Einführung in Theoretische Teilchenphysik

Lecture: PD Dr. S Gieseke – Exercises: Dr. D. López-Val, Dr. S. Patel, PD Dr. M. Rauch

Exercise Sheet 8

Submission: Mo, 18.12.17, 12:00

Discussion:	Mon, 18.12.17	14:00	Room 11/12
	Wed, 20.12.17	09:45	Room 10/1

Exercise 1: Complex scalar field

The Hamiltonian of a complex-valued scalar field obeying the Klein-Gordon Equation is given at the classical level by

$$H = \int d^3\vec{x}\,[\Pi^*\,\Pi + (\vec{\nabla}\varphi^*)\cdot(\vec{\nabla}\varphi) + m^2\,\varphi^*\,\varphi]\,. \tag{1}$$

The field variables φ, φ^* are treated as independent quantities, with $\Pi \equiv \frac{\partial\mathcal{L}}{\partial(\partial_0\,\varphi)}$, $\Pi^* \equiv \frac{\partial\mathcal{L}}{\partial(\partial_0\,\varphi^*)}$ denoting their respective canonically conjugated momenta. The theory is symmetric under real phase transformations $\varphi \to \varphi' = \varphi\,e^{i\alpha}$, $\varphi^* \to \varphi'^* = \varphi^*\,e^{-i\alpha}$ ($\alpha \in \mathbb{R}$). This global $U(1)$ symmetry leads at the classical level to the Noether current j^0 and the corresponding Noether charge

$$Q = \int d^3\vec{x}\,j^0 \; = i\int d^3\vec{x}\,(\Pi^*\,\varphi^* - \Pi\,\varphi)\,. \tag{2}$$

In agreement with canonical quantization, the field operators are required to fulfill the following commutation relations at equal time:

$$[\hat{\varphi}(\vec{x},t), \hat{\Pi}(\vec{x}',t)] = [\hat{\varphi}^\dagger(\vec{x},t), \hat{\Pi}^\dagger(\vec{x}',t)] = i\,\delta^{(3)}(\vec{x}-\vec{x'})\,,$$

while the commutators involving other combinations are required to vanish.

(a) Prove that the Heisenberg Equations,

$$i\,\frac{\partial\,\varphi}{\partial t} = [\varphi, \hat{H}]\,; \qquad\qquad i\,\frac{\partial\,\hat{\Pi}}{\partial t} = [\hat{\Pi}, \hat{H}] \tag{3}$$

imply that φ satisfies the Klein-Gordon equation.

(b) Introducing the field decomposition into the quantized normal modes,

$$\varphi(\vec{x},t) = \int \frac{d^3\vec{p}}{(2\pi)^3\,2E_{\vec{p}}}\left[a_{\vec{p}}\,e^{-ip\cdot x} + b^\dagger_{\vec{p}}\,e^{ip\cdot x}\right]\,;$$

$$\varphi^\dagger(\vec{x},t) = \int \frac{d^3\vec{p}}{(2\pi)^3\,2E_{\vec{p}}}\left[a^\dagger_{\vec{p}}\,e^{ip\cdot x} + b_{\vec{p}}\,e^{-ip\cdot x}\right]\,,$$

show that the classical Noether charge (2) leads at the quantum level to the charge operator

$$\hat{Q} = i\int \frac{d^3\vec{p}}{(2\pi)^3\,2E_{\vec{p}}}\,[\hat{N}^{(a)}_{\vec{p}} - \hat{N}^{(b)}_{\vec{p}}]\;,\;\;, \tag{4}$$

with $\hat{N}^{(a)}_{\vec{p}} \equiv a^\dagger_{\vec{p}}\,a_{\vec{p}}$, $\hat{N}^{(b)}_{\vec{p}} \equiv b^\dagger_{\vec{p}}\,b_{\vec{p}}$, and $E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$.

(c) Check that $[\hat{Q}, \hat{H}] = 0$, namely, that the charge is a conserved quantity in the quantized theory as well. For that, recall the expression of the quantized Hamiltonian (up to the zero-point energy term):

$$\hat{H} = \int \frac{d^3\vec{p}}{(2\pi)^3\, 2E_{\vec{p}}} E_{\vec{p}}\, [\hat{N}^{(a)}_{\vec{p}} + \hat{N}^{(b)}_{\vec{p}}]\,. \tag{5}$$

To investigate the action of $\hat{Q}$ on the field states, let us assume $|\alpha\rangle$ to represent an eigenstate of $\hat{Q}$ with eigenvalue q, $\hat{Q}\,|\alpha\rangle = q\;|\alpha\rangle$.

(d) Show that

$$\hat{Q}\,\varphi^\dagger = \varphi^\dagger\,(\hat{Q}+1) \qquad \text{and thereby} \qquad \hat{Q}\,\varphi^\dagger\,|\alpha\rangle = (q+1)\varphi^\dagger\;|\alpha\rangle\;, \tag{6}$$

which means that $\hat{Q}$ increases the charge of a state by one unit.

Exercise 2: Conserved charges for multiple complex fields

Consider two complex scalar fields φ_a, $a = 1, 2$ with equal masses satisfying the Klein-Gordon Lagrangian,

$$\mathcal{L} = \sum_a (\partial^\mu\,\varphi_a^*)\,(\partial_\mu\,\varphi_a) - m^2\,\varphi_a^*\,\varphi_a\;.$$

(a) Show that, at variance with the single-field case, there are now four conserved charges,

$$\hat{Q} = \sum_a \int d^3\vec{x}\, i\,\left[\varphi_a^*\,\hat{\Pi}_a^* - \varphi_a\,\hat{\Pi}_a\right] \qquad \hat{Q}^{(k)}\!= \sum_{a,b} \int d^3\vec{x}\,\frac{i}{2}\left[\varphi_a^*\,(\sigma^k)_{ab}\,\hat{\Pi}_b^* - \hat{\Pi}_a\,(\sigma^k)_{ab}\,\varphi_b\right]\;,$$

where σ^k denote the thee Pauli matrices, viz. the generators of the $SU(2)$ algebra.

Hint: Check that $[\hat{Q}, \hat{H}] = [\hat{Q^{(k)}}, \hat{H}] = 0$.

Notice that the first conserved charge $\hat{Q}$ is a straight generalization of the single-field case, while $\hat{Q}^{(k)}$ are genuine of the double-field structure.

(b) Could you guess what is the underlying symmetry that leads to the conserved quantities $\hat{Q}$ and $\hat{Q}^{(k)}$?

(c) Any intuition on how these results generalize to an arbitrary number N of complex scalar fields?