Sheet 2, problem 3

(b) - you can find alternative, more rigorous derivations - but this is the one which is rather straight and intuitive, as you were suggested -

Starting from the Jacobi identity,

$$[T^a T^k] f^{bck} + \underbrace{[T^b T^k]}_{i f^{bkm} T^m} f^{cak} + \underbrace{[T^c T^k]}_{i f^{ckm} T^m} f^{abk} = 0$$

hence $$[T^a T^k] f^{bck} = -i \left[f^{bkm} f^{cak} + f^{ckm} f^{abk} \right] T^m$$

given that $\mathrm{Tr}(T^m) = 0 \Rightarrow \mathrm{Tr}[T^a T^k] f^{bck} = 0$

$$\Longleftrightarrow \quad \mathrm{Tr}(T^a T^k) = \mathrm{Tr}(T^k T^a)$$

The trace must be a symmetric object in $(a \leftrightarrow k)$ therefore it can only be proportional to the identity,

$$\mathrm{Tr}(T^a T^k) = C(F)\, \delta^{ak}$$

c) Starting point:

$$\underbrace{[T^a \underbrace{[T^b T^c]}_{i f^{bck} T^k}]}_{}+ [T^b \underbrace{[T^c T^a]}_{i f^{cak} T^k}] + [T^c \underbrace{[T^a T^b]}_{i f^{abk} T^k}] = 0$$

$$\underbrace{i f^{bck}}_{-(F_b)_{ck}} \qquad \underbrace{i f^{cak}}_{-(F_c)_{ak}} \qquad \underbrace{i f^{abk}}_{-(F_a)_{bk}}$$

applying again $[T^a T^b] = i f^{abc} T^c$, we find

$$(F_b)_{ck} (F_a)_{km} + \underbrace{(F_c)_{ak}}_{-(F_a)_{ck}} (F_b)_{km} + (F_a)_{bk} (F_c)_{km} = 0$$

(using the antisymmetric property with respect to index exchange)

Hence: $$\underbrace{(F_b F_a)_{cm} - (F_a F_b)_{cm}}_{} = -(F_a)_{bk} (F_c)_{km}$$

$$- [F_a F_b]_{cm} = -i f_{abk} (F_k)_{cm}$$

$$\hookrightarrow \quad [F_a F_b]_{cm} = i f_{abk} (F_k)_{cm}$$

which is the same algebra as satisfied by the fundamental generators T.

<u>Note</u> $F_a(A)$ is just notation: "A" stands for "adjoint".

d) $\mathrm{Tr}\, F_a F_b = (F_a)_{ij}(F_b)_{ji} = -f_{aij}f_{bji} = f_{aij}f_{bij}$

as $f_{aij} \in \mathbb{R}$, the product is of course a real number.

e) $[F^2, F_a] = [F_b F_b, F_a]$

$\downarrow$

$F_b[F_b F_a] + [F_b F_a]F_b = iF_b f_{bac} T_c + i f_{bac} T^c F_b =$

$= i\underbrace{f_{abc}}_{\text{antisymmetric}}(F_b T_c + T_c F_b) = i f_{abc}\{F_b T_c\} = 0$

antisymmetric $\times$ symmetric $= 0$

Therefore, $[F^2, F_a] = 0 \Rightarrow F^2$ is an $SU(2)$ Casimir