Einführung in Theoretische Teilchenphysik

Lecture: PD Dr. S Gieseke – Exercises: Dr. D. López-Val, Dr. S. Patel, PD Dr. M. Rauch

Exercise Sheet 9

Submission: Mo, 08.01.18, 12:00

Discussion:

Mon, 08.01.18	14:00	Room 11/12
Wed, 10.01.18	09:45	Room 10/1

Exercise 1: Chiral Representation of Dirac Spinors

An explicit form of the spinors (in the chiral representation of the Dirac matrices) with mass m and four-momentum $(E, p_x, p_y, p_z)^T$ can be written as

$$u_\lambda(p) = \begin{pmatrix} \sqrt{E - \lambda|\vec{p}|}\ \chi_\lambda(p) \\ \sqrt{E + \lambda|\vec{p}|}\ \chi_\lambda(p) \end{pmatrix}, \qquad v_\lambda(p) = \begin{pmatrix} -\lambda\sqrt{E + \lambda|\vec{p}|}\ \chi_{-\lambda}(p) \\ \lambda\sqrt{E - \lambda|\vec{p}|}\ \chi_{-\lambda}(p) \end{pmatrix}$$

with

$$\chi_+(p) = \frac{1}{\sqrt{2|\vec{p}|(|\vec{p}| + p_z)}} \begin{pmatrix} |\vec{p}| + p_z \\ p_x + ip_y \end{pmatrix}, \qquad \chi_-(p) = \frac{1}{\sqrt{2|\vec{p}|(|\vec{p}| + p_z)}} \begin{pmatrix} -p_x + ip_y \\ |\vec{p}| + p_z \end{pmatrix}.$$

The spinors of the particle, u, and the antiparticle, v, thereby are solutions of the Dirac equation in momentum space

$$(\gamma^\mu p_\mu - m)u_\lambda(p) = 0\,, \qquad \bar{u}_\lambda(p)(\gamma^\mu p_\mu - m) = 0\,,$$
$$(\gamma^\mu p_\mu + m)v_\lambda(p) = 0\,, \qquad \bar{v}_\lambda(p)(\gamma^\mu p_\mu + m) = 0\,.$$

Show by using their explicit forms that u and v fulfil the following orthogonality relations

$$u^\dagger_\lambda(p)u_{\lambda'}(p) = v^\dagger_\lambda(p)v_{\lambda'}(p) = 2E\delta_{\lambda\lambda'}\,,$$
$$u^\dagger_\lambda(p)v_{\lambda'}(p_-) = v^\dagger_\lambda(p)u_{\lambda'}(p_-) = 0$$

with $p_- = (E, -\vec{p})^T$.

Hints:

Show first that

$$\chi^\dagger_\lambda(p)\chi_{\lambda'}(p) = \delta_{\lambda\lambda'}$$

by considering the two cases $\lambda = \lambda'$ and $\lambda \neq \lambda'$ separately.

Show also that

$$\chi^\dagger_\lambda(p)\chi_\lambda(p_-) = 0\,.$$

(The case for unequal λ, λ' is not needed here.)

These relations are sufficient for the rest of the sheet and the explicit form is not used in the following.

Exercise 2: Energy and Momentum of the Dirac Propagator

In analogy to exercise 1 on sheet 7, where we have calculated energy and momentum of the real Klein-Gordon field, we now consider the Dirac field. Its Lagrangian is given by

$$\mathcal{L} = \bar{\psi}(x)\,(i\gamma^\mu \partial_\mu - m)\,\psi(x)\,,$$

where ψ and $\bar{\psi}$ are considered as independent variables.
For the Dirac field we make the ansatz

$$\psi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E} \sum_{\lambda=-1,+1} a_\lambda(p) u_\lambda(p) e^{-ipx} + b^\dagger_\lambda(p) v_\lambda(p) e^{+ipx}\,.$$

(a) What is the corresponding energy-momentum tensor $T^{\mu\nu}$? Why does the term proportional to $g^{\mu\nu}$ vanish?

(b) Use this and the results of exercise 1 to calculate the 4-momentum vector

$$P^\mu = \int \mathrm{d}^3 x T^{0\mu}$$

and show that this leads to the form given in the lecture

$$: P^\mu : = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} p^\mu \sum_{\lambda=-1,+1} \left(\tilde{N}^a_\lambda(p) + \tilde{N}^b_\lambda(p) \right)$$

(c) Show that the current

$$j^\mu = \bar{\psi}(x)\gamma^\mu \psi(x)$$

is conserved.
Hint: This can be done without using the explicit form of ψ.

(d) The corresponding charge is given by

$$Q = \int \mathrm{d}^3 x j^0(x)\,.$$

Write the charge in terms of the variables a, $a^\dagger$, b and $b^\dagger$.