Exercise 1: Complex scalar field

The Hamiltonian of a complex-valued scalar field obeying the Klein-Gordon Equation is given at the classical level by

\[
H = \int d^3\vec{x} \left[ \Pi^* \Pi + (\vec{\nabla} \phi^*) \cdot (\vec{\nabla} \phi) + m^2 \phi^* \phi \right].
\] (1)

The field variables \( \phi, \phi^* \) are treated as independent quantities, with \( \Pi \equiv \frac{\partial L}{\partial (\partial_0 \phi)} \), \( \Pi^* \equiv \frac{\partial L}{\partial (\partial_0 \phi^*)} \) denoting their respective canonically conjugated momenta. The theory is symmetric under real phase transformations \( \phi \rightarrow \phi' = \phi e^{i\alpha}, \phi^* \rightarrow \phi'^* = \phi^* e^{-i\alpha} (\alpha \in \mathbb{R}) \). This global \( U(1) \) symmetry leads at the classical level to the Noether current \( j^0 \) and the corresponding Noether charge

\[
Q = \int d^3\vec{x} j^0 = i \int d^3\vec{x} (\Pi^* \phi^* - \Pi \phi).
\] (2)

In agreement with canonical quantization, the field operators are required to fulfill the following commutation relations at equal time:

\[
[\hat{\varphi}(\vec{x}, t), \hat{\Pi}(\vec{x}', t)] = [\hat{\varphi}^\dagger(\vec{x}, t), \hat{\Pi}^\dagger(\vec{x}', t)] = i \delta^{(3)}(\vec{x} - \vec{x}'),
\]

while the commutators involving other combinations are required to vanish.

(a) Prove that the Heisenberg Equations,

\[
i \frac{\partial \phi}{\partial t} = [\phi, \hat{H}]; \quad i \frac{\partial \Pi}{\partial t} = [\Pi, \hat{H}]
\] (3)

imply that \( \phi \) satisfies the Klein-Gordon equation.

(b) Introducing the field decomposition into the quantized normal modes,

\[
\varphi(\vec{x}, t) = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_\vec{p}} \left[ a_\vec{p} e^{-i\vec{p} \cdot \vec{x}} + b_\vec{p}^* e^{i\vec{p} \cdot \vec{x}} \right];
\]

\[
\varphi^\dagger(\vec{x}, t) = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_\vec{p}} \left[ a_\vec{p}^\dagger e^{i\vec{p} \cdot \vec{x}} + b_\vec{p} e^{-i\vec{p} \cdot \vec{x}} \right],
\]

show that the classical Noether charge \( Q \) leads at the quantum level to the charge operator

\[
\hat{Q} = i \int \frac{d^3\vec{p}}{(2\pi)^3 2E_\vec{p}} \left[ \hat{N}^{(a)}_\vec{p} - \hat{N}^{(b)}_\vec{p} \right],
\] (4)

with \( \hat{N}^{(a)}_\vec{p} \equiv a_\vec{p}^\dagger a_\vec{p} \), \( \hat{N}^{(b)}_\vec{p} \equiv b_\vec{p}^\dagger b_\vec{p} \), and \( E_\vec{p} = \sqrt{\vec{p}^2 + m^2} \).
(c) Check that $[\hat{Q}, \hat{H}] = 0$, namely, that the charge is a conserved quantity in the quantized theory as well. For that, recall the expression of the quantized Hamiltonian (up to the zero-point energy term):

$$\hat{H} = \int \frac{d^3 \vec{p}}{(2\pi)^3 2E_{\vec{p}}} E_{\vec{p}} \left[ \hat{N}^{(a)}_{\vec{p}} + \hat{N}^{(b)}_{\vec{p}} \right].$$

To investigate the action of $\hat{Q}$ on the field states, let us assume $|\alpha\rangle$ to represent an eigenstate of $\hat{Q}$ with eigenvalue $q$, $\hat{Q} |\alpha\rangle = q |\alpha\rangle$.

(d) Show that

$$\hat{Q} \varphi^\dagger = \varphi^\dagger (\hat{Q} + 1)$$

and thereby

$$\hat{Q} \varphi^\dagger |\alpha\rangle = (q + 1) \varphi^\dagger |\alpha\rangle,$$

which means that $\hat{Q}$ increases the charge of a state by one unit.

Exercise 2: Conserved charges for multiple complex fields

Consider two complex scalar fields $\varphi_a, a = 1, 2$ with equal masses satisfying the Klein-Gordon Lagrangian,

$$\mathcal{L} = \sum_a \left( \partial_{\mu} \varphi_a^* \partial^\mu \varphi_a - m^2 \varphi_a^* \varphi_a \right).$$

(a) Show that, at variance with the single-field case, there are now four conserved charges,

$$\hat{Q} = \sum_a \int d^3 \vec{x} i \left[ \varphi_a^* \hat{\Pi}_a - \varphi_a \hat{\Pi}_a^* \right] \quad \hat{Q}^{(k)} = \sum_{a,b} \int d^3 \vec{x} \frac{i}{2} \left[ \varphi_a^* (\sigma^k)_{ab} \hat{\Pi}_b^* - \hat{\Pi}_a (\sigma^k)_{ab} \varphi_b \right],$$

where $\sigma^k$ denote the the Pauli matrices, viz. the generators of the $SU(2)$ algebra.

Hint: Check that $[\hat{Q}, \hat{H}] = [\hat{Q}^{(k)}, \hat{H}] = 0$.

Notice that the first conserved charge $\hat{Q}$ is a straight generalization of the single-field case, while $\hat{Q}^{(k)}$ are genuine of the double-field structure.

(b) Could you guess what is the underlying symmetry that leads to the conserved quantities $\hat{Q}$ and $\hat{Q}^{(k)}$?

(c) Any intuition on how these results generalize to an arbitrary number $N$ of complex scalar fields?