where the source term is absent, the homogeneous Maxwell's Equations (2) be derived from

$$\partial^{[\mu} F^{\nu\rho]} \equiv \partial^\mu F^{\nu\rho} + \partial^\nu F^{\rho\mu} + \partial^\rho F^{\mu\nu} = 0 \, .$$

(c) A formulation alternative to Eq. (1), originally proposed by Fermi, is based on the Lagrangian

$$\mathcal{L} = -\frac{1}{2} \left(\partial^\mu A^\nu\right) \left(\partial_\mu A_\nu\right) - J^\mu A_\mu \, .$$

Obtain the Equations of motion in this case, and determine which condition is necessary to reproduce the standard form of Maxwell's Equations (2)-(3).

(d) In the following we consider a speculative extension of Classical Electrodynamics, by adding a new term to the field-strength tensor through

$$F'^{\mu\nu} = F^{\mu\nu} - \partial_\rho \epsilon^{\mu\nu\rho\sigma} A'_\sigma \, ,$$

as a function of a second 4-potential $A'^\mu = (\varphi', \vec{A}')^T$.
Show that the inhomogeneous Maxwell's Equations in this theory are equivalent to those of standard Electrodynamics.

(e) How would Maxwell's Equations differ from their standard form, if we also introduced an additional source term J'^μ,

$$\partial_\mu \tilde{F}'^{\mu\nu} = J'^\nu \, ?$$

Could you attribute a physical interpretation to the new term in the modified equations?