Einführung in Theoretische Teilchenphysik

Lecture: PD. Dr. S Gieseke – Exercises: Dr. D. López-Val, Dr. S. Patel, PD. Dr. M. Rauch

Exercise Sheet 7

Submission: Mo, 10.12.17, 12:00

Discussion: Mon, 10.12.17 14:00 Room 11/12
Wed, 12.12.17 09:45 Room 10/1

Exercise 1: Real scalar field: canonical quantization

A real scalar field is governed by the Klein-Gordon Lagrangian $\mathcal{L}_{\rm KG} = \frac{1}{2}(\partial_\mu \varphi(x))(\partial^\mu \varphi(x)) - \frac{m^2}{2}\varphi(x)^2$. The Fourier transformation of the quantized field can be written as

$$\varphi(\vec{x},t) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p}\left[a(\vec{p})e^{-ipx} + a^\dagger(\vec{p})e^{+ipx}\right] ,$$

where the plane wave solutions obey the orthogonality condition:

$$\int \mathrm{d}^3x e^{-ipx}e^{ip'x} = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{p}\,')$$

(notice that, due to the energy-momentum relation $E_p = \sqrt{\vec{p}^{\,2}+m^2}$, the above condition implies $p_0' = p_0$).

(a) Show that the annihilation operator takes the form

$$a(\vec{p}) = \int \mathrm{d}^3x\, e^{ipx}\left(i\dot{\varphi}(x) + E_p\varphi(x)\right) .$$

Given in terms of the fields, the canonical commutation relations yield

$$[\varphi(\vec{x},t),\varphi(\vec{x}\,',t)] = [\dot{\varphi}(\vec{x},t),\dot{\varphi}(\vec{x}\,',t)] = 0\,, \qquad [\varphi(\vec{x},t),\dot{\varphi}(\vec{x}\,',t)] = i\delta^{(3)}(\vec{x}-\vec{x}\,')\,.$$

(b) Obtain the corresponding relations for the operators $a(\vec{p}), a^\dagger(\vec{p})$,

$$[a(\vec{p}),a(\vec{p}\,')] = [a^\dagger(\vec{p}),a^\dagger(\vec{p}\,')] = 0\,, \qquad [a(\vec{p}),a^\dagger(\vec{p}\,')] = (2\pi)^3 2E_p\delta^{(3)}(\vec{p}-\vec{p}\,')\,.$$

(c) Evaluate the Stress-Energy-Momentum tensor $T^\mu{}_\nu$ of the scalar field $\varphi(\vec{x},t)$.

The energy spectrum of the scalar field is given by the eigenvalues of the Hamilton operator $\hat{H} = \int \mathrm{d}^3x\mathcal{H} = \int \mathrm{d}^3x T^0_0$.

(d) Using the Fourier transformation of the field, along with the commutation relations for $a(\vec{p}), a^\dagger(\vec{p})$, prove that $\hat{H}$ is diagonalized and may be written as

$$\hat{H} = \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}E_p\left[2\tilde{N}(\vec{p}) + C\right] .$$

Here we have introduced the particle number operator

$$N = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p}a^\dagger(\vec{p})a(\vec{p}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\tilde{N}(\vec{p})\,,$$

while C stands for a constant term. What is its physical interpretation?

(e) Similarly, show that the 3-momentum of the field $P_i = \int \mathrm{d}^3x T^0{}_i$ yields

$$\vec{P} = \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\vec{p}\left[2\tilde{N}(\vec{p}) + C\right] .$$

Exercise 2: Real scalar field: Feynman propagator

The Feynman propagator of a real scalar field yields

$$i\Delta_F(x-x') := \langle 0\,|T(\varphi(x)\varphi(x')|\,0\rangle\,,$$

where $T(\varphi(x)\varphi(x')) = \Theta(t-t')\varphi(x)\varphi(x') + \Theta(t'-t)\varphi(x')\varphi(x)$ defines the time-ordering operator. Upon several manipulations (which we do not consider here) the above expression takes the following form in momentum space:

$$\Delta_F(x) = \lim_{\epsilon\to 0^+}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{e^{-ipx}}{p^2-m^2+i\epsilon}\,. \tag{1}$$

Show that Eq. (1) fulfills the inhomogeneous Klein-Gordon Equation,

$$(\partial_\mu\partial^\mu + m^2)\Delta_F(x) = -\delta^{(4)}(x)\,.$$